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A comparison of correlation and Lyapunov dimensions

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Abstract

This paper investigates the relation between the correlation (D_2) and the Kaplan–Yorke dimension (D_{KY}) of three-dimensional chaotic flows. Besides the Kaplan–Yorke dimension, a new Lyapunov dimension (D_Σ), derived using a polynomial interpolation instead of a linear one, is compared with D_{KY} and D_2. Various systems from the literature are used in this analysis together with some special cases that span a range of dimension $2 < D_{KY} \leq 3$. A linear regression to the data produces a new fitted Lyapunov dimension of the form $D_{fit} = \alpha - \beta\lambda_1/\lambda_3$, where λ_1 and λ_3 are the largest and smallest Lyapunov exponents, respectively. This form correlates better with the correlation dimension D_2 than do either D_{KY} or D_Σ. Additional forms of the fitted dimension are investigated to improve the fit to D_2, and the results are discussed and interpreted with respect to the Kaplan–Yorke conjecture.
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1. Introduction

The dimension of a strange attractor is a measure of its geometric scaling properties or its "complexity" and has been considered the most basic property of an attractor. Numerous methods have been proposed for characterizing the fractional dimension of the strange attractors produced by chaotic flows. These methods fall into two categories, those derived from the topology, and those derived from the dynamics. Perhaps the most common of the former metrics is the correlation dimension, popularized by Grassberger and Procaccia [1], and the most common of the latter type is the Lyapunov dimension, proposed by Kaplan and Yorke [2]. The relation between these two dimensions has never been systematically studied for a wide variety of systems for three-dimensional flows, in part because the topological measures are very difficult to calculate accurately. Ledrappier has verified that the Kaplan–Yorke dimension D_{KY} is generically equal to the information dimension D_1 [13]. The latter is also verified for two-dimensional diffeomorphisms [16]. In this letter, we focus on the Kaplan–Yorke and the correlation dimension, and we try to verify via a statistical computational

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study the relation between these two for a wide range of three-dimensional chaotic systems.

What follows is such a detailed comparison, representing a computationally intensive study of 46 different three-dimensional chaotic systems with fractional dimensions that span the entire range from 2 to 3. A modified form of the Kaplan–Yorke dimension is tested, as well as a form derived using a polynomial fit to the spectrum of Lyapunov exponents.

The aim of this work is to investigate the relation between these two dimensions, since such a systematic study has never been done for three-dimensional chaotic flows to the best of our knowledge. The second goal of this paper is to construct a new Lyapunov dimension that better correlates with the correlation dimension D_2 than the Kaplan–Yorke dimension D_{KY} or a new Lyapunov dimension D_Σ (introduced in the next section) does. This study will examine the connection between the Lyapunov spectrum (the two nonzero Lyapunov exponents) of a three-dimensional chaotic flow and the fractional dimension that is derived from the topology of the strange attractor D_2.

2. Lyapunov and correlation dimensions

The Kaplan–Yorke dimension [2] can be defined as the fractional dimension in which a cluster of initial conditions will neither expand nor contract as it evolves in time. The rate of expansion is the sum of the Lyapunov exponents, and this sum will necessarily be negative for an attractor of any kind. By ordering the Lyapunov exponents from the largest (most positive) to the smallest (most negative), it is a simple matter to count the maximum number of exponents whose cumulative sum is positive, and this number represents a lower bound on the attractor dimension, since the cluster of initial conditions will still expand in this dimension. However, in the next higher integer dimension, the Lyapunov exponents will sum to a negative value, and hence the cluster contracts in that dimension, which thus represents an upper bound on the attractor dimension. The Kaplan–Yorke dimension can be considered as a linear interpolation between these two integer values to estimate the fractional dimension for which neither expansion nor contraction will occur.

Sprott [3] has suggested that a more accurate interpolation would result from fitting the sum of the first D exponents $\sum \lambda$ to a $(D-1)$-degree polynomial and finding the attractor dimension D_Σ from its zero crossing. For a three-dimensional chaotic flow with only three exponents, one of which (λ_2) is zero, the best one can do is to use a quadratic fit of the form $\sum \lambda = \lambda_3 D_\Sigma^2/2 - 3\lambda_3 D_\Sigma/2 + \lambda_1 + \lambda_3$ whose root for $\sum \lambda = 0$ is

$$D_\Sigma = 1.5 + 0.5\sqrt{\frac{1-8\lambda_1}{\lambda_3}}.$$

This formula gives a larger prediction than the usual

$$D_{KY} = 2 - \frac{\lambda_1}{\lambda_3}.$$

These two dimensions have $D_\Sigma \geq D_{KY}$ for $-1 \leq \lambda_1/\lambda_3 \leq 1$ with a maximum difference of $D_\Sigma - D_{KY} = 1/8$ at $\lambda_1/\lambda_3 = -3/8$. The next section will examine this difference in more detail.

The correlation dimension [1] is calculated from the correlation integral $C(r)$, which is the probability that two randomly chosen points on the attractor are separated by a distance less than r and is given by

$$D_2 = \lim_{r\to 0} \frac{\mathrm{d}\,\log\, C(r)}{\mathrm{d}\,\log\, r}.$$

Accurate calculation of D_2 is notoriously difficult because the value of the derivative often converges very slowly for $r \to 0$ where the number of data points is too small to permit an accurate determination of $C(r)$ and because the lacunarity [4] of the fractal attractor causes $C(r)$ to oscillate. For many cases, it can be shown [5] that $C(r)$ is of the form

$$\log C(r) = A + D_2 \log r + B \log(-\log r),$$

where B is typically in the range of 0–1 and measures the rate of convergence.

The values of D_2 quoted here were derived by a least-squares fit to the above formula using a minimum of 2×10^{12} correlations. The computed values tend to be slightly larger than those typically quoted in the literature for cases where D_2 has been estimated because of the slow convergence. Errors due to lacunarity of the attractor are reflected in the quoted precision of the fit and are the main source of uncertainty.

A significant identity for the correlation dimension D_2, the information dimension D_1 and the capacity dimension D_0 [14,15,17] that comes directly from their definitions, is

$$D_2 \leq D_1 \leq D_0.$$

A later conjecture held that the Kaplan–Yorke dimension is generically equal to a probabilistic dimension that appears to be identical to the information dimension D_1 [12]. This conjecture is partially verified by Ledrappier for any ergodic invariant measure of a smooth map [13]. In terms of the above, the Kaplan–Yorke conjecture (verified also for two-dimensional diffeomorphisms [16]) asserts that the Kaplan–Yorke dimension and the information dimension should generally coincide for natural invariant measures and also that the information dimension can coincide with the correlation dimension regardless of the spectrum of Lyapunov exponents. Hence, from the above, it can be conjectured that the Kaplan–Yorke dimension D_{KY} should be larger than or equal to the correlation dimension D_2 and that one could not conclude anything about the connection between D_2 and the spectrum of Lyapunov exponents for a chaotic system. However, a statistical study as described herein can provide insight into the differences in the various dimensions for typical three-dimensional chaotic flows. By selecting a variety of chaotic systems that span attractor dimensions between 2 and 3, a statistical study of the Lyapunov exponents and the correlation dimension D_2 suggests new Lyapunov dimensions that correlate better to D_2 than D_{KY} does, and verifies the conjectures described above.

3. Chaotic flows

This section concerns the relation between the correlation dimension D_2 and the two Lyapunov dimensions D_{KY} and D_Σ that were introduced above. To have a consistent and valid statistical result, a wide variety of chaotic systems must be used with dimensions spanning the range of 2–3. Most of these systems from the literature, like the Lorenz [6], the Rossler [7] attractors and many others, have dimensions only slightly greater than 2.0. Hence, three more systems were used that better span the range of dimension from 2 to 3. The first system is a chaotic flow with eight nonlinearities that is used to model semiconductor lasers optically driven by a monochromatic light beam, and whose dynamical properties are well known [8–10]. This rate-equation model will be named system A and is given by

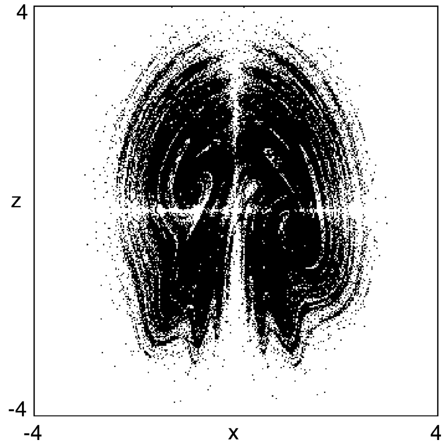

Fig. 1. Poincaré plot $\{y=0\}$ in the x–z plane for the A_8 case ($D_{KY}=2.764$) in Table 1.

$$\frac{dx}{dt} = K + \frac{1}{2}xz + \left(\omega - \frac{1}{2}\alpha z\right) y,$$

$$\frac{dy}{dt} = -\left(\omega - \frac{1}{2}\alpha z\right) x + \frac{1}{2}yz,$$

$$\frac{dz}{dt} = -2\Gamma z - (1 + 2Bz)(x^2 + y^2 - 1).$$

Typical values for the constants of system A are: $0<K<3$, $-3<\omega<3$, $0<\alpha<15$, $0<B<0.03$ and $B<\Gamma<0.1$. This system is capable of producing chaotic attractors with large values of dimension reaching even $2.9<D_{KY}<3$. A Poincaré plot in the x–z plane for $y=0$ is given in Fig. 1 for the A_8 case given in Table 1. It was also found in [9] that as α is increased, the largest Lyapunov exponent of the system increases linearly, resulting in a higher dimension.

A new simpler chaotic flow was found by modifying system A with $B=\Gamma=\omega=0$ and by adding a damping constant ε in the dy/dt equation. Also, the "1/2" expressions were removed from the dx/dt and dy/dt equations. This system will be named system B and is given by

$$\frac{dx}{dt} = K + z(x - \alpha y), \qquad \frac{dy}{dt} = z(\alpha x - \varepsilon y),$$

$$\frac{dz}{dt} = 1 - x^2 - y^2.$$

By varying the system's parameters and especially the parameter ε, this system can produce high-dimensional

Table 1
The 21 chaotic cases with the calculated Lyapunov exponents, D_{KY}, D_{Σ} and D_2

System	Parameters	λ_1	λ_3	D_{KY}	D_{Σ}	D_2
B_6	$K=0.4$, $\alpha=1.5$, $\varepsilon=0.86$	0.11	−0.461	2.239	2.353	2.147 ± 0.115
A_7	$K=1.10$, $\omega=0.56$, $\alpha=6.6$, $B=0.015$, $\Gamma=0.035$	0.2254	−0.91	2.248	2.363	2.202 ± 0.095
A_6	$K=0.80$, $\omega=0.56$, $\alpha=6.6$, $B=0.015$, $\Gamma=0.035$	0.2144	−0.6867	2.312	2.435	2.41 ± 0.108
A_2	$K=1.10$, $\omega=1.1$, $\alpha=9.0$, $B=0.00667$, $\Gamma=0.0079$	0.325	−0.88	2.37	2.494	2.33 ± 0.13
C_1	$\alpha=9.0$, $\gamma=0.18$	0.111	−0.2933	2.378	2.503	2.49 ± 0.13
A_1	$K=0.451$, $\omega=1.1$, $\alpha=2.6$, $B=0.0295$, $\Gamma=0.0973$	0.1206	−0.308	2.391	2.516	2.2 ± 0.11
C_3	$\alpha=10.0$, $\gamma=0.18$	0.141	−0.32	2.44	2.564	2.532 ± 0.117
B_5	$K=0.4$, $\alpha=3.0$, $\varepsilon=0.0$	0.1487	−0.333	2.447	2.57	2.433 ± 0.109
C_2	$\alpha=14.3$, $\gamma=0.18$	0.1972	−0.38	2.52	2.635	2.72 ± 0.13
B_9	$K=0.4$, $\alpha=4.0$, $\varepsilon=-1.66$	0.2155	−0.4133	2.521	2.637	2.367 ± 0.117
B_1	$K=0.4$, $\alpha=3.0$, $\varepsilon=-0.10$	0.187	−0.3526	2.535	2.645	2.353 ± 0.12
B_3	$K=0.4$, $\alpha=4.0$, $\varepsilon=0.052$	0.258	−0.4363	2.591	2.697	2.352 ± 0.114
B_8	$K=0.4$, $\alpha=4.0$, $\varepsilon=-1.55$	0.239	−0.3936	2.607	2.71	2.548 ± 0.128
A_3	$K=0.65$, $\omega=1.1$, $\alpha=9.0$, $B=0.00667$, $\Gamma=0.0079$	0.3956	−0.592	2.668	2.76	2.52 ± 0.15
B_4	$K=0.4$, $\alpha=4.0$, $\varepsilon=0.38$	0.2986	−0.4072	2.733	2.81	2.499 ± 0.15
A_8	$K=0.30$, $\omega=0.0$, $\alpha=8.0$, $B=0.015$, $\Gamma=0.035$	0.312	−0.408	2.764	2.834	2.557 ± 0.15
B_2	$K=0.4$, $\alpha=4.0$, $\varepsilon=-0.66$	0.3	−0.3828	2.784	2.85	2.703 ± 0.159
B_7	$K=0.5$, $\alpha=7.0$, $\varepsilon=0.23$	0.476	−0.556	2.856	2.9	2.719 ± 0.156
A_4	$K=0.20$, $\omega=1.1$, $\alpha=9.0$, $B=0.00667$, $\Gamma=0.0079$	0.2625	−0.2943	2.892	2.926	2.787 ± 0.184
A_5	$K=0.11$, $\omega=0.10$, $\alpha=9.0$, $B=0.00667$, $\Gamma=0.0079$	0.152	−0.16	2.95	2.966	2.983 ± 0.18
A_9	$K=0.10$, $\omega=0.0$, $\alpha=15.0$, $B=0$, $\Gamma=0$	0.19	−0.192	2.99	2.993	3.013 ± 0.202

chaotic attractors reaching $D_{KY}\approx2.9$. A plot of D_{KY} versus ε is given in Fig. 2, and a Poincaré plot is shown in Fig. 3 in the x–z plane for $y=0$ for the B_1 case in Table 1. In Fig. 2, as ε is varied, D_{KY} covers the range from 2 to 2.82 with a maximum value of D_{KY} given by the dotted line. With a suitable combination of ε and α, the whole range of $0<|\lambda_1/\lambda_3|<1$ is easily covered. The largest values of D_{KY} require $\alpha>7$.

Another new chaotic flow (named system C) was found and used in this work, and it is given by

$$\frac{dx}{dt}=xz+\gamma\cos(y),\qquad \frac{dy}{dt}=\delta z-\gamma y,$$

$$\frac{dz}{dt}=1-x^2.$$

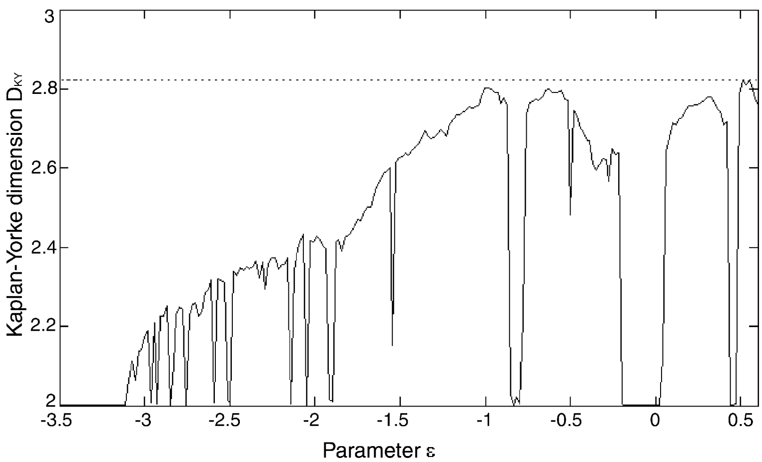

Fig. 2. Variation of D_{KY} with control parameter ε for $K=0.4$ and $\alpha=4$ for system B.

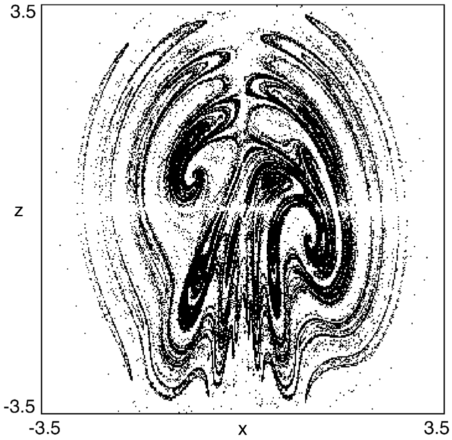

Fig. 3. Poincaré plot $\{y = 0\}$ in the x–z plane for the B_1 case ($D_{KY} = 2.535$) in Table 1.

Typical values for this system that produce chaotic solutions are $0 < \gamma < 1$ and $0 < \delta < 100$. Within this range, system C can produce a dimension reaching $D_{KY} \approx 2.8$. This value was found to be a maximum, in contrast to systems A and B that can cover the whole range $0 < |\lambda_1/\lambda_3| < 1$. For system B to produce $D_{KY} > 2.9$, a very high value of δ is required; i.e., $\delta = 500$ and $\gamma = 0.07$ gives $D_{KY} = 2.9$. A Poincaré plot is shown in Fig. 4 in the x–z plane for $y = 0$ for the C_2 case in Table 1.

Systems B and C are abstract chaotic vector fields, constructed for the purpose described in Section 1, producing an attractor with dimension almost anywhere between 2 and 3 to be used in the statistical study of

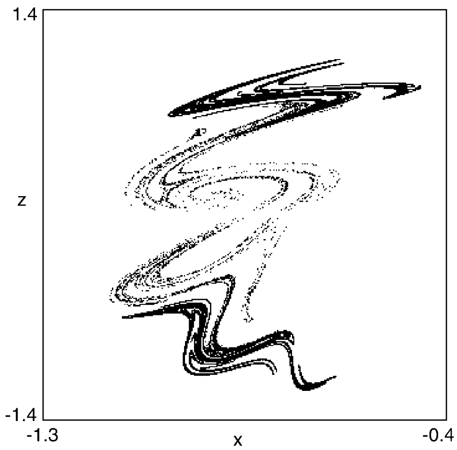

Fig. 4. Poincaré plot $\{y = 0\}$ in the x–z plane for the C_2 case ($D_{KY} = 2.52$) in Table 1.

the relation between the Lyapunov exponents and the correlation dimension D_2. The latter is introduced in the following section.

4. Comparison of the Lyapunov and correlation dimensions

These three systems A, B and C were used here to investigate the relation between the correlation dimension D_2 and the two Lyapunov dimensions D_{KY} and D_Σ in the range $0.2 < |\lambda_1/\lambda_3| < 1$. In Table 1 we present 21 cases of these three systems ordered from the lowest to the highest D_{KY}. The lowest value used was $D_{KY} = 2.239$, and the highest was $D_{KY} = 2.99$. Together with these cases, 24 more low-dimensional cases were used from Appendix A.5 and A.6 of [3], including four Hamiltonian systems and one more system [11] that has $D_{KY} = 2.897$ and $D_2 = 2.84 \pm 0.28$. Hence, 46 total chaotic systems were used in this analysis to cover the whole range $0 < |\lambda_1/\lambda_3| \leq 1$.

The calculation of D_{KY} and D_2 for these 21 cases of Table 1 for the systems A, B and C, reveal mixed relations between these two dimensions. For example, system B always has $D_{KY} > D_2$ as expected, whereas system C has $D_{KY} < D_2$ presumably as a result of numerical uncertainty in the calculation of D_2. For system A these relations are mixed. The unavoidable large uncertainty in the calculation of D_2 as explained in Section 2 is the reason why many systems with large attractor dimension were used in this analysis, so that the final statistical result is more valid and general.

The aim of this work is to construct a new Lyapunov dimension that better correlates with the correlation dimension D_2. Also by applying a multivariable regression, it will be determined whether D_{KY} or D_Σ correlates better with D_2 by measuring the weights α and β in the expression

$$D_2 = \alpha D_{KY} + \beta D_\Sigma, \quad \alpha + \beta \leq 1.$$

The summation $\alpha + \beta$ should be less than unity since both of these Lyapunov dimensions tend to be larger than D_2 as shown in Table 1 and in Fig. 5 where the scatter plot of D_2 versus λ_1/λ_3 is given for all 46 cases reported in Table 1. Fig. 5 clearly shows how the whole space is covered for $0 < |\lambda_1/\lambda_3| \leq 1$. In this plot, D_{KY} and D_Σ are included with solid and dashed

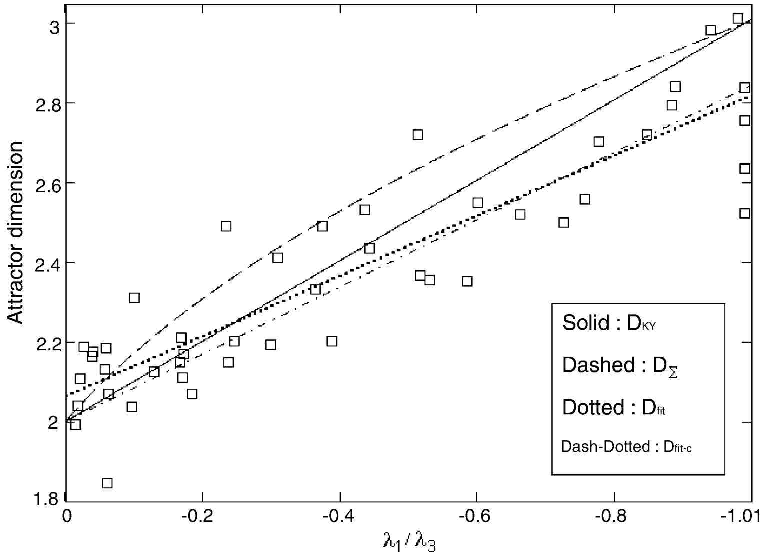

Fig. 5. Scatter plot for D_2 vs. λ_1/λ_3. The solid line is D_{KY}, the dashed line is D_Σ, the dotted line is D_{fit} of Eq. (1), and the dash-dotted line is $D_{fit\text{-}C}$ of Eq. (4). Squares stand for D_2.

lines, respectively, and the values of D_2 are denoted with squares in the same plot.

By applying a least-squares linear regression, a new Lyapunov dimension is calculated and shown in Fig. 5 with the dotted line given by

$$D_{fit} = 2.061 - 0.749\frac{\lambda_1}{\lambda_3}. \quad (1)$$

In contrast to D_{KY} (which is equal to $2 - \lambda_1/\lambda_3$), we note a small increase in the first parameter ($2 \to 2.061$) and a decrease in the second parameter ($1 \to 0.749$). The first of these shifts produces a better correlation of D_{fit} to D_2 for the low-dimensional systems with $|\lambda_1/\lambda_3| < 0.2$ according to Fig. 5. The second shift ($1 \to 0.749$) produces a better correlation for the higher-dimensional systems since most of them lie below the solid line of D_{KY}. To show that D_{fit} is a better approximation to D_2 than either D_{KY} or D_Σ, two multivariable regressions produced

$$D_2 = -0.00077D_{KY} + 1.001D_{fit}, \quad (2)$$

$$D_2 = 0.019D_\Sigma + 0.979D_{fit}. \quad (3)$$

However, Eq. (1) gives $D_{fit} = 2.061$ when $\lambda_1 = 0$, whereas the correlation dimension should be $D_2 \leq 2$. This is purely a mathematical artifact since this shift of 0.061 is made in order for D_{fit} to correlate better with the low-dimensional systems as discussed above. Hence, a new regression was applied with a constant term that now has a physical meaning ($D_{fit} = 2$ for $\lambda_1 = 0$) and the result was

$$D_{fit\text{-}C} = 2 - 0.836\frac{\lambda_1}{\lambda_3}. \quad (4)$$

Following the notation from Eqs. (2) and (3), this new $D_{fit\text{-}C}$ was found to correlate better with D_2 than D_{KY} and D_Σ do. However, Eq. (1) was derived from a multivariable regression for chaotic systems ($\lambda_1 > 0$), and thus it is valid only for $\lambda_1 > 0$ and not for $\lambda_1 = 0$. Hence, it is concluded that it has a physical meaning only for chaotic flows.

Next, we will examine the correlation of D_2 with D_{KY} and D_Σ. A multivariable regression gives

$$D_2 = 0.14D_{KY} + 0.815D_\Sigma. \quad (5)$$

From this expression, D_2 seems to correlate better with D_Σ than with D_{KY}. When the four Hamiltonian systems (clearly shown in Fig. 5 with $D_{KY} = 3.0$) are omitted from the regression, D_2 is still better correlated to D_{fit} than to the other two Lyapunov dimensions, but Eq. (4) is replaced by

$$D_2 = 1.17D_{KY} - 0.18D_\Sigma, \quad (6)$$

and we note that $1.17 - 0.18 < 1$ as expected.

This result was expected by looking at Fig. 5 where most of the points lie nearer the D_{KY} line, although the source of the discrepancy is still under investigation. One explanation for the difference between Eqs. (5) and (6) could be that a Hamiltonian system (with $|\lambda_1/\lambda_3| = 1$) has $D_{KY} = D_\Sigma = 3$, and this causes a numerical error in the regression. There is no point in comparing D_{KY} with D_Σ for Hamiltonian systems since they produce exactly the same dimension. Therefore, Eq. (6) is the more plausible result. Furthermore, in order to clarify this discrepancy whether the Hamiltonian systems are included or omitted, a regression with a constant term was made

$$D_2 = 0.55 + 0.68 D_{KY} + 0.07 D_\Sigma,$$

indicating that D_2 correlates better with D_{KY} than with D_Σ. Much of the difficulty of getting good fits for this specific regression arises from the large uncertainties in estimating D_2 for the Hamiltonian cases, and it might be resolved by examining more attractors with dimension close to 3. The result though of the D_{fit} remained unaltered whether the Hamiltonian systems were included or not.

The result above could also be deduced from the Kaplan–Yorke conjecture that implies $D_2 \le D_{KY}$. Since $D_\Sigma \ge D_{KY}$, we conclude that

$$D_2 \le D_{KY} \le D_\Sigma.$$

Thus our result is a verification of the Kaplan–Yorke conjecture. However, a careful inspection of Fig. 5 shows that many cases have $D_2 > D_{KY}$ or even $D_2 > D_\Sigma$. This result arises from the unavoidable large uncertainties in D_2 given in Table 1 since many of the systems used did not show good convergence in the numerical calculations of the correlation dimension D_2.

In this analysis, while more and more cases were being added to the calculations, the change of D_{fit} was negligible, suggesting that D_{fit} is a robust result. By comparing D_{fit} with D_{KY}, one can easily see that for $|\lambda_1/\lambda_3| < 1/4$, we have $D_{fit} > D_{KY}$. A daring suggestion could be made from the latter, that since D_2 is closer to D_{fit} than to the other dimensions, then if one could find with an accurate calculation a case where $D_2 > D_{KY}$, then this would happen for $|\lambda_1/\lambda_3| < 1/4$. However, this is a suggestion deduced from our results and not a conclusion due to the large uncertainties in the correlation dimension D_2. The same analysis can be made for D_Σ, where $D_{fit} > D_\Sigma$ for $|\lambda_1/\lambda_3| < 1/20$.

The main conclusion of the above analysis and from Fig. 5 is that the two new fitted Lyapunov dimensions D_{fit} and $D_{fit\text{-}C}$ are better approximations to D_2 than D_{KY} is. The general Kaplan–Yorke conjecture is also verified, since our calculations for the fitted dimensions indicate that $D_{KY} \ge D_2$ and since these two fitted dimensions are approximations to D_2, it should be concluded that $D_{KY} \ge D_{fit}$ and $D_{KY} \ge D_{fit\text{-}C}$. This is verified in Fig. 5, although some cases are reported where $D_2 \ge D_{KY}$ and also $D_{fit} \ge D_{KY}$. The latter is explained by the unavoidable large uncertainties in the calculation of D_2 as described in Section 2.

5. New forms of the fit dimension

Besides the regression form of $D_{fit} = \alpha - \beta\lambda_1/\lambda_3$, other forms were tested. A power series polynomial regression is a good candidate for the kind of data presented in Fig. 5. This new expression $D_{fit\text{-}q}$ (quadratic) is given by

$$D_{fit\text{-}q} = 2.055 - 0.796\frac{\lambda_1}{\lambda_3} - 0.047\left(\frac{\lambda_1}{\lambda_3}\right)^2. \tag{7}$$

However, the result did not improve much the linear D_{fit} since $-0.006 \le D_{fit\text{-}q} - D_{fit} \le 0.0057$. The greatest difference of $D_{fit\text{-}q} - D_{fit}$ (although it is very small) is for $0.4 < |\lambda_1/\lambda_3| \le 0.6$ and $|\lambda_1/\lambda_3| > 0.8$. This is logical from Fig. 5 because the quadratic fit improves the linear fit, since in these two ranges there are significant discrepancies. For example, there are two Hamiltonian systems with $D_{KY} = 3$ (or $|\lambda_1/\lambda_3| = 1$), one with $D_2 = 2.521 \pm 0.146$, and the other with $D_2 = 2.837 \pm 0.173$. A higher-order (cubic or quartic) regression is probably not justified because of the large uncertainties in D_2.

Another fit could be an exponential one to correlate better with the low-dimensional cases and try to align with the moderate and high-dimensional ones. It is given by

$$D_{fit\text{-}exp} = -2.102\frac{\lambda_1}{\lambda_3} + 2.15\exp\left(\frac{\lambda_1}{\lambda_3}\right). \tag{8}$$

This new dimension $D_{fit\text{-}exp}$ was found to correlate better to D_2 than D_{KY} and D_Σ do, but not better than the linear D_{fit}. Furthermore, a disadvantage of the $D_{fit\text{-}exp}$ is that it has a minimum at $\lambda_1/\lambda_3 = -0.00225$, whereas the

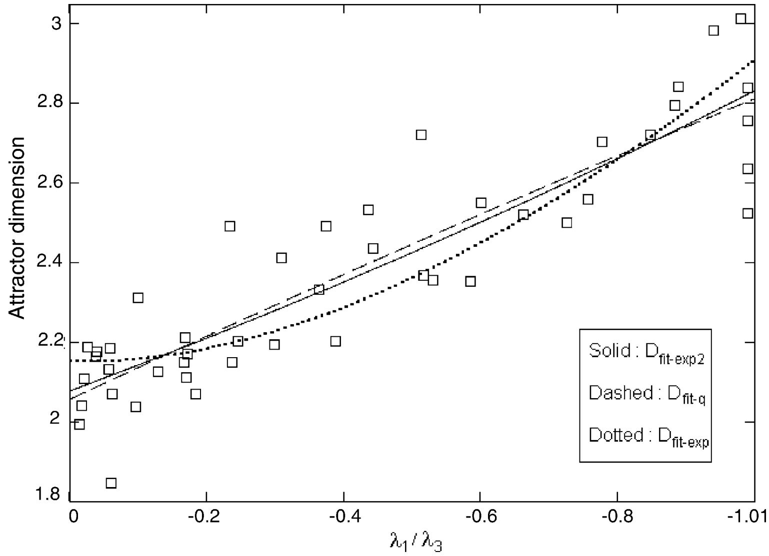

Fig. 6. Scatter plot for D_2 vs. λ_1/λ_3. The solid line is $D_{\text{fit-exp2}}$ of Eq. (9), the dashed line is $D_{\text{fit-q}}$ of Eq. (7), and the dotted line is $D_{\text{fit-exp}}$ of Eq. (8). Squares stand for D_2.

dimension should increase monotonically with λ_1/λ_3 and have no local extrema for $0 < |\lambda_1/\lambda_3| \leq 1$. This is true for D_{KY}, D_Σ, D_{fit} and $D_{\text{fit-q}}$ for the range $0 < |\lambda_1/\lambda_3| \leq 1$, whereas it is not true for a higher-order (cubic or quartic) regression, because its second derivative would change sign in this range.

Another fit, which gave results almost identical to D_{fit}, is given by

$$D_{\text{fit-}\exp 2} = 2.076 \exp\left(-0.307 \frac{\lambda_1}{\lambda_3}\right). \tag{9}$$

Due to the small parameter (0.307) in the exponent, $D_{\text{fit-exp2}}$ is similar to the linear D_{fit}. The latter is still preferable mostly due to the large uncertainties in D_2.

All of the above fitted dimensions from Eqs. (7)–(9) are given in the scatter plot in Fig. 6. These new fits have $D > 2$ for $\lambda_1 = 0$. This can easily be corrected with the same multivariable analysis as discussed in Section 4, and it was found that the effects on the correlation were similar.

6. Conclusions

In this paper we demonstrated for the first time to the best of our knowledge a comparison of the correlation and the Kaplan–Yorke dimensions for three-dimensional chaotic flows, both of which are used widely in experimental and theoretical work. For the results to be consistent, three systems were used that can span dimensions between 2 and 3, with two of these introduced here for the first time. A total of 46 chaotic systems were used, including four Hamiltonian systems covering the whole range $0 < |\lambda_1/\lambda_3| \leq 1$. For these 46 cases, the correlation dimension and the Lyapunov exponent spectrum were calculated using the best methods available. By fitting these results with different forms of regression, we constructed new Lyapunov dimensions that were found to correlate better to D_2 than D_{KY} does. The linear regression in the general form $D = \alpha - \beta\lambda_1/\lambda_3$ was found to be the best due to the large uncertainties in D_2. The Kaplan–Yorke conjecture (described in Section 2) was verified, and hence the best approximation to D_2 according to our calculations was found to be Eq. (4): $D_{\text{fit-C}} = 2 - 0.836\lambda_1/\lambda_3$ since the two-parameter fit in Eq. (1): $D_{\text{fit}} = 2.061 - 0.748\lambda_1/\lambda_3$ violates the Kaplan–Yorke conjecture for $|\lambda_1/\lambda_3| < 1/4$ although these two dimensions $D_{\text{fit-C}}$ and D_{fit} coincide for $|\lambda_1/\lambda_3| > 1/4$. The latter was explained in this letter as a result of the unavoidable uncertainties in the calculation of the correlation dimension D_2.

Besides the Kaplan–Yorke dimension D_{KY}, a new Lyapunov dimension D_Σ that uses a quadratic interpolation instead of the linear one used for the derivation of the D_{KY} was tested. This new Lyapunov dimension was found to correlate less well to D_2 than D_{KY} or D_{fit} and $D_{fit\text{-}C}$ do, by applying a multivariable regression in the form $D_2 = \alpha D_{KY} + \beta D_\Sigma$, where the parameters α and β are the weights of correlation. From this result, D_Σ was found not to correlate better with D_2. Furthermore, it was found that D_Σ is always greater than D_{KY} in the whole range $0 < |\lambda_1/\lambda_3| < 1$ and that it has a maximum difference from D_{KY} equal to $D_\Sigma - D_{KY} = 1/8$ at $\lambda_1/\lambda_3 = -3/8$, making it the largest of all the dimensions used to characterize three-dimensional chaotic flows.

An obvious extension of this work would be to compare D_{KY} with the entire spectrum of generalized dimensions. However, the computation required to determine D_2 for this large collection of systems amounted to many CPU-years on state-of-the-art personal computers, and so such an ambitious project will have to await further advances in computational capabilities.

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