Maximally complex simple attractors

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(Received 12 July 2007; accepted 17 August 2007; published online 28 September 2007)

A relatively small number of mathematically simple maps and flows are routinely used as examples of low-dimensional chaos. These systems typically have a number of parameters that are chosen for historical or other reasons. This paper addresses the question of whether a different choice of these parameters can produce strange attractors that are significantly more chaotic (larger Lyapunov exponent) or more complex (higher dimension) than those typically used in such studies. It reports numerical results in which the parameters are adjusted to give either the largest Lyapunov exponent or the largest Kaplan-Yorke dimension. The characteristics of the resulting attractors are displayed and discussed. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2781570]

Countless papers have been published in the past few decades in which a small number of relatively common iterated maps and systems of ordinary differential equations are used as prototypical examples of low-dimensional chaos in discrete and continuous-time systems, respectively. Typically, these systems have a number of parameters for which standard values are generally taken, in most cases values that were chosen somewhat arbitrarily in the original papers and that have continued to be used. With the advent of fast computers, it is now possible to explore the entire parameter space of these systems with the goal of finding parameters that optimize some characteristic of the system such as its chaoticity or complexity. Here we use the largest Lyapunov exponent as a measure of chaoticity and the Kaplan-Yorke dimension of a measure of complexity (or strangeness). In some cases, the standard parameters are close to optimal for these quantities, but in other cases, quite different attractors result. These near optimal parameters might better serve some of the purposes for which these systems are often used.

I. GENERAL SYMMETRIC UNIMODAL MAPS

To illustrate the idea, consider one of the simplest chaotic systems, the one-dimensional unimodal map, of which the logistic map[1] is perhaps the best known example, but the tent map[2] is another common example. These are two of a wider class of general symmetric map[3] given by

$$X_{n+1} = A(1 - |2X_n - 1|^{\alpha}), \tag{1}$$

where A is the usual bifurcation parameter ($0 \leq A \leq 1$) and α is a measure of the smoothness of the map, with $\alpha=2$ corresponding to the logistic map, $\alpha=1$ corresponding to the tent map, and $\alpha=0.5$ corresponding to the cusp map. The map is symmetric about the critical point at $X_n=0.5$ and is noninvertible since each iterate has two possible preimages. For $\alpha \leq 1$, the derivative is discontinuous at the critical point. For $\alpha > 1$, the derivative is zero at the critical point and continuous, and the function is smooth. For $\alpha > m$ (an integer), the mth derivative is zero, and the critical point is degenerate for $m \geq 2$. The map does not have chaotic solutions for $\alpha < 0.5$ because the fixed point at $X_n=0$ is then stable for all A.

We can now ask where in the $A\alpha$ plane is the Lyapunov exponent the largest. Since A is a measure of the stretching, we expect for a given α that the largest Lyapunov exponent will occur at the largest value of A for which the solution is bounded, and that value occurs at $A=1$, where the unit interval $0 \leq X_n \leq 1$ is endomorphic. Thus the problem reduces to finding the value of α that maximizes the Lyapunov exponent, given in this case by $\lambda_1 = \ln(2\alpha|2X_n - 1|^{\alpha-1})$ averaged over the orbit. Note that all logarithms in this paper are base e .

Figure 1 shows the result for 10^8 iterations at each value of α. The Lyapunov exponent is nearly constant from α slightly less than 1 to almost 3, with clear peaks at $\alpha=1$ and 2, corresponding to the tent map and logistic map, respectively. This result confirms the fact that these two maps are conjugate with Lyapunov exponents of $\ln(2)=0.693\,14\ldots$ and that they are not conjugate to other maps of this class. What is less well known is that they are maximally chaotic for maps of the form of Eq. (1).

II. HÉNON MAP

More interesting is the case of two-dimensional dissipative maps for which the Hénon map[4] is probably the simplest such example since it is given by

$$X_{n+1} = 1 - aX_n^2 + bY_n, \quad Y_{n+1} = X_n. \tag{2}$$

This map is invertible for $b \neq 0$ with a constant area contraction (sum of the Lyapunov exponents) of $\ln|b|$. The usual parameters are taken as $a=1.4$ and $b=0.3$, for which the Lyapunov exponents[5] are $\lambda_1=0.419\,22$ and $\lambda_2=-1.623\,19$ with a Kaplan-Yorke dimension[6] of $D_{KY}=1-\lambda_1/\lambda_2 =1.258\,27$.

To find the location in the positive ab plane with the highest value of λ_1, a random search was performed starting with the above values of a and b and exploring a Gaussian two-dimensional (2D) neighborhood in parameter space with an initial fractional standard deviation of $\varepsilon=0.1$, taking 2×10^7 iterations at each set of parameters and calculating the Lyapunov exponents using the method in Ref. 3. Whenever a

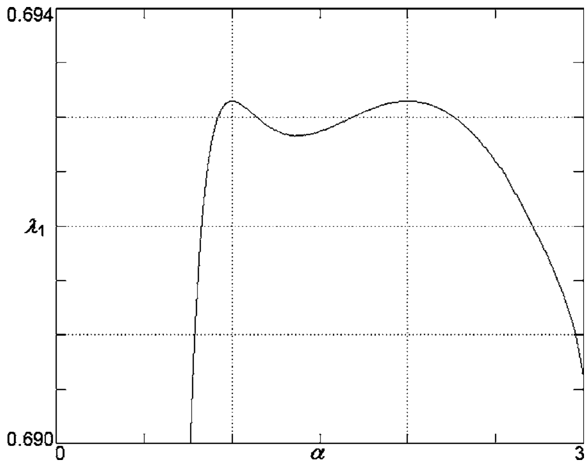

FIG. 1. Lyapunov exponent for the general symmetric map showing peaks at $\alpha=1$ (tent map) and $\alpha=2$ (logistic map) where its value is ln(2).

value of λ_1 was found that was higher than any previously found, the search neighborhood was moved to those coordinates, and ε was increased by a factor of 1.1. Otherwise ε was reduced by a factor of 0.999 and the search continued until ε became negligibly small. Initial conditions were taken as the values at the end of the previous best solution. Once an optimum was found, it was recalculated with many more iterations (typically 10^{11}) to verify the accuracy of the results.

Using this method, the solution rapidly and consistently converged to $a=2$ and $b=0$, which corresponds to a collapse of the Hénon map to the quadratic map, which is conjugate to the logistic map and hence has Lyapunov exponents of $\lambda_1=\ln(2)=0.693\,14\ldots$ and $\lambda_2=-\infty$ with a Kaplan-Yorke dimension of $D_{KY}=1$. In this limit, the map is infinitely dissipative and noninvertible. Said differently, any dissipation less than infinite only reduces the chaoticity.

More interesting and less obvious are the parameters that maximize the Kaplan-Yorke dimension. Using the same method, the values found (to five significant digits) are $a=1$ and $b=0.542\,72$, for which the Lyapunov exponents are $\lambda_1=0.271\,42$ and $\lambda_2=-0.882\,58$ with a Kaplan-Yorke dimension of $D_{KY}=1.307\,53$. Throughout this paper, numerical results are quoted to the number of digits that are thought to be significant, although the last digit is usually only an approximation. Figure 2 shows the attractor for this case, which is recognizably Hénon-like but with a bit more structure as befits its higher dimension, along with its basin of attraction. The basin boundary touches the attractor at numerous places, as is also the case for the general symmetric maps with $A=1$ and is a general feature of these optimized solutions. Attractors are usually most chaotic or most complex just before their orbits become unbounded. As an aside, note that if one uses the entropy as a measure of complexity, its value would be greatest where λ_1 is greatest, since by Pesin's identity,[7] the entropy is the sum of the positive Lyapunov exponents and all the cases considered in this paper have a single positive exponent. Unlike chaoticity, which is quantified by the largest Lyapunov exponent, there is no universally agreed upon definition of complexity, but the dimension

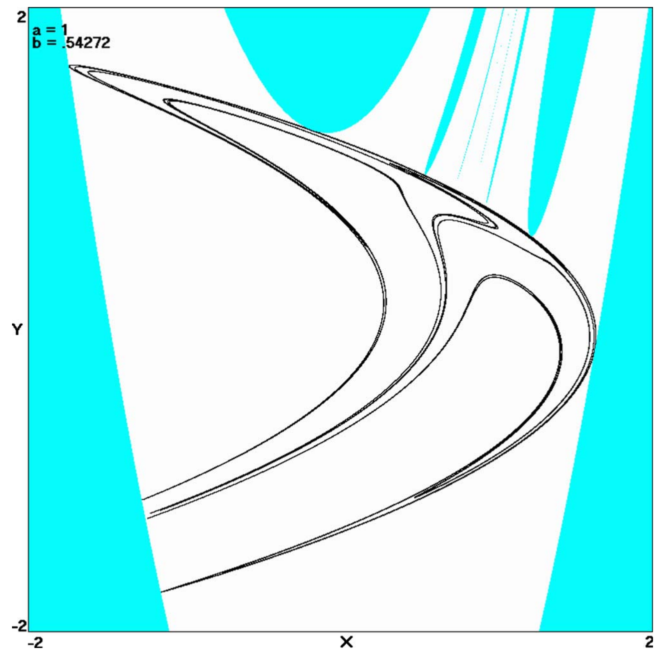

FIG. 2. (Color online) Attractor for the maximally complex Hénon map with $a=1$ and $b=0.542\,72$ along with its basin of attraction.

of the attractor (most accurately calculated using the Kaplan-Yorke conjecture) captures the notion of the minimum number of variables required for a system to exhibit the given behavior.

III. LOZI MAP

Closely related to the Hénon map is the Lozi map,[8] given by

$$X_{n+1} = 1 - a|X_n| + bY_n, \quad Y_{n+1} = X_n. \tag{3}$$

It can be viewed as a piecewise-linear approximation to the Hénon map in the same way that the tent map is a piecewise-linear approximation to the logistic map. Typical parameters are $a=1.7$ and $b=0.5$, where the Lyapunov exponents are $\lambda_1=0.470\,23$ and $\lambda_2=-1.163\,38$ with a Kaplan-Yorke dimension of $D_{KY}=1.404\,19$. Not surprisingly, it is also maximally chaotic for $a=2$ and $b=0$, where the Lyapunov exponents are $\lambda_1=\ln(2)=0.693\,14\ldots$ and $\lambda_2=-\infty$ with a Kaplan-Yorke dimension of $D_{KY}=1$.

Much more interesting are the parameters that maximize the Kaplan-Yorke dimension, which occur along the boundary $b=4-2a$ where the solutions become unbounded.[9] The greatest dimension occurs for $a=1.7052$ and $b=0.5896$, for which the Lyapunov exponents are $\lambda_1=0.448\,36$ and $\lambda_2=-0.976\,67$ with a Kaplan-Yorke dimension of $D_{KY}=1.459\,07$. Figure 3 shows the attractor for this case along with its basin of attraction, whose boundary touches the attractor.

One might wonder whether it is possible to obtain larger values of the Lyapunov exponent or Kaplan-Yorke dimension by changing the coefficients of the remaining two terms in Eqs. (2) and (3) to values other than unity. Such is not possible because one can in general linearly rescale the two variables X and Y to make two of the coefficients unity with-

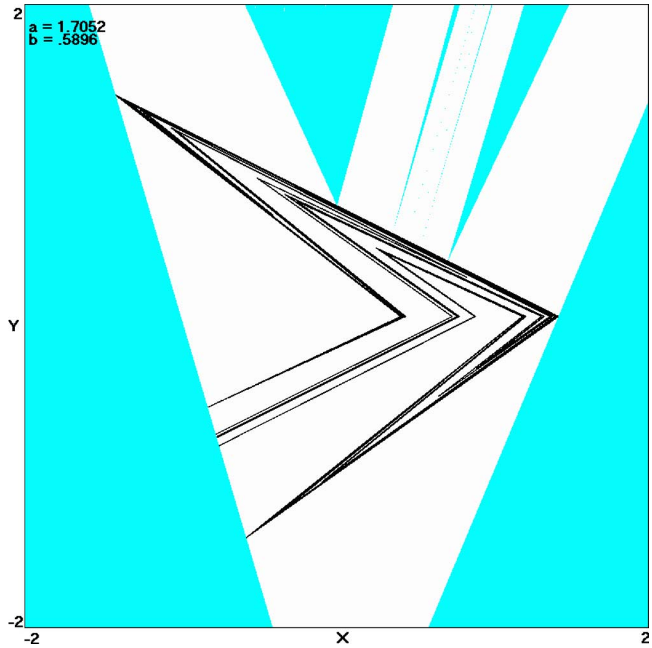

FIG. 3. (Color online) Attractor for the maximally complex Lozi map with $a=1.7052$ and $b=0.5896$ along with its basin of attraction.

out changing the Lyapunov exponent or Kaplan-Yorke dimension. Thus these equations are already in their most general form, although the choice of where to put the parameters is somewhat arbitrary. Indeed, Hénon and Lozi in their original papers put the parameter b as the coefficient of the X_n term in Eqs. (2) and (3), respectively.

IV. LORENZ SYSTEM

These methods can also be applied to autonomous chaotic flows, probably the most widely cited example of which is the Lorenz system[10] given by

$$\dot{x} = \sigma(y - x), \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz, \tag{4}$$

where $\dot{x}=dx/dt$, etc. The parameters suggested by Lorenz are $\sigma=10$, $r=28$, and $b=8/3$, which gives a Lyapunov exponent spectrum[5] of $\lambda=(0.905\,64,\ 0,\ -14.572\,31)$ and a Kaplan-Yorke dimension of $D_{\mathrm{KY}}=1-\lambda_1/\lambda_3=2.062\,15$. Note that for a three-dimensional flow as in Eq. (4), the coefficients of four of the seven terms can be set to unity by a renormalization of x, y, z, and t, so that three parameters suffice to define all the possible dynamics of the system.

However, the parameters chosen by Lorenz affect the time scale of the dynamics, which is given by $1/\sqrt{\sigma r}$, so that an arbitrarily large Lyapunov exponent can be obtained by taking σ and/or r sufficiently large. Indeed, a naive attempt to optimize the parameters for the greatest Lyapunov exponent leads to all three parameters growing without bound, while the dynamics become increasingly rapid. A similar but less rapid divergence of r occurs when attempting to optimize the Kaplan-Yorke dimension.

What is required is a transformation of Eq. (4) into a dimensionless form where the time scale and the attractor size are of order unity. Such a transformation is given by

$$\begin{aligned} &u = x/\sqrt{\sigma r}, \quad v = y/\sqrt{\sigma r}, \\ &w = (z-r)/\sqrt{\sigma r}, \quad \tau = t\sqrt{\sigma r}. \end{aligned} \tag{5}$$

Then in terms of the new variables, Eq. (4) becomes

$$\begin{aligned} &\dot{u} = \alpha(v-u), \quad \dot{v} = -uw - \gamma v, \\ &\dot{w} = uv - \beta/\alpha - \beta w, \end{aligned} \tag{6}$$

where $\dot{u}=du/d\tau$, etc., and the new parameters are given by

$$\alpha = \sqrt{\sigma/r}, \quad \gamma = 1/\sqrt{\sigma r}, \quad \beta = b/\sqrt{\sigma r}. \tag{7}$$

The standard Lorenz parameters then become $\alpha=0.5976$, $\gamma=0.0598$, and $\beta=0.1594$, giving Lyapunov exponents of $\lambda=(0.054\,12,\ 0,\ -0.870\,92)$.

The fact that all three terms in Eq. (6) have a linear damping explains why the basin of attraction for the Lorenz system is the entire state space. The Lorenz system is somewhat special in this sense, and it can be shown that there is an ellipsoid centered on the origin for which $d/dt(x^2+y^2+z^2)$ is everywhere negative, and that eventually traps all trajectories that begin outside it. The state space contraction (the sum of the Lyapunov exponents) is $-(\alpha+\gamma+\beta)$.

Equation (6) is amenable to optimization of both the Lyapunov exponent and the Kaplan-Yorke dimension. The values for α, γ, and β were determined as for the Hénon map except using 2×10^7 iterations of a fourth-order Runge-Kutta integrator with a fixed time-step size of 0.1. As before, the results quoted are from a much longer calculation (typically 10^{10} iterations with a step size of 0.05) to ensure five-digit accuracy of the Lyapunov exponents and Kaplan-Yorke dimension.

A numerical difficulty is that parameters close to those that maximize the Lyapunov exponent give transiently chaotic solutions, some of which are of very long duration, eventually settling to one of the equilibrium points at $(u^*,\ v^*,\ w^*)=(\pm\sqrt{\beta(1/\alpha-\gamma)},\ \pm\sqrt{\beta(1/\alpha-\gamma)},\ -\gamma)$. A sufficient condition for preventing such transients is $\alpha^2(\alpha+3\gamma+\beta)-\alpha+\gamma+\beta<0$, which ensures that these points are unstable. However, that condition is not necessary, and there exists a small region of parameter space where a strange attractor coexists with these stable equilibrium points.[11] Unfortunately, no analytical result is available for calculating this boundary where the largest Lyapunov exponent apparently occurs. This difficulty was addressed by increasing the duration of the computation to 10^8 iterations as the final solution is approached, testing the final solution with $>10^{10}$ iterations, and increasing the final value of α very slightly to give some margin for error. It is also necessary that the equilibrium at $(0,\ 0,\ -1/\alpha)$ be unstable, but this condition is easily satisfied by $\alpha\gamma<1$, which is well within the region of interest.

The maximum Lyapunov exponent calculated as described above apparently occurs for $\alpha=0.300$, $\gamma=0.028$, and $\beta=0.250$, for which the Lyapunov exponents are $\lambda=(0.071\,35,\ 0,\ -0.649\,35)$ and the Kaplan-Yorke dimension is $D_{\mathrm{KY}}=1-\lambda_1/\lambda_3=2.109\,87$. The attractor, shown in Fig. 4 projected onto the uw plane, resembles the familiar Lorenz butterfly attractor with the usual parameters, although

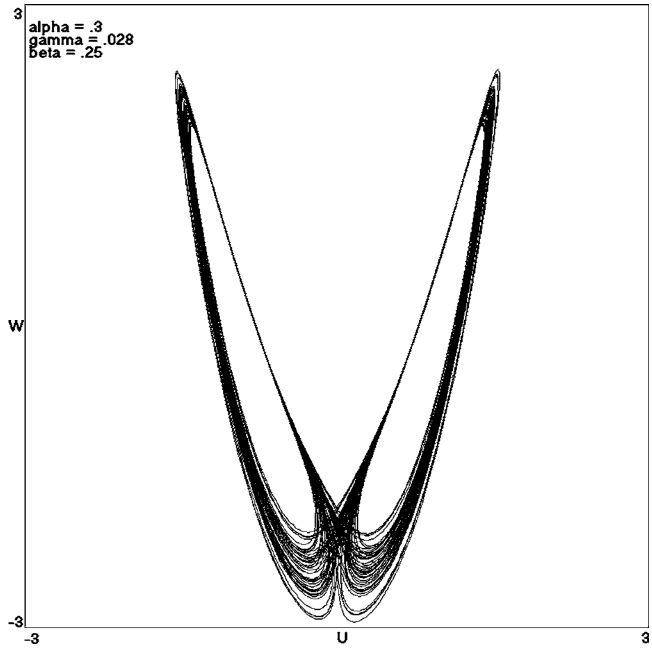

FIG. 4. Attractor for the maximally chaotic normalized Lorenz system with $\alpha=0.300$, $\gamma=0.028$, and $\beta=0.250$.

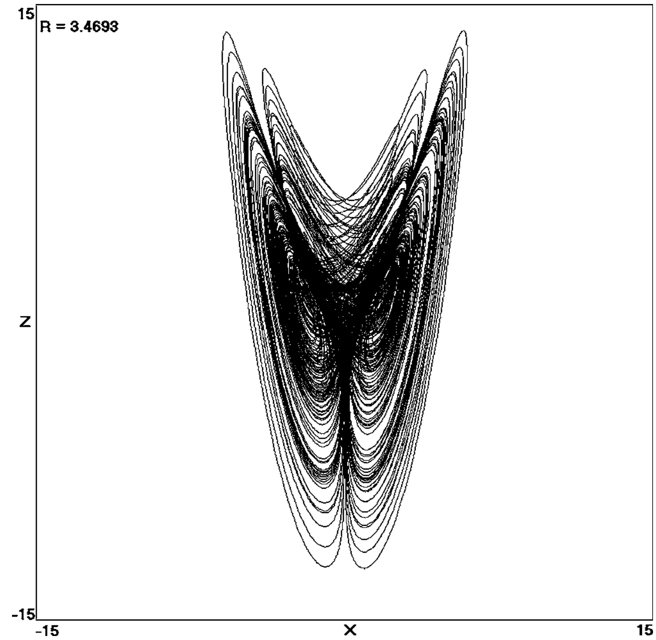

FIG. 5. Attractor for the maximally complex Lorenz system with $R=3.4693$.

it is somewhat thinner. The Lorenz parameters corresponding to this case are $\sigma=\alpha/\gamma=10.71$, $r=1/\alpha\gamma=119.0$, and $b=\beta/\gamma=8.93$, for which the Lyapunov exponents are $\lambda=(2.548\,80,\ 0,\ -23.188\,80)$.

A simple attempt to adjust the parameters of Eq. (6) for the largest Kaplan-Yorke dimension causes all three parameters to shrink to zero. Therefore, the γv and βw terms in Eq. (6) can be set to zero, and the variables rescaled according to $(x,\ y,\ z,\ t)=(\alpha u,\ \alpha v,\ \alpha w,\ \tau/\alpha)$, which results in the one-parameter system

$$\dot{x}=y-x, \quad \dot{y}=-xz, \quad \dot{z}=xy-R, \tag{8}$$

where $R=\beta/\alpha^3=br/\sigma^2$. Apart from signs, Eq. (8) is the diffusionless Lorenz system that has been previously reported[12] and studied.[13] It has equilibrium points at $(x^*,\ y^*\ z^*)=(\pm\sqrt{R},\ \pm\sqrt{R},\ 0)$ with eigenvalues that satisfy the characteristic equation $\lambda^3+\lambda^2+R\lambda+2R=0$.

A simple calculation shows that the maximum Kaplan-Yorke dimension occurs for $R=3.4693$ where the Lyapunov exponents are $\lambda=(0.307\,91,\ 0,\ -1.307\,91)$. For this value of R, the equilibrium points are spiral saddles with eigenvalues $-1.581\,18$, $0.290\,59\pm2.074\,56i$. The attractor for this case is shown in Fig. 5 projected onto the xz plane and has a dimension of $D_{\mathrm{KY}}=2.235\,42$. Equation (8) illustrates nicely how simplifying an equation such as Eq. (4) can increase the complexity of its solution.

As a consistency check, the parameters above can be converted into equivalent Lorenz parameters for $\gamma=\beta=0.01$ (small) as follows:

$$b=\beta/\gamma=1, \quad \sigma=\sqrt[3]{b/\gamma^2R}=\sqrt[3]{10^4/R}, \quad r=1/\gamma^2\sigma=\sqrt[3]{10^8\,R}. \tag{9}$$

For $R=3.4693$, the corresponding parameters are $\sigma=14.232$, $r=702.66$, and $b=1$, which gives Lyapunov exponents of $\lambda=(3.832\,50,\ 0,\ -20.064\,50)$ and a Kaplan-Yorke dimension of $D_{\mathrm{KY}}=2.191\,01$, which is only 2% smaller than the asymptotic value of 2.235 42 in the limit of infinite r.

Figure 6 shows the Poincaré section for the attractor of Fig. 5 in the $y=0$ plane. It shows fractal structure that is not usually visible for the Lorenz system because its dimension is very close to 2.0 for the standard parameters. The basin of attraction includes the entire $y=0$ plane except for the line at $x=0$, where the trajectory escapes to $z=-\infty$. Figure 7 shows how the Kaplan-Yorke dimension varies with R in the cha-

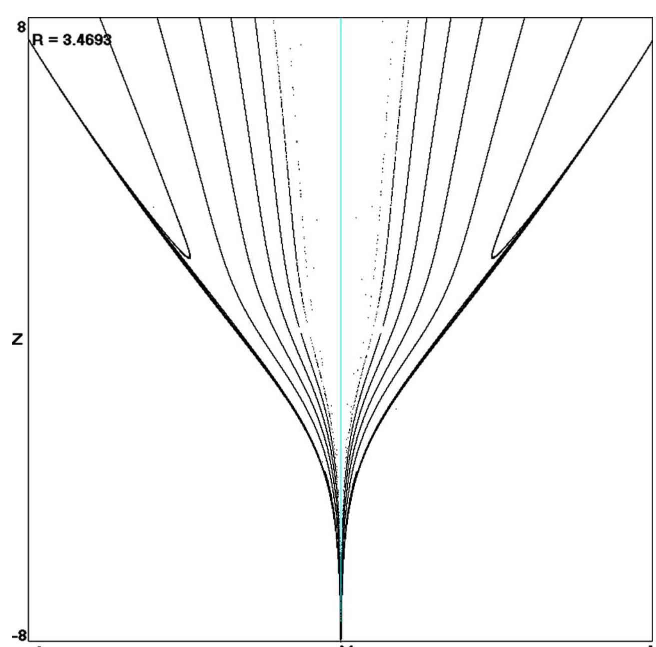

FIG. 6. (Color online) Poincaré section at $y=0$ for the maximally complex Lorenz system with $R=3.4693$.

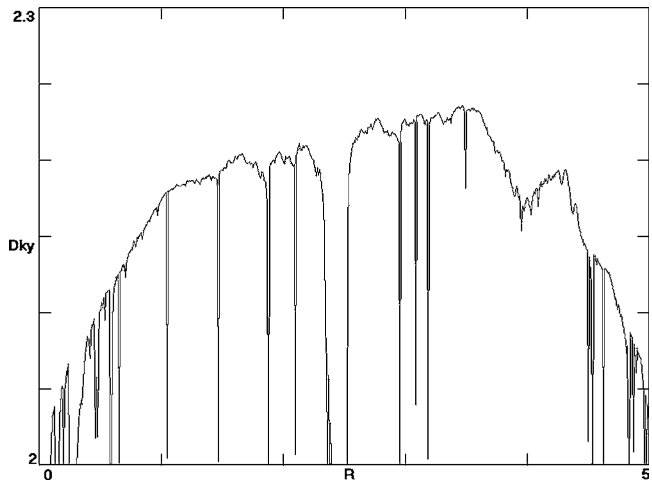

FIG. 7. Variation of the Kaplan-Yorke dimension with the parameter R for the diffusionless Lorenz system in Eq. (8).

otic region along with numerous periodic windows as is typical for low-dimensional chaotic systems. This figure also illustrates why a random search method is necessary to avoid getting caught in one of the many local maximums, which are even more numerous in a three-dimensional parameter space.

V. RÖSSLER SYSTEM

The last case to be considered is the Rössler system[14] given by

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c). \tag{10}$$

The parameters suggested by Rössler are $a=b=0.2$ and $c=5.7$, which gives a Lyapunov exponent spectrum of $\lambda=(0.0714,\ 0,\ -5.3943)$ and a Kaplan-Yorke dimension of $D_{\mathrm{KY}}=1-\lambda_1/\lambda_3=2.0132$. Fortunately, this system is immune to most of the difficulties encountered with the Lorenz system except that both the maximum Lyapunov exponent and the maximum Kaplan-Yorke dimension apparently occur just before the orbits become unbounded (for example, by increasing c slightly), although such orbits occur without very long-duration transients and are thus relatively easy to detect. The optimization is straightforward using the method described earlier except with a Runge-Kutta time step size of 0.02.

The Lyapunov exponent has its greatest value for $a=0.395$, $b=0.487$, and $c=8.164$ where the Lyapunov exponent spectrum is $\lambda=(0.248\,92,\ 0,\ -5.816\,60)$ and the Kaplan-Yorke dimension is $D_{\mathrm{KY}}=2.042\,80$. Its attractor projected onto the xy plane as shown in Fig. 8 resembles the familiar Rössler attractor with the usual parameters.

The Kaplan-Yorke dimension has its greatest value of $D_{\mathrm{KY}}=2.158\,70$ for $a=0.6276$, $b=0.7980$, and $c=2.0104$ where the Lyapunov exponent spectrum is $\lambda=(0.102\,99,\ 0,\ -0.648\,96)$. Its attractor projected onto the xy-plane as shown in Fig. 9 resembles the familiar Rössler attractor except that it is somewhat more compact.

These parameters offer the opportunity to display the fractal structure of the Rössler attractor in a Poincaré section,

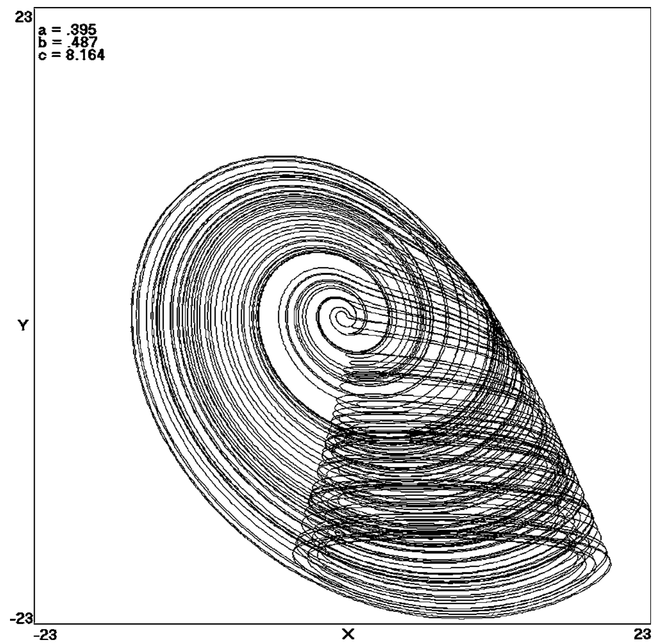

FIG. 8. Attractor for the maximally chaotic Rössler system with $a=0.395$, $b=0.487$, and $c=8.164$.

something that is almost never done because the dimension of the attractor is so very close to 2.0 for the usual parameters. There are infinitely many possible Poincaré sections from which to choose, but Fig. 10 shows a slice through the plane at $z=z_-^*$, which is the plane in which lies one of the equilibrium points given by $(x^*,\ y^*,\ z^*)$ with $x^*=c/2\pm\sqrt{c^2/4-ab}=az^*=-ay^*$. Also shown in the figure is the basin of attraction, whose boundary apparently touches the attractor at three points in this plane.

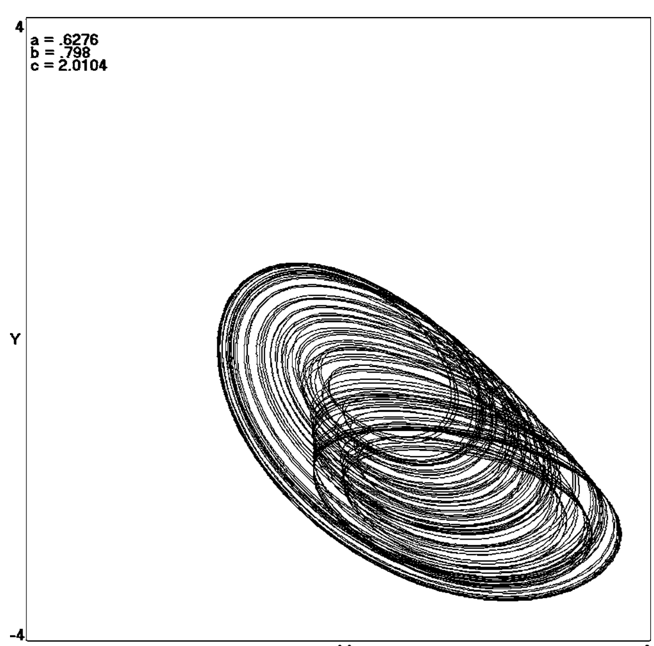

FIG. 9. Attractor for the maximally complex Rössler system with $a=0.6276$, $b=0.7980$, and $c=2.0104$.

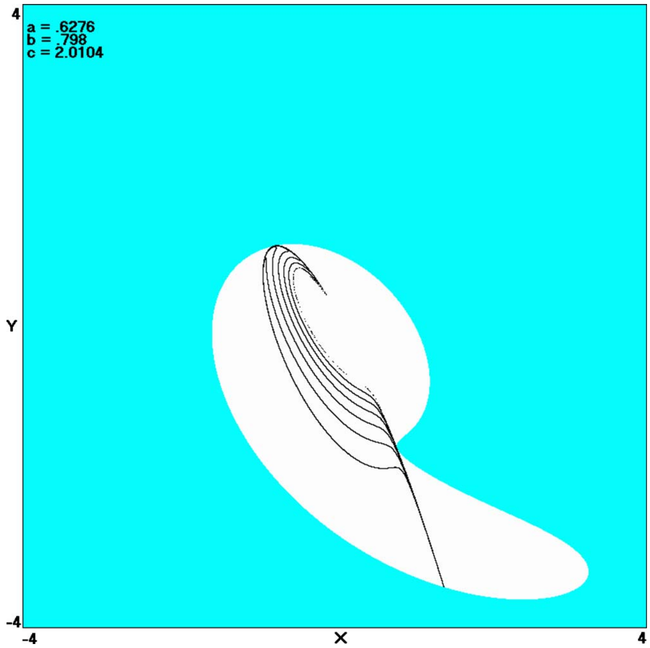

FIG. 10. (Color online) Poincaré section at $z=z_{-}^{*}$ for the maximally complex Rössler system with $a=0.6276$, $b=0.7980$, and $c=2.0104$.

VI. CONCLUSIONS

This paper has reported estimates of the parameters that optimize the chaoticity and complexity (or strangeness) of a number of common chaotic systems. For some purposes, these parameters might be more suitable than the ones usually used. It offers a view of the attractors that is rarely seen. The attractors resemble the conventional ones, but the higher dimension especially allows one to see their fractal structure much more easily. The method cannot guarantee that these are the absolute best such parameters, but they provide a close lower bound since the search involved many weeks of around-the-clock computation for each case and multiple instances of the random search converged on similar values. Also, only positive values of the parameters were explored since negative values in some sense represent a different system. The method could be applied to other chaotic systems for which there is an interest in such an optimization. In this way, one can answer the question of how chaotic or how complex the solution of a given mathematical system can be. It might be desirable when reporting new chaotic systems to perform such an optimization on their parameters as a standard and expected part of their characterization.

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