Physics Letters A 173 (1993) 21–24
North-Holland

PHYSICS LETTERS A

How common is chaos?

J.C. Sprott
Department of Physics, University of Wisconsin, Madison, WI 53706, USA

Received 1 July 1992; accepted for publication 18 November 1992
Communicated by A.R. Bishop

The solutions of about 4×10^7 low-dimensional, low-order polynomial maps and ordinary differential equations were classified as either fixed point, limit cycle, chaotic, or unstable. Of those cases for which the solutions are stable, representing candidate models for the real world, typically a few percent were found to be chaotic.

It is now widely understood that complex behavior in nature may have a simple underlying cause. Several examples of low-dimensional nonlinear maps [1,2] and ordinary differential equations [3,4] with chaotic solutions have been extensively studied. On the other hand, many familiar processes are known to have regular periodic solutions. It would thus be of interest to quantify the issue of how common chaos is in nature. As a modest first step, this paper quantifies the occurrence of chaos in certain simple systems of nonlinear equations that are known to characterize a wide range of natural phenomena.

Take, for example, the logistic equation [1], which is a simple one-dimensional quadratic map:

$$X_{n+1}=\lambda X_n(1-X_n)\,. \tag{1}$$

Upon repeated iteration, the solution will do one of four things depending upon the initial condition (X_0) and the value of λ: (1) it will converge to a stable fixed point (a point attractor), (2) it will converge to a periodic series of distinct values (a limit cycle), (3) it will yield a nonperiodic series of values within some bounded range of X (a chaotic strange attractor), or (4) it will diverge (attract to infinity). For this simple case, the stable solutions are all in the range of $-2<\lambda<4$, and it is only within this range that the logistic equation represents a candidate model for a real physical process. We can thus examine this restricted range of λ which represents stable solutions to see what proportion of these are chaotic. In order to do this, the Lyapunov exponent [5]

$$\mu=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log_2|1-2X_n| \tag{2}$$

was calculated versus λ, and the result is shown in fig. 1. A positive Lyapunov exponent implies chaos. The chaotic regions contain an infinite number of periodic windows, but chaos occurs over 13% of the range. Thus we might infer that for the subset of physical processes for which the logistic equation is an adequate model, we would expect chaos to occur a similar fraction of the time.

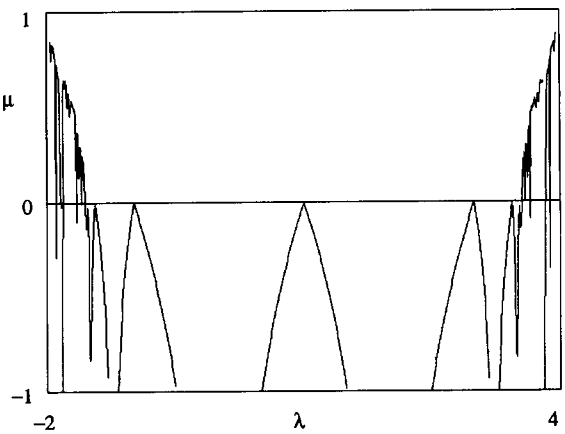

Fig. 1. Lyapunov exponent versus λ for the logistic map showing that the system is chaotic ($\lambda>0$) over 13% of the stable region from $-2<\lambda<4$.

To generalize the above result, it is necessary to examine a much wider class of functions [6]. Consider general, D-dimensional, Nth order polynomial maps given by

$$X_{d,n+1} = a_d + \sum_{i=1}^{D} a_{d,i} X_{i,n} + \sum_{i=1}^{D} \sum_{j=i}^{D} a_{d,i,j} X_{i,n} X_{j,n} + \dots . \quad (3)$$

The coefficients $a_{d,i,j,\dots}$ constitute a high-dimensional space, some portion of which contains chaotic solutions. For example, a three-dimensional cubic map has sixty coefficients. Since it is impractical to examine systematically such a sixty-dimensional space, a Monte Carlo technique was employed in which the coefficients were chosen randomly and uniformly over a hypercube centered on the origin of the space. Initial conditions were taken at the origin ($X_{d,0}=0$ for $1<d<D$), and the sign of the Lyapunov exponent was determined for each stable case after allowing 1000 iterations for the initial transient to disappear.

A practical difficulty is that of determining the appropriate size for the hypercube. Too small a value excludes all the chaotic solutions, and too large a value results in nearly all unstable solutions, which wastes computer time. Figure 2 shows the fraction of stable solutions and the portion of the stable solutions that are chaotic versus a_{max} (the maximum extent of the hypercube in each dimension) for a collection of about 10^7 two-dimensional quadratic maps whose coefficients form a twelve-dimensional space. The slope of the curve implies that the stable solutions occupy a (presumably fractal) subspace of dimension 5.6 ± 0.7. It appears to be a general feature of all the cases examined that the chaotic subspace has a dimension roughly half the dimension of the space of coefficients.

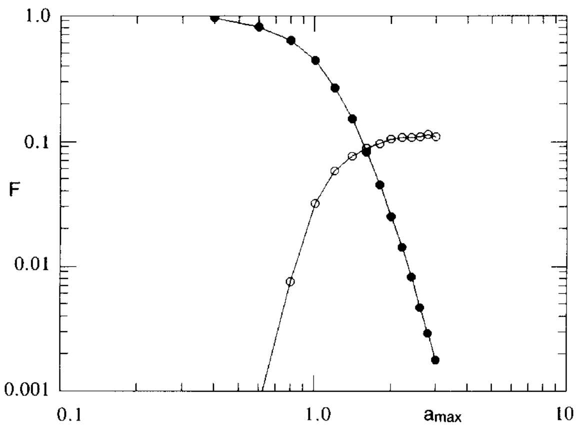

Fig. 2. The fraction of stable solutions within a hypercube of size $2a_{max}$ centered on the origin (solid circles) and the fraction of the stable solutions that are chaotic (open circles) for two-dimensional quadratic maps.

In the limit of large a_{max} where nearly all the stable solutions are sampled, the percent that are chaotic is asymptotic to a value of 11.10 ± 0.36 %. The relative estimated error is taken as the reciprocal of the square root of the number of chaotic cases. Successive runs with different random coefficients confirm that the results generally fall within the error bars. It was also verified that the results are insensitive to the chosen initial conditions.

The process described above was repeated for maps of various dimensions and orders, and the results are summarized in table 1. Note that whereas linear maps are never chaotic, cubic and higher-order maps are slightly less chaotic than are quadratic maps. Furthermore, there is a strong tendency for higher-dimensional maps to be less chaotic than low-dimensional maps. Both of these results are surprising and counter-intuitive. The chaotic percentage can be fit with an average error of about 10% to a power law of the form $34.9D^{-1.69}N^{-0.28}$.

Note also that limit cycles occur about a third of the time for all cases. The limit cycles were identified as those stable cases for which the largest Lyapunov exponent is less than 0.005 bits per iteration and successive iterates differ by more than 10^{-6}. Similar results were obtained by looking explicitly for periods up to 500; higher periods are relatively rare. Included in the limit cycles are a number of quasi-periodic solutions (tori) with negative Lyapunov exponents.

Since many processes in nature are better described by systems of ordinary differential equations (ODEs) than by difference equations (maps), it is of interest to ask how common chaos is in ODEs. The simplest ODEs that exhibit chaos are three-dimensional quadratic systems. The Lorenz and Rössler equations are of this class. Unfortunately, the study

Table 1
The percentage of stable solutions of various types for maps and ordinary differential equations of different dimensions and orders.

Type	Dimension	Order	Fixed point	Limit cycle	Chaotic
map	1	2	38.14±0.88 %	34.97±0.84 %	26.90±0.74 %
map	1	3	40.03±0.87 %	36.21±0.83 %	23.76±0.67 %
map	1	4	42.44±0.90 %	35.62±0.83 %	21.95±0.65 %
map	1	5	44.18±0.85 %	33.17±0.73 %	22.65±0.61 %
map	2	2	50.09±0.76 %	38.82±0.66 %	11.10±0.36 %
map	2	3	53.93±0.85 %	36.28±0.69 %	9.79±0.36 %
map	3	2	57.24±0.59 %	38.19±0.48 %	4.57±0.17 %
map	3	3	59.74±0.53 %	36.35±0.41 %	3.91±0.14 %
map	4	2	60.48±0.44 %	37.22±0.35 %	2.29±0.09 %
ODE	3	2	94.08±0.33 %	5.54±0.08 %	0.38±0.02 %
ODE	3	3	92.45±0.44 %	7.09±0.12 %	0.46±0.03 %
ODE	4	2	90.87±0.53 %	8.46±0.16 %	0.67±0.05 %

and classification of large numbers of ODEs by numerical methods is computationally demanding.

The behavior of ODEs was assessed by adding a term $X_{d,n}$ to the right hand side of eq. (3) and reducing $a_{\max}$ until the fraction of chaotic solutions no longer depends on $a_{\max}$. The result is a map whose successive iterates are nearly equal and whose behavior should mimic the behavior of the corresponding system of ODEs. This procedure is equivalent to solving the differential equations by the forward Euler numerical method with the largest fixed step size that gives statistically valid results. Figure 3 shows the

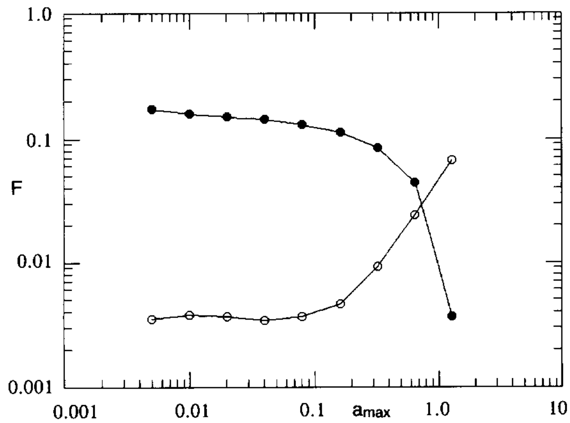

Fig. 3. The fraction of stable solutions within a hypercube of size $2a_{\max}$ centered on the origin (solid circles) and the fraction of the stable solutions that are chaotic (open circles) for three-dimensional quadratic ODEs.

variation in the fraction of stable solutions and the fraction of stable solutions that are chaotic versus $a_{\max}$ for three-dimensional quadratic ODEs. Other types are similar.

Using a value of $a_{\max}=0.01$, the results for three types of ODEs were calculated and summarized in table 1. Note that low-dimension, low-order ODEs are less chaotic than low-dimension, low-order maps but that ODEs tend to become more chaotic as the dimension and order increases, in direct contrast to the behavior of maps. For the three ODE cases considered, the chaotic percentage is given to within an average error of about 1% by $0.03D^2N^{0.5}$. Limit cycles are also relatively less common.

The above results may contain small systematic errors resulting from ambiguity in classifying marginal cases and uncertainty in the appropriate choice for $a_{\max}$, but the trends are believed to be significant. The extent to which natural processes are governed by low-dimension, low-order maps and ODEs with randomly chosen coefficients is a separate question not addressed in this Letter. However, a sampling of other maps involving trigonometric and other nonlinear functions produce similar results.

An interesting by-product of the study was the generation of about 35000 new examples of strange attractors which, like snowflakes, are all different and most of which are delightfully beautiful. A statistical analysis of this collection may provide further insight into the conditions under which chaos occurs in systems of nonlinear equations.

I am grateful to George Rowlands for kindling my interest in this subject and for many useful discussions. The manuscript was read by L. Durand, C. Goebel, M. Olsson, K. Symon, and L. Turner who each provided helpful comments.

References

[1] R. N. May, Nature 261 (1976) 459.
[2] M. Hénon, Commun. Math. Phys. 50 (1976) 69.
[3] E. N. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[4] O. E. Rössler, Phys. Lett. A 57 (1976) 397.
[5] A. M. Lyapunov, Ann. Math. Studies 17 (Princeton Univ. Press, Princeton, NJ, 1949).
[6] J.C. Sprott, Computers and graphics, accepted for publication.