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Predicting the dimension of strange attractors

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Abstract

The correlation dimension was calculated for a collection of 6080 strange attractors obtained numerically from low-degree polynomial, low-dimensional maps and flows. It was found that the average correlation dimension scales approximately as the square root of the dimension of the system with a surprisingly small variation. This result provides an estimate of the number of dynamical variables required to characterize an experiment in which a strange attractor has been found as well as an estimate of the dimension of attractors produced by chaotic systems in which the dimension of the state space is known.

It has become fashionable to search for simple determinism (chaos) in fluctuating, non-periodic, experimental data. This is often done by calculating the correlation dimension [1] from a time-series record using the method of time-delay reconstruction [2,3]. This method has been applied to systems as diverse as the stock market [4], sunspots [5], rainfall [6], electrocardiograms [7], electroencephalograms [8], and childhood epidemics [9]. Such studies are motivated by the hope that a strange attractor with low fractional dimension will be found, in which case it might be possible to model the dynamics using a number of variables as small as the next higher integer.

It is useful to consider whether the dimension of an attractor provides any additional information about the dimension of the system that produced it. This paper addresses this question by calculating the distribution of correlation dimensions of strange attractors produced by various systems of equations. This novel statistical approach should provide guidance in modeling chaotic data and in searching for low-dimensional attractors in systems whose state-space dimension is known.

Consider first the case of general iterated N-th degree polynomial D-dimensional maps given by

$$X_{d,n+1} = a_d + \sum_{i=1}^{D} a_{d,i} X_{i,n} + \sum_{i=1}^{D} \sum_{j=i}^{D} a_{d,i,j} X_{i,n} X_{j,n} + \dots . \qquad (1)$$

The real coefficients $a_{d,i,j,\dots}$ constitute a high-dimensional control space, some portion of which contains bounded solutions for initial conditions near the origin ($X_{i,0}=0$ for $1 \leqslant i \leqslant D$). This subspace of bounded solutions contains candidate models for a wide variety of physical processes. It has been shown [10] that the bounded solutions are clustered within a distance of order unity from the origin of the control space, and that within this region, the fraction of chaotic solutions approaches an asymptotic value that depends weakly on N and more strongly on D. Furthermore, the chaotic solutions obtained by random choices of control parameters are nearly always visually distinct

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strange attractors [11]. This technique thus provides a way of sampling the attractors that can arise from iterated polynomial maps and determining the distribution of their dimensions and other characteristics. To the extent that arbitrary rational functions can be Taylor expanded, polynomials model a wide range of physical phenomena. Furthermore, it will be shown that the results are not sensitive to the degree of the polynomial and that selected nonpolynomial forms give similar results.

The numerical procedure consists of choosing values of each control parameter randomly and uniformly over the interval $(-a_{max}, a_{max})$ and iterating Eq. (1) while testing the solution for boundedness ($|X|<10^6$) and sensitivity to initial conditions (positive Lyapunov exponent [12,13]). Thus the control space is a hypercube with linear dimension $2a_{max}$ centered on the origin and sampled uniformly over its hypervolume. The value of a_{max} was adjusted for each combination of D and N so that about 99% of the cases were unbounded. This criterion reduces the chance that the chaotic cases are somehow atypical. For example, it has been shown [10] that this criterion is sufficient for a statistically valid measure of the fraction of bounded cases that exhibits chaos. Values of a_{max} ranged from about 13 at $D=1$ to about 0.5 at $D=9$.

For those cases that were chaotic, the correlation dimension was calculated from the slope of the correlation integral using 10^5 points, each correlated with 50 randomly chosen points from the previous 900 [14]. The resulting 5×10^6 correlations are expected to provide an accurate measure of attractor dimensions up to about 3.4, which encompasses nearly all of the cases studied [15]. The slope was calculated at a scale on the order of 1% of the size of the attractor (the two most distant points). The calculated correlation dimension is accurate to a few percent when applied to standard cases such as the Logistic [16], Hénon [17], Lorenz [18], and Rössler [19] attractors, and to data from a generator of high-quality uniform pseudorandom numbers.

For a collection of 3840 chaotic maps equally distributed over 24 combinations of $1\leqslant D\leqslant 5$, $2\leqslant N\leqslant 5$ and $6\leqslant D\leqslant 9$, $N=2$, the average correlation dimension was determined by multiple linear regression to fit approximately the function $F\approx 0.84D^{0.45}N^{0.03}$. The dependence on N is weak and not statistically significant. The dependence on D is consistent with the square root.

The procedure described above was extended to systems of ordinary differential equations (ODEs) with dimension D and polynomial terms of degree N of the form

$$\mathrm{d}X_d/\mathrm{d}t = a_d + \sum_{i=1}^{D} a_{d,i}X_i + \sum_{i=1}^{D}\sum_{j=1}^{D} a_{d,i,j}X_iX_j + \dots \quad (2)$$

The equations were solved using a second-order Runge-Kutta technique [20] with a step size of $\delta t=0.1$, and initial conditions $X_i=0.05$ for $1\leqslant i\leqslant D$. The statistical results are insensitive to initial conditions, although values too far from the origin decrease the fraction of bounded solutions. A collection of 2240 chaotic flows equally distributed over 14 combinations of $3\leqslant D\leqslant 5$, $2\leqslant N\leqslant 5$ and $6\leqslant D\leqslant 7$, $N=2$ were examined. Chaotic flows with $D\leqslant 2$ are precluded by the Poincaré-Bendixon theorem, which states that in two dimensions, the most complicated bounded trajectory is a limit cycle. The absence of chaotic solutions in a search of two-dimensional flows is a confirmation of the numerical criterion used to identify chaos.

The correlation dimension fits the function $F\approx 1.13D^{0.30}N^{-0.05}$. Again, the dependence on N is weak and not statistically significant, and the dependence on D is only slightly weaker than in the case of maps. The results for both maps and ODEs are summarized in Fig. 1 in which the N-dependence has been ignored and the error bars indicate ± one standard deviation for each of the 3840 cases. Although the error bars are large, they are sufficient to preclude, for example, a linear dependence of F on D at about a 95% confidence level. Also note that the error bars represent a spread of dimensions about the average rather than an uncertainty in the determination of the average dimension, whose scaling is the subject of this paper.

A histogram was constructed of the relative probability of the correlation dimension normalized to the square root of the system dimension for all 6080 chaotic cases, including both maps and flows. The result as shown in Fig. 2 is strongly peaked with a mean of about 0.81 and a standard deviation of 0.21. Perhaps

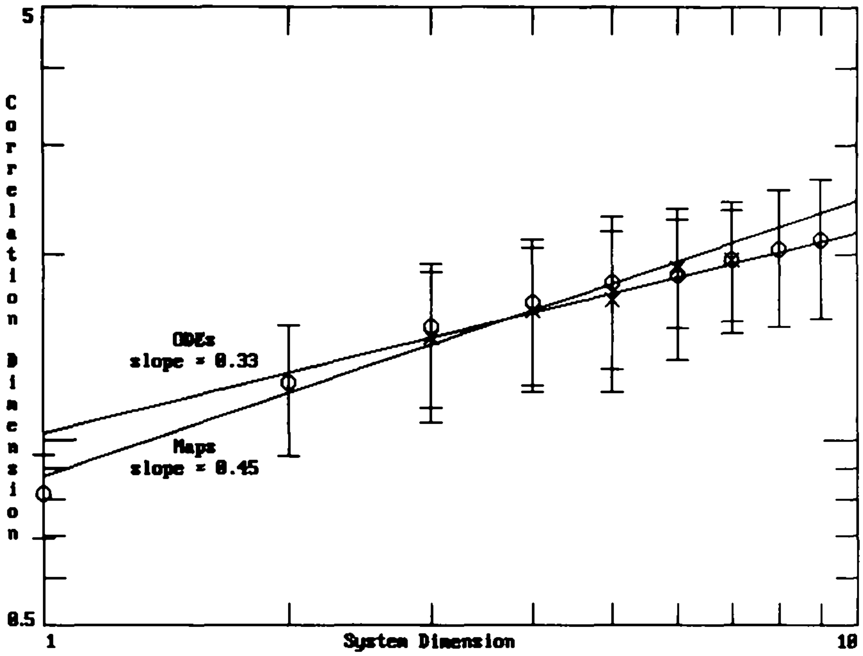

Fig. 1. The average correlation dimension of 6080 chaotic attractors scales approximately as the square root of the dimension of the system for low-order polynomial maps (○) and flows (×). The error bars represent the spread in dimensions, not an uncertainty in the average values.

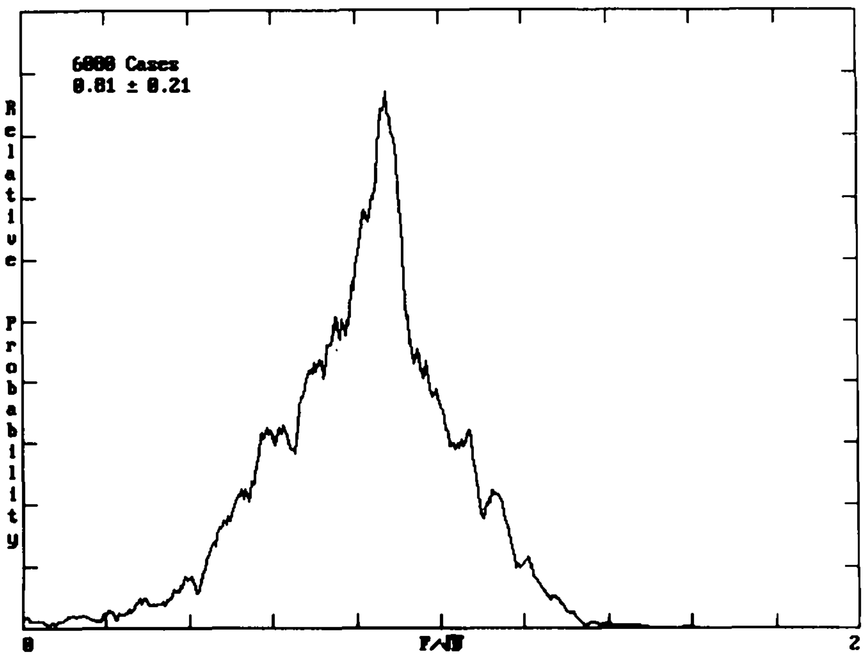

Fig. 2. The correlation dimension has a high probability of a value about 0.81 times the square root of the dimension of the system.

even more remarkable is the almost total absence of attractors with correlation dimensions greater than about $1.3\sqrt{D}$.

The probability distribution is so strongly peaked as to suggest that these attractors might in some sense all be the same attractor, perhaps transformed in some trivial way. This possibility seems unlikely since the various cases bear little visual resemblance to one another [21]. Furthermore, the Lyapunov exponents are much more uniformly distributed. The average value of the largest Lyapunov exponent for the maps scales as $L \approx 0.60 D^{-1.24} N^{-0.03}$ bits/iteration and for the flows scales as $L \approx 0.55 D^{-0.84} N^{1.01}$ bits/s. The dependence of Lyapunov exponent on N is weak for the maps, but not for the flows. Fig. 3 shows the Lyapunov exponent plotted versus D for the maps and ODEs with the N-dependence ignored.

The spread in Lyapunov exponents is best illustrated by plotting the probability of various combinations of L times D for the maps only. Such a plot is shown in Fig. 4. It is very weakly peaked with a mean of 0.45 bits/iteration and a standard deviation of 0.27 bits/iteration. Contrast this result with the much more sharply peaked correlation dimension in Fig. 2.

It is interesting to ask whether these results apply to systems that are not polynomials or that have much higher dimensions. A selection of 480 three-dimensional systems involving absolute values, non-integer powers, and sine functions was examined, and the results as shown in Table 1 are similar to the polynomial cases.

Also shown in Table 1 are the values of $F/\sqrt{D}$ for some standard chaotic systems. In addition to those previously mentioned, the list includes the Zaslovskii map [22], the Kaplan-Yorke map [23] and the ODE systems of Rabinovich-Fabrikant [24] and of Rössler hyperchaos [25]. The value of F for several of these cases is the Lyapunov dimension [26], which is technically an upper bound of the correlation dimension [27] but for our purposes is indistinguishable from it.

High-dimensional numerical examples in which the attractor dimension has been accurately calculated are relatively rare. One particularly interesting case [28] is a heroic solution of the Navier Stokes equation for turbulent Poiseuille flow at a Reynolds number of 3200, requiring 400 h of Cray 2 time. The infinite dimensional partial differential equation was ap-

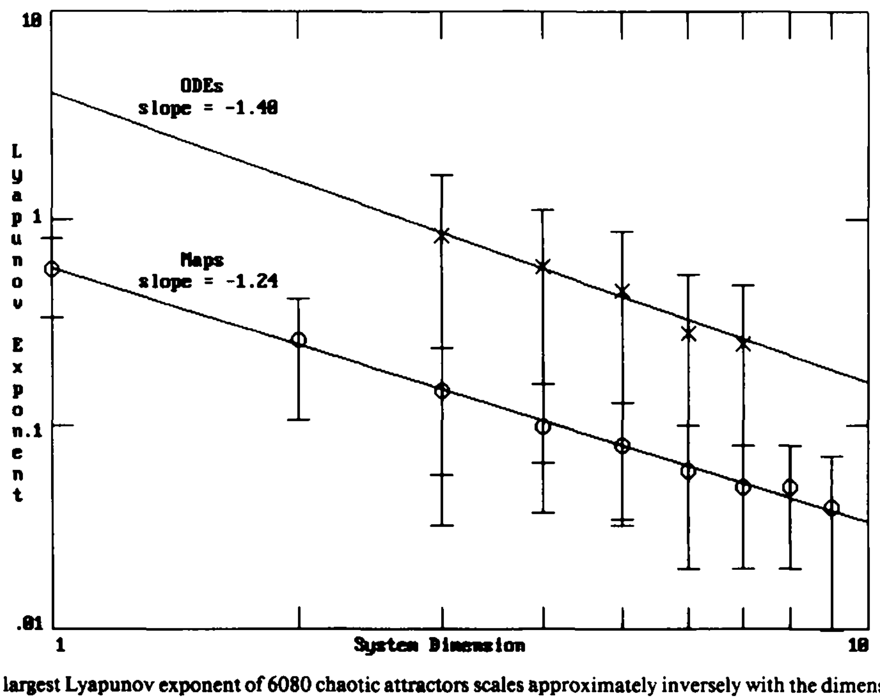

Fig. 3. The average largest Lyapunov exponent of 6080 chaotic attractors scales approximately inversely with the dimension of the system for low-order polynomial maps (○) and flows (×). The error bars represent the spread in exponents, not an uncertainty in the average values.

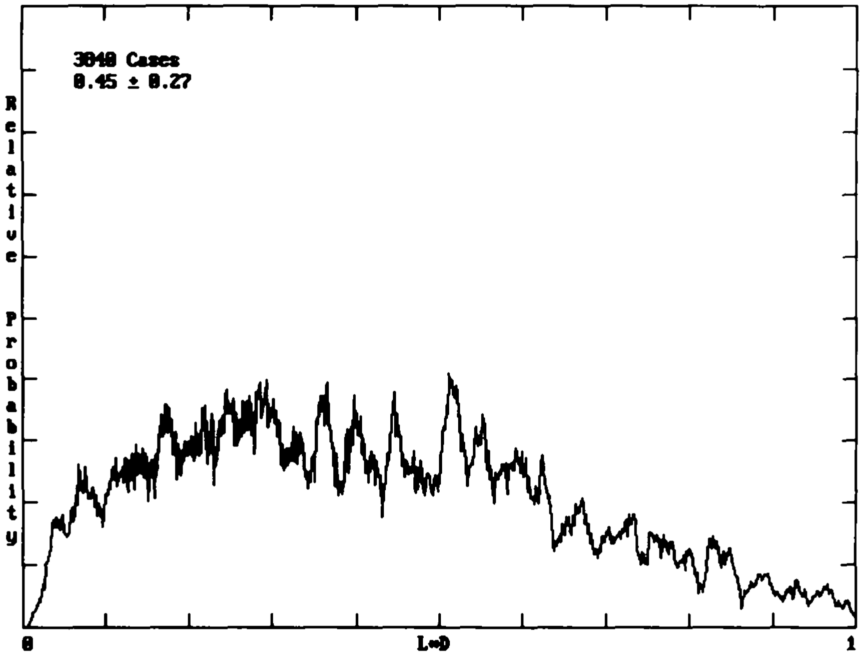

Fig. 4. The average value of the largest Lyapunov exponent for chaotic maps has a relatively uniform probability when multiplied by the system dimension.

Table 1
Average correlation dimensions (F) of various D-dimensional chaotic systems

System	D	F	$F/\sqrt{D}$
polynomials	1-9		0.81±0.21
absolute values	3	1.10	0.64±0.24
non-integer powers	3	1.40	0.80±0.24
sines	3	1.43	0.83±0.27
logistic map [16]	1	0.5–1	0.50–1.00
Hénon map [17]	2	1.21	0.85
Zaslovskii map [22]	2	1.38	0.98
Kaplan–Yorke map [23]	2	1.43	1.01
Lorenz attractor [18]	3	2.05	1.18
Rössler attractor [19]	3	2.05	1.18
Rabinovich–Fabrikant [24]	3	2.19	1.26
Rössler hyperchaos [25]	4	3.01	1.50
Navier–Stokes equation [28]	6930	360	4.32
earth's atmosphere [29]	1977	20–100	0.45–2.25

proximated by a system of 6930 ODEs, and the Lyapunov dimension was estimated to be about 360.

High-dimensional experimental examples in which the attractor dimension has been accurately measured are even more rare. Pierrehumbert [29] attempted to estimate the correlation dimension of the atmospheric temperature at the 500 mbar level over a grid of 1977 points in the northern hemisphere using winter monthly averaged data from the years 1950 to 1980. He estimated the correlation dimension in this 1977-D embedding to be in the range of 20 to 100.

It is remarkable that all of these examples have values of $F/\sqrt{D}$ of order unity. However, it is certainly possible to construct examples in which $F/\sqrt{D}$ differs significantly from unity. One can imagine a one-dimensional chaotic system such as the logistic map coupled very weakly to a high-dimensional non-chaotic system, in which case $F/\sqrt{D}$ could be made arbitrarily small. At the other extreme, a large collection of chaotic attractors can be weakly coupled to produce a system in which F/D is of order unity. Lorenz has proposed and examined such a system [30]. Coupled lattice maps [31,32] consisting of logistic maps or tent maps as well as neural networks [33] are generally assumed to have fractal dimensions that scale linearly with the system dimension, although recent calculations [34] using a modified version of the Grassberger–Procaccia algorithm produce results that are equally consistent with a square root dependence.

Although the results presented here suggest that a chaotic system with dimension D will likely produce an attractor of dimension near $\sqrt{D}$, it does not logically follow that an attractor of dimension F most probably came from a system with dimension near F^2. A given system can produce many different attractors as the control parameters are varied, and the same attractor can presumably be produced by more than one system of equations. However, the estimate of F^2 provides a reasonable starting point for modeling a natural system whose dynamics exhibit a strange attractor of dimension F.

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