

Can a Monkey with a Computer Create Art?

J. C. Sprott¹, University of Wisconsin

Abstract: *A computer can be programmed to search through the solution of millions of equations to find a few hundred whose graphical display is aesthetically pleasing to humans. This paper describes some methods for performing such an exhaustive search, criteria for automatically judging aesthetic appeal, and examples of the results.*

Key Words: art; aesthetics; chaos; strange attractor

INTRODUCTION

Aspects of chaos have been known and understood for hundreds of years. Sir Isaac Newton was said to get headaches contemplating the three-body problem, and the French mathematician Henri Poincaré (1890) won a prize in 1889 for showing that the three-body problem had no analytic solution and hence was unpredictable. However, the widespread appreciation of chaos had to await the advent of powerful and plentiful computers that can approximate the solution of nonlinear equations and display the results with colorful, high-resolution graphics.

The story is now well known how the meteorologist Edward Lorenz (1993) accidentally discovered sensitive dependence on initial conditions in the early 1960s while modeling atmospheric convection on a primitive digital computer (Lorenz, 1963). The Lorenz attractor became an emblem of chaos and for many years was the prototypical and almost the only such system that was widely studied. A few other

¹ Department of Physics, University of Wisconsin, Madison, Wisconsin 53706.
e-mail: sprott@physics.wisc.edu.

Editor's note: Figures 5 and 6 are available in color and may be viewed at the journal's web site: <http://www.societyforchaostheory.org/ndpls/> select Contents Vol. 8.

chaotic systems of differential equations were known, including those by Rössler (1976), Moore & Spiegel (1966), and Ueda (1979), shown in Fig. 1, but chaos was viewed as rather rare and exceptional. These objects were called “strange attractors” by Ruelle and Takens (1971), but neither can recall who coined the term (Ruelle, 1991).

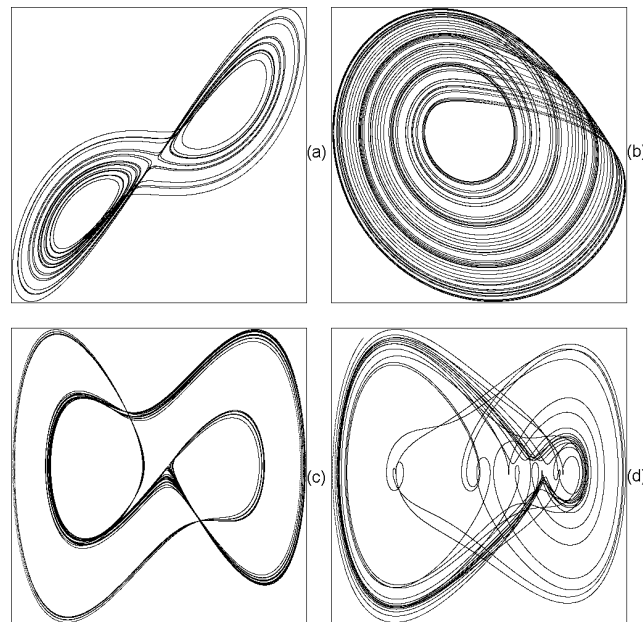


Fig. 1. Some early chaotic flows (a) Lorenz, (b) Rössler, (c) Moore & Spiegel, and (d) Ueda.

Concurrent with these developments was the understanding that chaos also occurs in discrete-time systems governed by finite difference equations, the most celebrated example of which is the logistic map

$$X_{n+1} = AX_n(1 - X_n) \quad (1)$$

which was brought to the attention of scientists by Sir Robert May (1976). Other one-dimensional chaotic maps were known much earlier, including the linear congruential generator (Knuth, 1997)

$$X_{n+1} = AX_n + B \pmod{C} \tag{2}$$

which had been used for years to produce pseudorandom numbers on the computer. One of the earliest and most widely studied two-dimensional chaotic map was due to Hénon (1976)

$$\begin{aligned} X_{n+1} &= 1 - aX_n^2 + bY_n \\ Y_{n+1} &= X_n \end{aligned} \tag{3}$$

although others were known, including those by Lozi (1978), Ikeda (1979), and Sinai (1972), shown in Fig. 2.

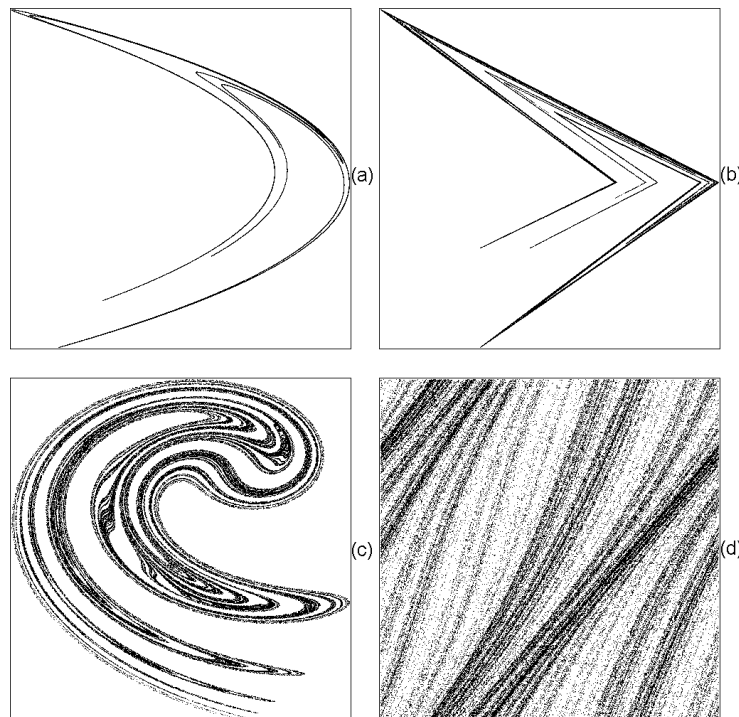


Fig. 2. Some early chaotic maps (a) Hénon, (b) Lozi, (c) Ikeda, and (d) Sinai.

COMPUTER SEARCH

It is clear from these early examples that a wide variety of complex visual patterns can be produced by extremely simple equations and that some of these images have aesthetic as well as mathematical appeal. The advent of modern computers raised the possibility of mass-producing unique images of this type despite the fact that most systems of dynamical equations produce rather simple and uninteresting solutions (Sprott, 1993). The key observations are that a simple system of dynamical equations with a small number of parameters can produce an almost unlimited variety of shapes as the parameters are varied and that the visually interesting cases are those whose solutions are chaotic. Unfortunately, there are no known general rules for predicting the conditions under which chaos will occur, and thus the only general approach entails an extensive search. Fortunately, such a search can be automated and performed relatively quickly with modern computers.

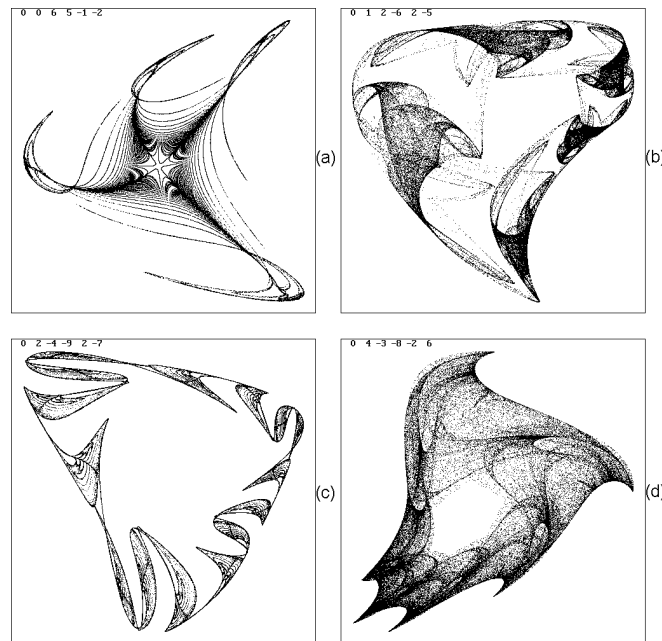


Fig. 3. Sample strange attractors from Eq. 4 for various values of the parameters a_1 through a_6 .

One of the simplest systems that can produce a large variety of such images is the general time-delayed quadratic map, which is a generalization of the Hénon map in Eq. 3

$$\begin{aligned} X_{n+1} &= a_1 + a_2 X_n + a_3 X_n^2 + a_4 X_n Y_n + a_5 Y_n + a_6 Y_n^2 \\ Y_{n+1} &= X_n \end{aligned} \quad (4)$$

where a_1 through a_6 are the parameters that govern the behavior. This six-dimensional parameter space is vast and admits an enormous variety of forms, some examples of which for integer values of the parameters are shown in Fig. 3. You can think of the six values as the settings on a combination lock, some of which open the door to visually interesting images.

To find visually interesting solutions to a system such as Eq. 4, at least two tests must be performed. Unbounded orbits that escape to infinity are excluded by monitoring the value of $|X_n|$ and moving on to a new set of parameters if it exceeds some large value such as $|X_n| > 1000$. Nonchaotic solutions are excluded in a similar way by testing for sensitive dependence on initial conditions. Formally, this is done by calculating the Lyapunov exponent (Sprott, 2003) and discarding cases for which it is not decidedly positive. More simply, perform two simultaneous calculations in which the initial conditions (typically taken as $X_0 = Y_0 = 0.05$) differ by some small amount such as 10^{-6} , and discard cases for which any subsequent iterate differs by less than this amount. It is also possible to discriminate against attractors that are too thin (line-like) or too thick (area-filling) by calculating the fractal dimension (Sprott 2003) or more simply by counting the number of screen pixels visited by the orbit and discarding cases for which this number is very small (less than about 10% of the number of pixels on the screen) or very large (more than about 50%).

It is also helpful to begin plotting only after some number of iterations, such as 1000, to be sure the orbit has reached the attractor and to allow calculation of the minimum and maximum values of X_n so that the plot can be appropriately scaled. Note that the scale for X and Y will be the same since $Y_n = X_{n-1}$ is the time delayed value of X_n . Even restricting the six parameters to integer values in the range $-10 < a < 10$ gives

$19^6 \sim 47$ million different values of which approximately 450 (0.001%) satisfy the above criteria and are nearly all different. Finer increments of the parameters produce an astronomical number of unique cases.

AESTHETIC EVALUATION

The above methods eliminate the vast majority of solutions that are of little aesthetic interest. Those that remain span a wide range from rather mundane to quite spectacular. In one experiment (Sprott 1993), eight volunteers rated a collection of 7500 strange attractors similar to those in Fig. 3 on a scale of 1 to 5 according to their aesthetic appeal. The results in Fig. 4 show a gray scale in which the darker gray indicates those combinations of largest Lyapunov exponent λ_1 and correlation dimension D with greatest appeal. Think of the dimension as a measure of the strangeness of the attractor, and the Lyapunov exponent as a measure of its chaoticity. All evaluators tended to prefer attractors with dimensions between about 1.1 and 1.5 and with small Lyapunov exponents. This range of dimensions characterizes many natural objects such as rivers and coastlines. The Lyapunov exponent preference is harder to understand since it is a dynamical rather than geometrical measure, but it suggests that strongly chaotic systems are too unpredictable to be appealing. There is some evidence that scientists and nonscientists have different preferences (Aks & Sprott, 1996), but the differences are small.

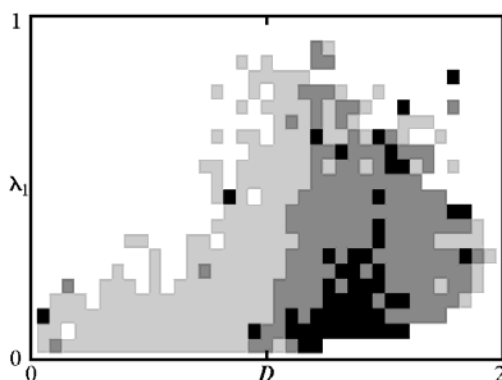


Fig. 4. Values of the largest Lyapunov exponent λ_1 and fractal dimension D that give the most aesthetically pleasing images are shown in darker gray.

The discovery of mathematical metrics that quantify aesthetics is interesting and even disturbing to many people. However, it does raise the possibility of programming the computer to evaluate its own art and to discard cases that it judges would not be appealing to a human. The procedure is a bit like having a monkey press computer keys and then having a program that saves those few gems of prose that would be produced after millions of trials. Fortunately, one's tolerance for visual art is less demanding than for the written word, and monkeys are capable of producing some quite respectable paintings (Lenain, 1997).

EMBELLISHMENTS

There are countless ways to embellish the methods described above. An obvious extension is to add color. One simple way to do this is to introduce a third variable $Z_{n+1} = Y_n$ to Eq. 4 and use the value of Z to choose the color plotted at each (X, Y) position from some palette such as a rainbow. Color figures cannot be shown here, but a Java applet that produces a new and different pattern of this type every five seconds is at <http://sprrott.physics.wisc.edu/java/attract/attract.htm>.

The method can be extended to more than three variables, subject only to finding ways to display the additional variables (Sprrott 1993). For example, the four-dimensional system

$$\begin{aligned} X_{n+1} &= a_1 X_n + a_2 X_n^2 + a_3 Y_n + a_4 Y_n^2 + a_5 Z_n + a_6 Z_n^2 + a_7 C_n + a_8 C_n^2 \\ Y_{n+1} &= X_n \\ Z_{n+1} &= Y_n \\ C_{n+1} &= Z_n \end{aligned} \tag{5}$$

has been used to produce images with depth Z displayed with shadows and occlusion, and the fourth dimension C displayed as color. Eight parameters are used so that the information needed to reconstruct the image can be encoded into an 8-byte string and used as the DOS file name. An example with color displayed as a gray scale is in Fig. 5, but much more dramatic high-resolution color samples are at <http://sprrott.physics.wisc.edu/fractals/pubqual/>. When printed at large size, these images are quite stunning and artistic by most criteria, except that a computer produced them without human intervention (other than to write the program that searches for them).

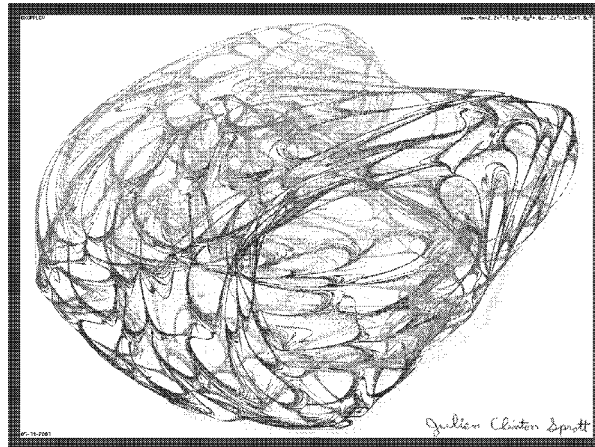


Fig. 5. Sample four-dimensional strange attractor from Eq. 5 in which one dimension is displayed by shadows and another by color (gray scale here).

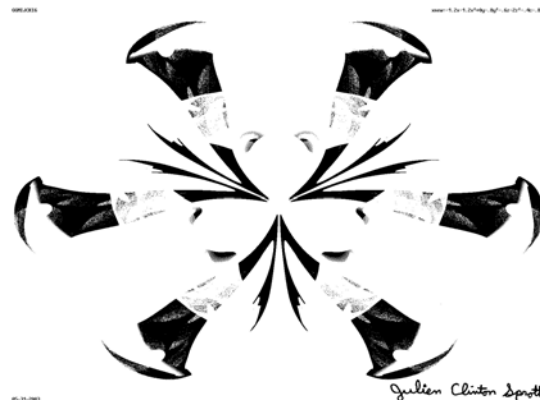


Fig. 6. Sample four-dimensional symmetric icon from Eq. 5 in which the attractor has been replicated six times around a circle.

Images such as these are delightful, but they typically lack global symmetry. It is possible to choose equations whose solutions are symmetric (Field & Golubitsky, 1992), but a simpler method is to impose the symmetry afterwards by distorting the attractor into a pie-shaped wedge and replicating it some number of times, typically between

two and nine, perhaps with alternate repetitions reversed (Sprott, 1996). Such images are called “symmetric icons,” and an example of one, resembling the petals of a flower, is in Fig. 6. Many more such examples in color are at <http://sprott.physics.wisc.edu/fractals/icons/>.

Although the methods described above pertain to strange attractors, they can be extended to other types of mathematical fractals. For example, iterated function systems (Barnsley, 1988) consist of two or more linear affine mappings of the form

$$\begin{aligned} X_{n+1} &= a_1X_n + a_2Y_n + a_5 \\ Y_{n+1} &= a_3X_n + a_4Y_n + a_6 \end{aligned} \tag{6}$$

chosen randomly at each time step. Even with as few as two such mappings, a variety of images emerge as shown in Fig. 7, which have been selected aesthetically by their correlation dimension and Lyapunov exponent (Sprott 1994).

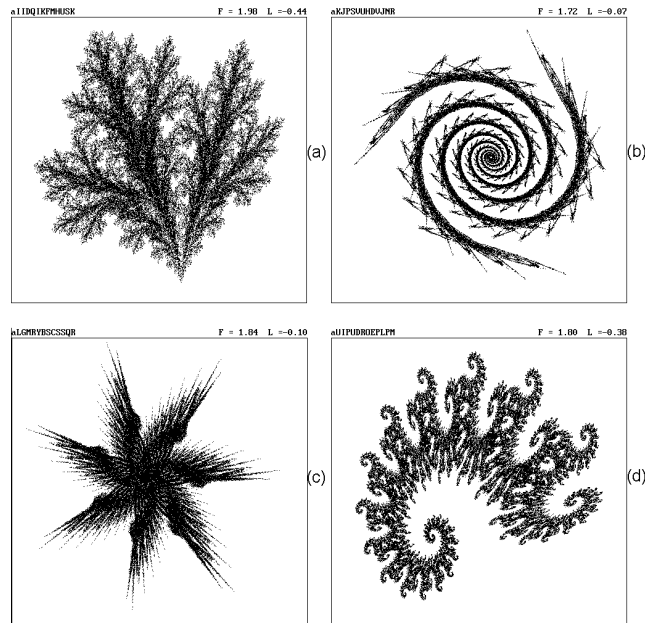


Fig. 7. Sample iterated function systems from two randomly chosen affine mappings as given by Eq. 6.

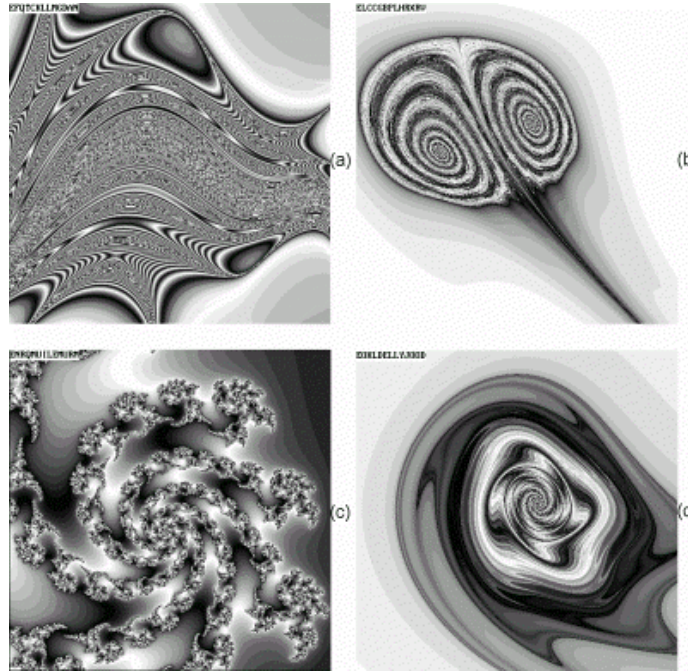


Fig. 8. Sample escape-time contours for the general quadratic map basins in Eq. 8.

Some of the most beautiful examples of mathematical fractals come from Julia sets, which are the basin of attraction of bounded solutions of the complex map

$$Z_{n+1} = Z_n^2 + c \quad (7)$$

where $Z = X + iY$ and c is a complex constant $c = a + ib$. Much human effort has been expended in finding values of a and b that produce visually interesting images. That process can be automated by programming the computer to search through thousands of parameter values, searching for cases whose iterates of $Z_0 = 0$ escape ($|Z_n| > 2$), but only slowly (such as $100 < n < 1000$). Then the value of n can be plotted in a color or gray scale for those orbits that escape for a range of starting values of X_0 and Y_0 (such as -1 to 1). Figure 8 shows some examples of

this technique using the generalized quadratic map (Sprott & Pickover, 1995)

$$\begin{aligned} X_{n+1} &= a_1 + a_2X_n + a_3X_n^2 + a_4X_nY_n + a_5Y_n + a_6Y_n^2 \\ Y_{n+1} &= a_7 + a_8X_n + a_9X_n^2 + a_{10}X_nY_n + a_{11}Y_n + a_{12}Y_n^2 \end{aligned} \quad (8)$$

with 16 shades of gray that cycle forward and then backward as n (for $X_n^2 + Y_n^2 > 1 \times 10^6$) increases. Many stunning color examples can be found at <http://sprott.physics.wisc.edu/fractals/autoquad/>. There is a special visual appeal for those cases that satisfy the Cauchy-Riemann conditions (Arfken 1985), which imply $a_6 = -a_3$, $a_8 = -a_5$, $a_9 = -a_4/2$, $a_{10} = 2a_3$, $a_{11} = a_2$, and $a_{12} = a_4/2$. A variety of other rendering methods for systems of this type can also be easily implemented (Carlson 1996).

Finally, it is possible to overlay two or more images produced as described above by combining the pixels at a given location in some way. For example, one can plot the larger or smaller of the two color values or perform an exclusive-or or other binary operation on the values. This method works best if the two images are the same size and use the same color palette.

CONCLUSIONS

Simple nonlinear dynamical equations can produce an enormous variety of forms, a small fraction of which are visually appealing. Simple rules have been developed that enable the computer to search through the vast space of possibilities and single out those cases that are likely to appeal to humans. In this way the computer is both the artist and the critic of its own work. Carefully tuned programs can generate thousands of unique and highly appealing images in this way. The method is used to produce a “fractal of the day” at <http://sprott.physics.wisc.edu/fractals.htm> and to produce the cover art that will adorn this journal in this and subsequent issues.

ACKNOWLEDGMENTS

I am grateful to George Rowlands for kindling my interest in chaos, to Ted Pope for assuring me that these objects are artistic, and to Cliff Pickover for inspiration and advice over the years.

REFERENCES

- Aks, D. J. & Sprott, J. C. (1996). Quantifying aesthetic preference for chaotic patterns. *Empirical Studies of the Arts*, 14, 1-16.
- Arfken, G. (1985). *Mathematical methods for physicists* (2nd edn). Orlando, FL: Academic Press.
- Barnsley, M. F. (1988). *Fractals everywhere*. Boston: Academic Press.
- Carlson, P. W. (1996). Pseudo-3-D rendering methods for fractals in the complex plane. *Computers & Graphics*, 20, 751-758.
- Field, M. & Golubitsky, M. (1992). *Symmetry in chaos*. New York: Oxford University Press.
- Knuth, D. E. (1997). *Sorting and searching* (3rd edn), Vol. 3 of *The art of computer programming*. Reading, MA: Addison-Wesley-Longman.
- Lenain, T. (1997). *Monkey painting*. London: Reaktion Books.
- Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20, 130-141.
- Lorenz, E. N. (1993). *The essence of chaos*. Seattle, WA: University of Washington Press.
- May, R. (1976). Simple mathematical models with very complicated dynamics. *Nature*, 261, 45-67.
- Moore, D. W. & Spiegel, E. A. (1966). A thermally excited non-linear oscillator. *Astrophysical Journal*, 143, 871-887.
- Poincaré, H. (1890). Sur le problème des trois corps et les equations de la dynamique. *Acta Mathematica*, 13, 1-270.
- Rössler, O. E. (1976). An equation for continuous chaos. *Physics Letters A*, 57, 397-398.
- Ruelle, D. (1991). *Chance and chaos*. Princeton, NJ: Princeton University Press.
- Ruelle, D. & Takens, F. (1971). On the nature of turbulence. *Communications in Mathematical Physics*, 20, 167-192.
- Sprott, J. C. (1993). *Strange attractors: creating patterns in chaos*. New York: M&T Books.
- Sprott, J. C. (1994). Automatic generation of iterated function systems. *Computers & Graphics*, 18, 417-425.
- Sprott, J. C. (1996). Strange attractor symmetric icons. *Computers & Graphics*, 20, 325-332.
- Sprott, J. C. (2003). *Chaos and time-series analysis*. Oxford: Oxford University Press.
- Sprott, J. C. & Pickover, C. A. (1995). Automatic generation of general quadratic map basins. *Computers & Graphics*, 19, 309-313.
- Ueda, Y. (1979). Randomly transitional phenomena in the system governed by Duffing's equation. *Journal of Statistical Physics*, 20, 181-196.