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Chaos and Graphics

AUTOMATIC GENERATION OF STRANGE ATTRACTORS

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Abstract—A pair of coupled quadratic difference equations with randomly chosen coefficients is repeatedly iterated by computer to produce a two-dimensional map. The map is tested for stability and sensitivity to initial conditions. The process is repeated until a chaotic solution is found. In this way a computer can generate a large collection of strange attractors that are all different, and most of which have considerable aesthetic appeal. A simple computer program and examples of its output are provided. Many of the attractors have been systematically evaluated for visual appeal, and a correlation is found with the Lyapunov exponent and correlation dimension.

1. INTRODUCTION

Art and music derive much of their aesthetic appeal from a juxtaposition of order and unpredictability. In recent years it has come to be widely understood that simple mathematical equations can have solutions that, over the long term, are for all practical purposes unpredictable. And yet the simplicity of the equations ensures that the unpredictability is accompanied by a degree of determinism and order. Such equations are said to exhibit chaos[1], and their solutions usually form a strange attractor[2]. A strange attractor is an example of a fractal[3], a geometrical object of non-integer dimension and structure on all size scales, although the structure in general is not self-similar. Strange attractors are thus powerful generators of new visual new art forms[4].

A single equation with different coefficients can produce an almost endless variety of strange attractors, most of which have considerable beauty. Even simple personal computers can easily generate these patterns. The difficulty is that no one knows how to predict the conditions under which chaos will result, and thus the same standard examples are generally exhibited.

Whereas most previous work in the production of strange attractors starts with a system known to be chaotic, this paper proposes a way for a computer to search a large class of potentially chaotic equations for visually interesting solutions. The visual appeal of the resulting patterns is shown to correlate with mathematical quantities that characterize the attractors, suggesting that it might be possible to refine further the automatic selection of patterns with strong visual appeal.

2. TWO-DIMENSIONAL QUADRATIC MAPS

For a system to exhibit chaos, the governing equations must be nonlinear. A quadratic equation is perhaps the simplest such example. However, a single equation in a single variable has solutions that lie along segments of a curve and thus tend to be rather uninteresting. The graph of a quadratic equation is a parabola.

With a pair of equations involving two variables x and y, the solutions are more interesting and are well suited for display on a computer monitor or sheet of paper. The simplest such example is the two-dimensional quadratic map given in its most general form by

$$x_{n+1} = a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n y_n + a_5 y_n + a_6 y_n^2$$

$$y_{n+1} = a_7 + a_8 x_n + a_9 x_n^2 + a_{10} x_n y_n + a_{11} y_n + a_{12} y_n^2$$

The character of the solution is determined by the values of the twelve coefficients a_1 through a_{12} and the initial values x_0 and y_0.

With some initial value of x_0 and y_0 at $n = 0$, successive values of x and y are determined by repeatedly iterating the above equations. The iterates are plotted as points on a two-dimensional surface. After a number of iterations, the solution will do one of four things: (a) It will converge to a single fixed point; (b) it will take on a succession of values that eventually repeat, producing a limit cycle; (c) it will be unstable and diverge to infinity; (d) it will exhibit chaos and gradually fill in some often complicated but bounded region of the x-y plane.

The visually interesting solutions are the chaotic ones. Most of these solutions are strange attractors in that a range of starting values of x and y, within the basin of attraction, yield the same eventual solution. The first few iterates should be discarded, since they almost certainly lie off the attractor. Occasionally solutions are chaotic and visually appealing but are not attractors since each pair of initial values produces a different shape. The boundary of the basin of attraction of a strange attractor may itself be a fractal.

3. SENSITIVITY TO INITIAL CONDITIONS

To search automatically for chaotic solutions, it is necessary to have a criterion for detecting chaos. One such criterion is the sensitivity to initial conditions. Imagine iterating the two-dimensional quadratic map described above with two initial conditions that differ by a small amount. If successive iterates approach a fixed point or limit cycle, the difference between the two solutions will on average grow smaller with each iteration. If the solution is unstable or chaotic, the difference will tend to grow larger with each iteration. Unstable solutions can be eliminated by discarding

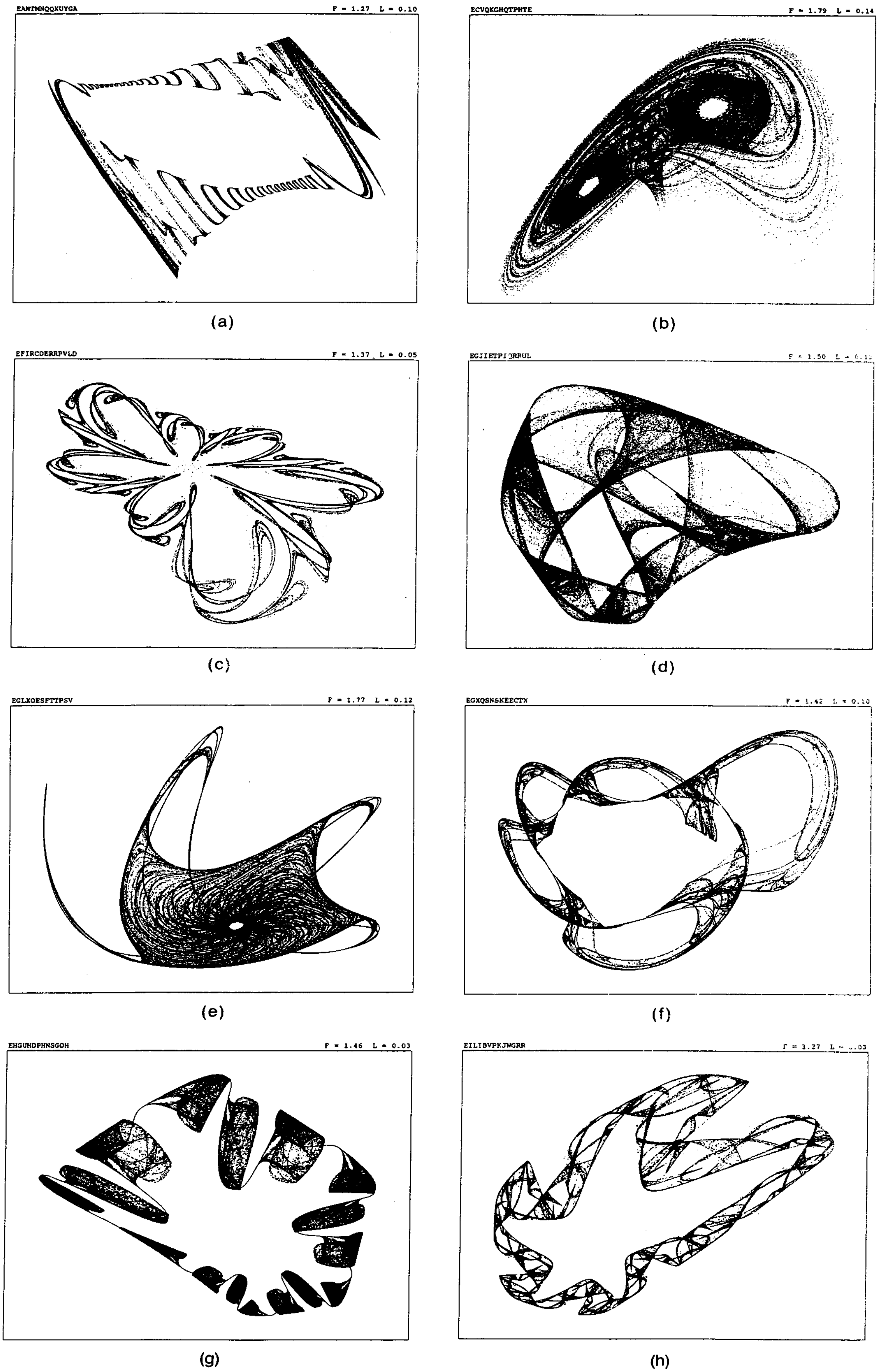

Fig. 1. Examples of strange attractors produced by two-dimensional iterated quadratic maps.

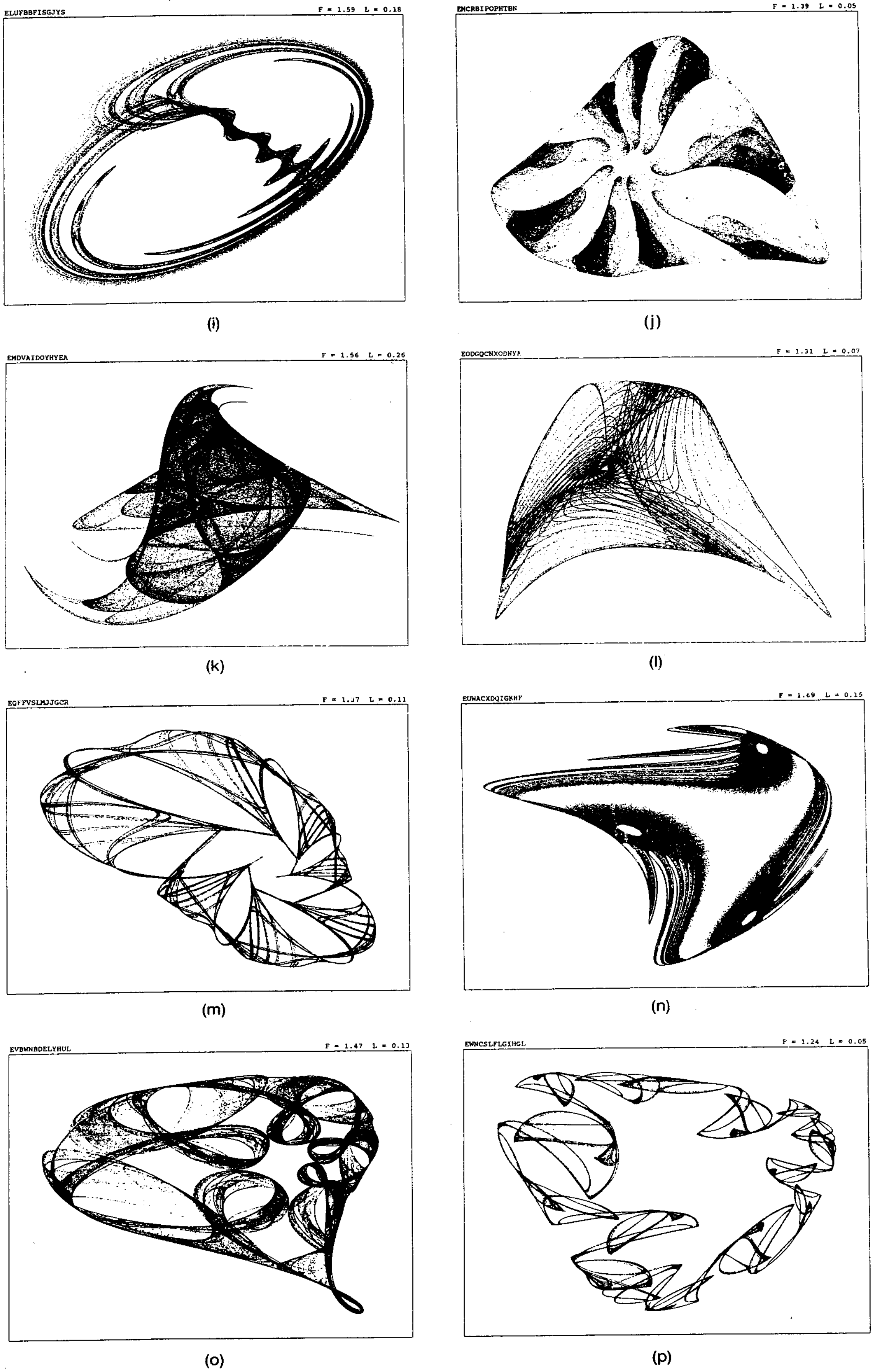
ELUFBBFISGJYS
F = 1.59 L = 0.18
(i)
EMCRBIPOPHTBN
F = 1.39 L = 0.05
(j)
EMDVAIDOYHYEA
F = 1.56 L = 0.26
(k)
EODGQCNXODNYA
F = 1.31 L = 0.07
(l)
EQFFVSLMJJGCR
L = 0.11
(m)
EUWACXDQIGKHF
F = 1.69 L = 0.15
(n)
EVBWNBDELYHUL
F = 1.47
(o)
EWNCSLFLGIHGL
F = 1.24 L = 0.05
(p)

Fig. 1. (*cont'd.*)

cases in which x or y grow beyond some arbitrary large value such as 10^6.

The difference between the two solutions initially grows on average at an exponential rate for a chaotic system. The rate of divergence is characterized by the Lyapunov exponent[5], which can be thought of as the power of 2 by which the separation increases on average for each iteration. Thus, if the separation doubles with each iteration, the Lyapunov exponent is 1 bit per iteration. The Lyapunov exponent can be thought of as the rate at which information about the initial condition is lost, or, equivalently, the rate at which the accuracy of a prediction declines as one projects farther into the future.

A two-dimensional map actually has two Lyapunov exponents, since a cluster of nearby initial points may expand in one direction and contract in another, stretching out like a cigar. The more positive one is the one that signifies chaos, and it is the one that dominates after a few iterations using the above procedure[6]. It has been conjectured[7] that the fractal dimension F is related to the two Lyapunov exponents through the relation

$$F = 1 - L_1/L_2$$

where L_1 is the more positive of the two exponents.

A further difficulty is that the two solutions eventually get far apart, on the order of the size of the attractor, and the growth saturates. This problem can be remedied if after each iteration the points are moved back to their original separation along the direction of the separation. The Lyapunov exponent is then determined by the average of the distance they must be moved for each iteration in order to maintain a constant separation. If the two cases are separated by a distance d_n after the nth iteration and the separation after the next iteration is d_{n+1}, the Lyapunov exponent is determined from

$$L = \Sigma \log_2(d_{n+1}/d_n)/N$$

where the sum is taken over all iterations from $n = 0$ to $n = N - 1$. After each iteration, the value of one of the iterates is changed to make $d_{n+1} = d_n$. For the cases here, d_n is taken equal to 10^{-6}.

Calculation of the Lyapunov exponent is only one possible way to identify chaotic attractors. One could also visually inspect all the stable solutions or look for cases with non-integer fractal dimension. Visual inspection is inefficient since only about 7% of the stable solutions of the two-dimensional quadratic maps are chaotic. Calculation of the fractal dimension is relatively time-consuming, typically requiring several thousand iterations. The Lyapunov dimension calculation is very fast and reliable. The exponential growth of the separation ensures that for most cases only a few iterations are required to determine the sign of the largest exponent.

4. COMPUTER SEARCH PROCEDURE

The procedure for implementing a computer search for strange attractors is straightforward. Choose the 12 coefficients a_1 through a_{12} randomly over some interval, choose initial conditions x_0 and y_0, iterate the equations for the map while calculating the Lyapunov exponent and checking for divergence, and keep only those solutions that are bounded and have a positive Lyapunov exponent.

A computer program* that repetitively performs these operations is listed in the Appendix. It is written in a primitive version of BASIC so as to be widely accessible and easily understood. The program should run without modification under Microsoft BASICA, GW-BASIC, QBASIC, QuickBASIC, or Visual BASIC for DOS, Borland International Turbo BASIC, and Spectra Publishing PowerBASIC on IBM PC or compatibles. It assumes VGA (640 × 480 pixel) graphics. If the hardware or BASIC compiler do not support this graphics mode, change the SCREEN 12 command line 130 to a lower number (*i.e.*, SCREEN 2 for CGA mode). A compiled BASIC and a computer with a math coprocessor are strongly recommended.

The coefficients are chosen in increments of 0.1 over the range −1.2 to 1.2 (25 possible values) in line 320. Smaller coefficients result in missing many chaotic solutions, and larger coefficients produce mostly unstable solutions. The increment was chosen so that each attractor is visibly different and coefficients can be coded into letters of the alphabet A through Y ($A = -1.2$, $B = -1.1$, *etc.*) for easy reference and replication. Thus each attractor is uniquely identified by a 12-letter name. The number of possible cases is thus 25^{12} or about 6×10^{16}. Of these, approximately 1.6% are chaotic or about 10^{15} cases[8]. Viewing them all at a rate of one per second would require over 30 million years! Thus it is very unlikely that any patterns produced by the program will ever have been seen before, and like snowflakes, nearly all of them are different.

Initial conditions are set arbitrarily to $x = y = 0.05$ in line 310. Other small initial values produce the same result for most cases as expected for an attractor. The Lyapunov exponent is calculated using an initial condition in which x is increased by 10^{-6} (line 310). The program performs 100 iterations before considering the Lyapunov exponent (line 640). After 100 iterations, the program begins keeping track of the minimum and maximum values of x and y (lines 520–550) so that after 1000 iterations the screen can be cleared and resized to allow a 10% border around the attractor (line 560). If 11,000 iterations are reached with a positive Lyapunov exponent and a bounded solution, the result is assumed to be a strange attractor (line 620). The search immediately resumes after each attractor

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* An IBM DOS disk containing the BASIC source code in the Appendix, an executable version of the code, and a more versatile menu-driven strange attractor program with 3-D glasses are available for $30 postpaid from the author. Specify 3.5 or 5.25-inch disk.

is confirmed and continues until a key is pressed (line 650).

The search procedure is surprisingly fast. On a 33 MHz 80486 computer running QuickBASIC 4.5, the program finds about 1200 strange attractors per hour. The listing in the Appendix only displays the attractors on the screen. A more versatile program would call a subroutine from line 620 to print the attractors, perhaps after user confirmation or evaluation, or would save the coded coefficients in a disk file for later analysis.

5. SAMPLE STRANGE ATTRACTORS

Figure 1 shows samples of the shapes that arise from the iteration of such two-dimensional quadratic maps. These cases are all strange attractors and were selected for their beauty and diversity from a much larger collection. However, they are by no means atypical, and there are many others that would have served equally well. It is remarkable that such a diversity of shapes comes from the same simple set of equations with only different numerical values of the coefficients.

The cases shown were produced on a laser printer with 300 dots per inch resolution on an 8.5 × 11-inch page after about 500,000 iterations. Of course, the program needs modification to output the plots to a printer at high resolution. However, satisfactory results can be obtained by any of the various utilities that allow one to print a screen image.

Also shown on each figure is the code name preceded by the letter E to denote a two-dimensional quadratic map, the Lyapunov exponent L (in bits per iteration) and the fractal dimension F. F is actually the correlation dimension [9] and is somewhat ill-defined because the dimension of a strange attractor varies somewhat with scale. The dimension is taken here at a scale of about 1% of the largest diameter of the attractor.

Normally, correlation dimension calculations are very slow because they involve determining the spatial separation between every pair of points that constitute the attractor. A much faster technique that entails only a slight loss in accuracy was used here. The method requires that the coordinates of the last N iterates be retained. A value of $N = 500$ is generally sufficient. With each new iteration, one of the previous N points is chosen randomly, and its separation from the new point is calculated. A count is kept of those cases for which the separation is less than each of two values, which differ by a factor of 10, and whose geometric mean is the size scale for which the dimension is to be calculated. If the respective counts are N_1 (for the smaller value) and N_2 (for the larger value), the correlation dimension is given by

$$F = \log_{10}(N_2/N_1)$$

With little computational penalty, the value of F can be updated whenever N_1 or N_2 is incremented. The accuracy of the dimension estimate is of order $N_1^{-1/2}$.

6. AESTHETIC EVALUATION

A collection of about 7500 such attractors was systematically examined by the author and seven volunteers, including two graduate art students, a former art history major, three physics graduate students, and a former mathematics major. All evaluators were born and raised in the United States. The evaluations were done by choosing attractors randomly from a collection of about 18,000 and displaying them sequentially on the computer screen without any indication of the quantities that characterize them. The volunteers were asked to evaluate each case on a scale of one to five according to its aesthetic appeal. It only took a few seconds for each evaluation.

At the end of the session a graph similar to Fig. 2 was produced in which the average rating is displayed using a gray scale on a plot in which the largest Lyapunov exponent (L) and correlation dimension (F) are the axes. The darkness of each box increases with the average rating of those attractors whose values of L and F fall within the box. Figure 2 shows a summary of all the evaluations, although the cases examined by the various individuals show a similar trend. In particular, all evaluators tended to prefer attractors with a dimension between about 1.1 and 1.5 and a Lyapunov exponent between zero and about 0.3. Some of the most interesting cases have Lyapunov exponents below about 0.1.

The dimension preference is perhaps not surprising since many natural objects have dimensions in this range. The Lyapunov exponent preference is harder to understand, but it suggests that strongly chaotic systems are too unstructured to be appealing. For the 443 cases that were rated five (best) by the evaluators, the average correlation dimension was $F = 1.30 \pm 0.20$, and the average Lyapunov exponent was $L = 0.21 \pm 0.13$ bits per iteration, where the errors represent plus or minus one standard deviation. About 28% of the cases evaluated fall within the error bars.

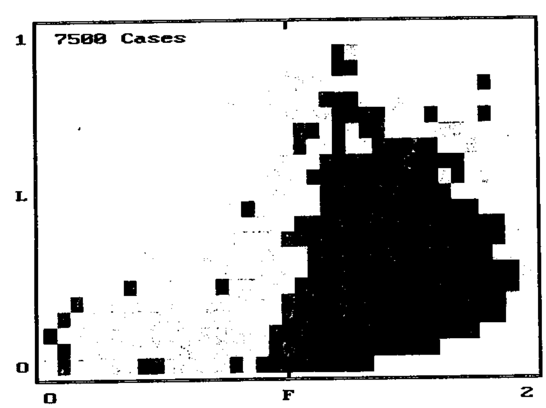

Fig. 2. Results of evaluating 7500 strange attractors, showing that the most visually appealing cases are those with small Lyapunov exponents (L) and with correlation dimensions (F) somewhat greater than one.

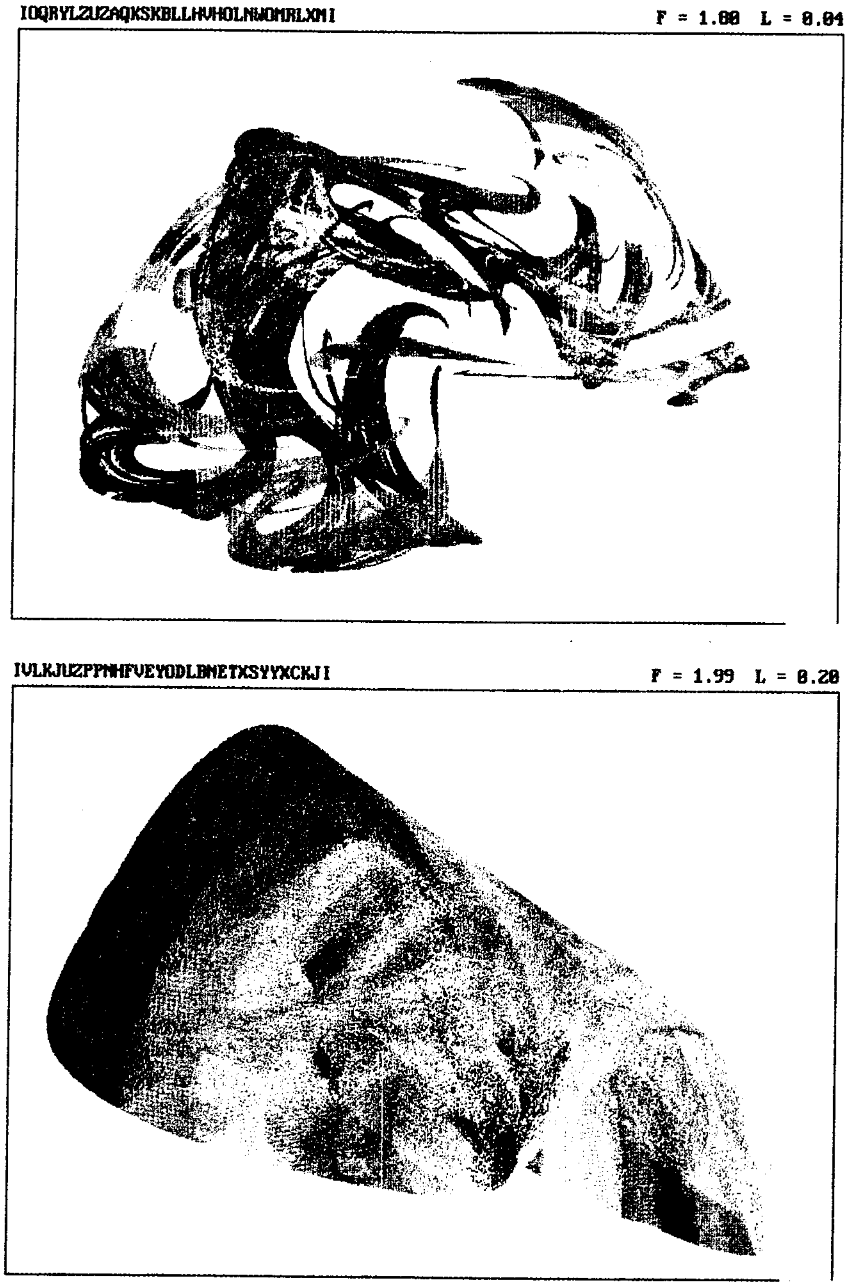

Fig. 3. Examples of strange attractors produced by three-dimensional iterated quadratic maps in which the color is determined by one of the variables.

7. SUGGESTIONS FOR FURTHER WORK

The method described above can be easily extended in a number of ways[10]. There is nothing special about two-dimensional quadratic maps, other than perhaps simplicity and consequent computational speed. There's an infinity of other nonlinear maps and flows. For example, Pickover has produced nice two-dimensional sculptures using trigonometric maps[11]. It is straightforward to add cubic and higher order terms to the equations. More complicated nonlinearities do not significantly enhance the occurrence of chaotic solutions, but they do somewhat increase the variety of patterns. The number of coefficients increases rapidly with the order of the polynomial, and the variety of cases becomes even larger.

Having found a visually appealing attractor, one can make small variations of the coefficients to optimize even further its appearance. The attractors can be animated by producing a succession of frames, each with a slightly different value of one or more of the parameters.

Adding a third dimension (x, y, and z) increases the number of coefficients to 30 for quadratic maps. It also raises interesting possibilities for new display modes. The simplest case is to plot x and y, but to ignore z, which is equivalent to viewing the projection (or shadow) of the attractor on the x-y plane. Alternately, the attractor can be projected onto the x-z or y-z plane or rotated through an arbitrary angle. A gray scale can be used to represent the number of iterates that fall within a given rectangle on the screen[12].

Another possibility is to code the third dimension in color. Examples of three-dimensional quadratic maps using 16 colors are shown in Fig. 3. These figures were produced directly from VGA screen images using a color ink-jet printer. Some computer languages allow one to cycle through a variety of color palettes to find the most pleasing combination of colors or to produce a kind of animated color display. Modern versions of BASIC have such a PALETTE command.

It is also possible to produce an anaglyph[13] in which each x-y value is plotted twice, once in red and once in cyan, displaced horizontally by a distance proportional to z so as to produce a three-dimensional monochrome image when viewed through red/blue glasses. Color three-dimensional images can be produced by plotting the two colored views side-by-side and either viewing them cross-eyed or through an inexpensive prism stereoscope.†

These techniques can also be applied to two- and even one-dimensional maps by using a previous value of one of the variables as the third variable. The attractors can be rotated to provide a view from the most pleasing angle or animated with successively rotated images.

Chaotic maps can also be used to produce a crude kind of computer music. For a two-dimensional map, x might be used to control the pitch and y the duration of each note. The result is a not-displeasing though alien-sounding form of music that might appeal to those with exotic musical tastes.

The method described above can also be applied to systems of nonlinear ordinary differential equations whose solutions are continuous flows rather than discrete maps. In such a case, chaos requires at least three equations and three variables. Differential equations can be solved approximately on a digital computer by reducing them to appropriate finite difference equations[14]. Long computing times are required for high accuracy, which fortunately is not essential in this application.

Plots of the basin of attraction for strange attractors are sometimes very beautiful, especially when multiple nearby attractors compete and produce a fractal boundary. The popular Mandelbrot and Julia sets are basins of attraction. It is traditional to plot in different colors the number of iterations required for each unstable initial condition to reach some large value. Such plots require a large amount of computer time, however.

† Stereoscopes and other 3-D supplies are available from Reel 3-D, P.O. Box 2368, Culver City, CA 90231.

Much more could be done with correlating the aesthetic appeal of the attractors with the various numerical quantities that characterize them. The Lyapunov exponent and correlation dimension are only two such quantities; there are infinitely many others[15]. One could determine if there are discernible differences between the preferences of scientists and artists. Preliminary indications suggest that complexity might appeal more to artists than to scientists, who tend to see beauty in simplicity. There may be discernible cultural differences. One could determine whether the results are the same for more complicated systems of equations and for different methods of displaying the results, such as color versus monochrome.

If such correlations exist, then it should be possible to program the computer to be even more selective and to become a critic of its own art[16]. Like the infinite number of monkeys with an infinite number of typewriters who will eventually reproduce all the works of Shakespeare, so too the computer starting with random numbers might evolve into something of an artist with unparalleled stamina and productivity.

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APPENDIX

```
110 DEFDBL A-Z: DIM A (12)
120 RANDOMIZE TIMER                     'Reseed random number generator
130 SCREEN 12                           'Assume VGA graphics
140 GOSUB 300                           'Set parameters
150 GOSUB 400                           'Iterate equations
160 GOSUB 500                           'Display results
170 GOSUB 600                           'Test results
180 ON T% GOTO 130, 140, 150
190 END

300 REM Set parameters
310 X = .05: Y = .05: XE = X + .000001: YE = Y
320 For I% = 1 to 12: A(I%) = .1 * (INT(25 * RND) - 12): NEXT I%
330 T% = 3: LSUM = 0: N = 0
340 XMIN = 1000000#: XMAX = -XMIN: YMIN = XMIN: YMAX = XMAX
390 RETURN

400 REM Iterate equations
410 XNEW = A(1) + X * (A(2) + A(3) * X + A (4) * Y) + Y * (A(5) + A(6) * Y)
420 YNEW = A(7) + X * (A(8) + A(9) * X + A(10) * Y) + Y * (A(11) + A(12) * Y)
430 N= N + 1
490 RETURN

500 REM Display results
510 IF N < 100 or N > 1000 THEN GOTO 560
520    IF X < XMIN  THEN XMIN  = X
530    IF X > XMAX THEN XMAX = X
540    IF Y < YMIN  THEN YMIN  = Y
550    IF Y > YMAX THEN YMAX = Y
560 IF N = 1000 THEN GOSUB 800
570 IF X > XL AND X < XH AND Y > YL AND Y < YH AND N > 1000 THEN PSET (X, Y)
590 RETURN

600 REM Test results
610 GOSUB 700                                        'Calculate Lyapunov exponent
620 IF N > 11000 THEN T% = 2                         'Strange attractor found
630 IF ABS (XNEW) + ABS (YNEW) > 1000000# THEN T% = 2  'Unstable
640 IF N > 100 AND L < .005 THEN T% = 2              'Limit cycle
650 IF LEN(INKEY$) THEN T% = 0                       'User key press
660 X = XNEW: Y = YNEW
690 RETURN

700 REM Calculate Lyapunov exponent
710 XSAVE = XNEW: YSAVE = YNEW: X = XE: Y = YE: N = N - 1
720 GOSUB 400                                        'Reiterate equations
730 DLX = XNEW - XSAVE: DLY = YNEW - YSAVE: DL2 = DLX * DLX + DLY * DLY
740 DF = 1000000000000# * DL2: RS = 1# / SQR (DF)
750 XE = XSAVE + RS * (XNEW - XSAVE) : YE = YSAVE + RS * (YNEW - YSAVE)
760 XNEW = XSAVE: YNEW = YSAVE
770 LSUM = LSUM + LOG(DF) : L = .721347 * LSUM / N
790 RETURN

800 REM Resize the screen (and discard the first thousand iterates)
810 DX = .1 * (XMAX - XMIN) : DY = .1 * (YMAX - YMIN)
820 XL = XMIN - DX: XH = XMAX + DX: YL = YMIN - DY: YH = YMAX + DY
830 IF XH - XL < .000001 OR YH - YL < .000001 THEN GOTO 890
840    WINDOW (XL, YL) - (XH, YH) : CLS
850    LINE (XL, YL) - (XH, YH) , , B
890 RETURN
```