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Simplest dissipative chaotic flow

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Abstract

Numerical examination of third-order, autonomous ODEs with one dependent variable and quadratic nonlinearities has uncovered what appears to be the algebraically simplest example of a dissipative chaotic flow, $\dddot{x} + A\ddot{x} - \dot{x}^2 + x = 0$. This system exhibits a period-doubling route to chaos for $2.017 < A < 2.082$ and is approximately described by a one-dimensional quadratic map. © 1997 Published by Elsevier Science B.V.

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Many of the general features of chaotic systems have been understood through study of the quadratic map

$$x_{n+1} = x_n^2 - A \tag{1}$$

and its variants such as the logistic map [1], $x_{n+1} = Ax_n(1 - x_n)$. The quadratic map is the algebraically simplest example of a chaotic map in the sense that it contains the smallest number of terms (2) and the simplest differentiable nonlinearity (x^2). Eq. (1) is chaotic over the range $1.4011\ldots < A < 2$ except for an infinite number of periodic windows with small but finite measure ($\sim 10\%$).

With chaotic flows, governed by differential equations, there is no single correspondingly simple prototypical chaotic system. The Poincaré–Bendixson theorem [2] requires that autonomous first-order ordinary differential equations with continuous functions be at least three-dimensional to have bounded chaotic solutions. Standard examples include the autonomous Lorenz [3] and Rössler [4] attractors and various periodically driven systems such as the Ueda [5] oscillator which can be recast into autonomous form. Here we propose what appears to be the algebraically simplest example of a dissipative chaotic flow and demonstrate that it exhibits continuous dynamics analogous to the quadratic map. This work follows Lorenz [3] who showed that his attractor is governed approximately by a tent map and Hénon [6] whose 2-D dissipative map was constructed to model the features of the Lorenz attractor. Olsen and Degn [7] also did a similar calculation for the Rössler attractor.

An earlier paper [8] described a computer search that revealed nineteen examples of chaotic flows that are algebraically simpler than the Lorenz and Rössler systems. Recently, Gottlieb [9] suggested examining

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a subclass of such systems given by $\dddot{x} = j(x, \dot{x}, \ddot{x})$, where j is a jerk function (time derivative of acceleration). We report here the result of such a search in which j contains a minimum number of terms and is at most a quadratic function of $\ddot{x}$, $\dot{x}$ and x. The most general such expression is

$$j = (a_1 + a_2 x + a_3 \dot{x} + a_4 \ddot{x})\ddot{x} + (a_5 + a_6 x + a_7 \dot{x})\dot{x} + (a_8 + a_9 x)x + a_{10}, \tag{2}$$

corresponding to a force F per unit mass that satisfies $\mathrm{d}F/\mathrm{d}t = j$.

Since functions of the form of Eq. (2) would not represent a simplification of cases previously found [8] unless fewer than four of the terms are non-zero, the numerical procedure [10] was to search all cubic subsets of the ten-dimensional control space of coefficients (a_1 through a_{10}) for bounded chaotic solutions as evidenced by a positive Lyapunov exponent [11]. The calculations were performed using a fourth-order Runge–Kutta integrator with a step size of $\Delta t = 0.05$ and initial conditions of $x = \dot{x} = \ddot{x} = 0.05$. The three non-zero coefficients were chosen randomly and assigned uniform random values in the range -5 to 5 in increments of 0.1, giving the order of 10^9 cases, of which about 10^7 were randomly chosen for examination. Even so, the calculation took several months of 66 MHz CPU time.

By this method, a chaotic case was found with all coefficients equal to zero except for a_1, a_7, and a_8. It is generally possible to normalize two of the coefficients by rescaling the variables x and t. Thus there remains only one parameter, which was arbitrarily taken as $a_1 \equiv -A$, leading to the equation

$$\dddot{x} + A\ddot{x} - \dot{x}^2 + x = 0. \tag{3}$$

It is unlikely that any algebraically simpler form of an autonomous chaotic flow exists because the above equation has the minimum number of terms that allows an adjustable parameter and it has only a single quadratic nonlinearity. It can be equivalently written as three, first-order, ordinary differential equations with a total of five terms. This is one fewer term or nonlinearity than in any of the nineteen cases previously found and two fewer than in the Rössler equations.

An alternate form of Eq. (3) was found in which the $\dot{x}^2$ term was replaced with $x\dot{x}$, but this case is equivalent to Eq. (3) to within a constant as can be seen by differentiating Eq. (3) with respect to time and defining a new variable $v \equiv \dot{x}$. No other dissipative forms were found with three terms and a single quadratic nonlinearity, although several other slightly more complicated examples were found [12] as well as a conservative case of the form $\dddot{x} + \dot{x} - x^2 + B = 0$ with $0 < B \lesssim 0.05$. These cases were apparently missed in the earlier search [8] because of the narrow range of parameters over which chaos occurs.

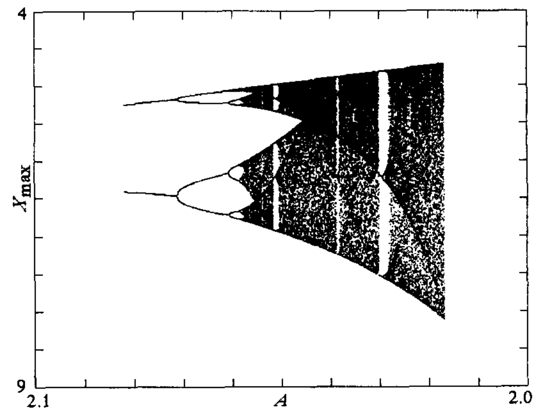

Fig. 1. Bifurcation diagram. Note that the scales have been reversed to emphasize the similarity to the familiar Feigenbaum diagram.

With the above normalization, the parameter A was scanned over the range 0 to 10 in increments of 0.01 to identify the chaotic regions, which were found to fall in the range $2.0168\ldots < A < 2.0577\ldots$. Throughout most of the remainder of the range, the solutions are unbounded. A more careful scan of the chaotic region, recording the successive local maxima of x, after transients have decayed, resulted in the bifurcation diagram shown in Fig. 1. Note that the scales have been reversed to emphasize the similarity to the familiar Feigenbaum diagram for the logistic equation in which a period-doubling route to chaos is observed.

The similarity of Fig. 1 to a Feigenbaum diagram suggests that the sequence of local maxima should approximately follow a quadratic map. Fig. 2 shows such a map in which each maximum is plotted versus the previous maximum for $A = 2.017$, which is in the region of greatest chaos (largest Lyapunov exponent). At low resolution, the return map appears to be one-dimensional, and it resembles a slightly

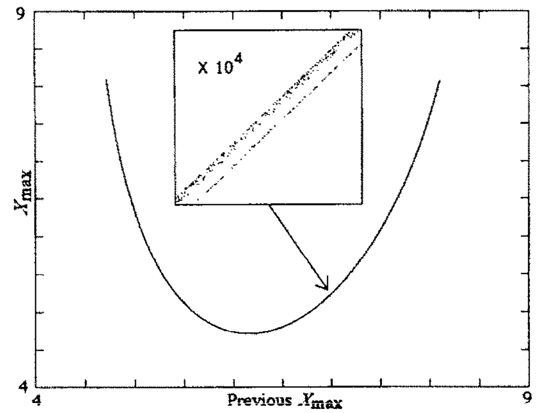

Fig. 2. Return map showing each value of x_{max} versus the previous value of x_{max}. The insert shows fractal structure at a magnification of 10^4.

skewed parabola. The curve is inverted with respect to the usual logistic map, but it has the same sense as the quadratic map in Eq. (1).

The insert in Fig. 2, showing a short segment of the curve magnified by a factor of 10^4, reveals that what appeared to be a simple curve at low resolution is actually a two-dimensional map with fractal structure as expected for a strange attractor. A dissipative chaotic flow in three dimensions must have an attractor with dimension greater than 2 but less than 3. Hence the corresponding map must have dimension greater than 1, although only slightly so in this case. What looks like a single line at low resolution is actually a pair of lines, the upper one of which (at least) consists of multiple lines in what is presumably an infinite self-similar structure.

Eq. (3) has a single fixed point at the origin with eigenvalues λ that satisfy the characteristic equation $\lambda^3 + A\lambda^2 + 1 = 0$. For the range of A over which solutions are bounded, λ is given to within about 1% by $\lambda = -2.24, 0.10 \pm 0.66i$. Thus the origin is an unstable saddle-focus with an instability index of 2; the stable manifold is a line, and the unstable manifold is a surface. For $A = 2.017$, the Lyapunov exponents (base-e) as determined numerically are $L \simeq 0.0550, 0, -2.0720$, and the corresponding Kaplan–Yorke dimension [13] is $D_{KY} = 2 - L_1/L_3 \simeq 2.0265$. Note that the sum of the Lyapunov exponents is the rate of volume contraction and is given by $\Sigma L = \partial j/\partial \ddot{x} = -A$. Thus A is a measure of the damping, as is evident from $\partial F/\partial \dot{x} = -A$.

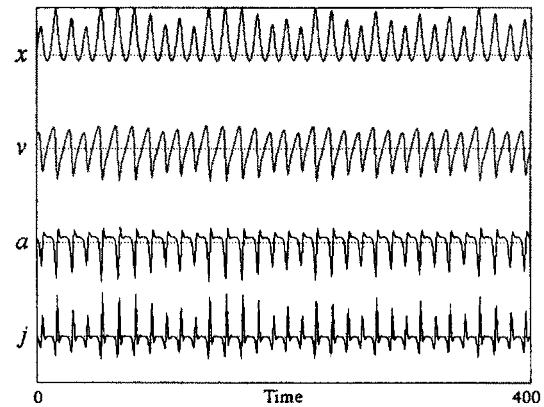

Fig. 3. Chaotic time variation of x, v, a, and j for $A = 2.017$.

Fig. 3 shows the time history of x and its first three time-derivatives after the trajectory has settled onto the attractor for $A = 2.017$. The behavior is nearly periodic but with an apparent chaotic component. The period over the entire bounded region is within about 1% of $T \simeq 11.8$, corresponding to a dominant angular frequency of $\omega \simeq 0.53$, which is about 20% lower than the linear frequency for rotation about the fixed point at the origin ($\omega_0 \simeq 0.66$). The contraction per cycle is given by $e^{L_3 T} \cong 2.5 \times 10^{-11}$, which helps to explain why the return map is so nearly one-dimensional. Note that whereas x is approximately sinusoidal, the higher derivatives are successively less so, with the jerk function consisting mostly of recurrent spikes.

Fig. 4 shows a stereoscopic view of the trajectory for $A = 2.017$. In this plot, you are looking down on the x–$\dot{x}$ plane from the $+\ddot{x}$ direction. As is most

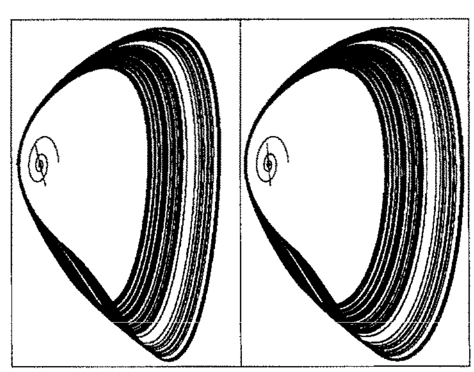

Fig. 4. Stereoscopic view of the attractor, which is approximately a Möbius strip. Also shown is the fixed point at the origin with its stable and unstable manifold.

evident from animated rotational views of the attractor [14], the trajectory lies approximately on a Möbius strip, which accounts for the lack of period-1 solutions. The fixed point at the origin is shown toward the left center of the figure with a line and spiral signifying the stable and unstable manifolds, respectively.

To ensure that the results are not numerical artifacts, the calculation was done in various precisions up to 80-bit, and it was verified that the result is not sensitive to the precision, iteration step size, initial conditions (within the basin of attraction), or the number of iterations. Note that with a step size of 0.05, there are over 200 iterations per cycle, and the trajectory was followed for times as long as 10^7 cycles. Thus the solution is apparently stable and not a chaotic transient.

The basin of attraction for the three-dimensional flow has been examined in some detail. Far from the attractor, the flow is in the $+\ddot{x}$ direction and is highly sheared. The basin of attraction intersects the x axis in the range $4.9046\ldots < x < 8.2571\ldots$ and in an infinite number of segments in the range $-1.0311\ldots < x < 1.2490\ldots$. A 2-D slice of the basin with $\ddot{x} = 0$ shows a spiral structure near the origin. Animated 3-D views [14] show that the basin is shaped like a tadpole with a tail that apparently extends to infinity along the $-\ddot{x}$ axis. When time is run backwards beginning with an initial condition on the attractor, the trajectory typically escapes to infinity within a few thousand iterations, while remaining close to the $-\ddot{x}$ axis.

This particularly simple example of a dissipative chaotic flow illustrates the minimum requirements for chaos in such systems and invites further detailed study.

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[14] Animated rotational views of the attractor and its basin are available at http://sprott.physics.wisc.edu/simpjerk.htm.