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A Unified Piecewise Smooth Chaotic Mapping that Contains the Hénon and the Lozi Systems

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Abstract

In this paper we introduce a new piecewise smooth mapping of the plane as a unified discrete-time chaotic system that contains the original Hénon and Lozi systems as two extremes and other systems as a transition in between and that has robust homoclinic chaos over a portion of its key system parameters. Dynamical behaviors of the unified system are investigated in some detail.

Keywords: A unified piecewise smooth map, transition Hénon-like and Lozi-like chaotic attractors, robust homoclinic chaos.

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1 Introduction

Discrete mathematical models arise directly from experiment or by the use of the Poincaré section for the study of continuous models. Two of these models are the Hénon [1] and Lozi [2] maps given by, respectively:

$$H(x,y) = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} \text{ and } L(x,y) = \begin{pmatrix} 1 - a\,|x| + y \\ bx \end{pmatrix}. \tag{1}$$

The H mapping gives a chaotic attractor called the *Hénon attractor*, which is obtained for $a = 1.4$ and $b = 0.3$ as shown in Fig. 1(a). There are many papers that discuss the original Hénon and Lozi maps such as [3-6]. Moreover, it is possible to change the form of the Hénon mapping H to obtain other chaotic attractors [2-7-8]. Applications of these maps include secure communications using the notions of chaos [11-12]. The Lozi map L is a 2-D invertible iterated map that gives a chaotic attractor called the *Lozi attractor*, which is obtained for $a = 1.4$ and $b = 0.3$ as shown in Fig. 1(b). It is therefore interesting to ask if there is a chaotic system that can unify these two chaotic systems and realize the continued transition from one to the other. This paper provides a positive answer to this question and reveals a surprising property of the transitional systems.

2 Robust chaos and its applications

Robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space, since the existence of these windows in some chaotic regions implies that small changes of the parameters would destroy the chaos. This effect implies the fragility of this type of chaos. Contrary to this situation, there are many practical applications such as in communications and spreading the spectrum of switch-mode power supplies to avoid electromagnetic interference [13-14] where it is necessary to obtain reliable operation in the chaotic mode and thus where robust chaos is required. A practical example can be found from electrical engineering to demonstrate robust chaos as shown in [10]. The occurrence of robust chaos in a smooth system is proved and discussed in [16] along with a general theorem and a practical procedure for constructing S-unimodal maps that generate robust chaos. This result is contrary to the hypothesis that robust chaos cannot exist in smooth systems [10]. On the other hand, many methods are used to search for a smooth and robust chaotic map, such as in [15], where a one-dimensional smooth map that generates robust chaos in a large domain of the parameter space is presented. In [17], simple polynomial unimodal maps that show robust chaos are constructed. Other methods are given in [16-18].

3 The unified chaotic system that contains the Hénon and the Lozi mappings

Since practical applications of chaos require the chaotic orbit to be robust, we introduce in this paper a new unified chaotic system that reduces to the original Hénon and Lozi systems [1-2] as two extremes and to other systems as a transition in between, and which has robust homoclinic chaos over a portion of its key system parameters. The proposed unified chaotic model is a piecewise smooth map of the plane defined by:

$$U(x,y) = \begin{pmatrix} 1 - 1.4 f_{\alpha}(x) + y \\ 0.3x \end{pmatrix}, \tag{2}$$

where $0 \leq \alpha \leq 1$ is the bifurcation parameter and the function f_{α} is given by:

$$f_{\alpha}(x) = \alpha |x| + (1-\alpha) x^2. \tag{3}$$

It is easy to remark that for $\alpha = 0$, one has the original Hénon map, and for $\alpha = 1$, one has the original Lozi map, and for $0 < \alpha < 1$, the unified chaotic map (2) is chaotic with different kinds of attractors. The Lyapunov exponents and bifurcation diagram are shown in Fig. 2. We remark that the unified chaotic map (2) is a piecewise smooth map and due to the shape of the vector field U of the unified chaotic map (2), the plane can be divided into two regions denoted by:

$$D_1 = \left\{(x, y) \in \mathbb{R}^2 /\ x < 0\right\} \tag{4}$$

$$D_2 = \left\{(x, y) \in \mathbb{R}^2 /\ x > 0\right\}. \tag{5}$$

Let us define:

$$A = \left\{(x, y) \in \mathbb{R}^2 /\ x = 0\right\}, \tag{6}$$

which denotes a smooth curve that divides the phase plane into two regions D_1 and D_2, so that the unified chaotic map (2) can be rewritten as follow:

$$U(x, y) = \begin{pmatrix} \begin{cases} 1.4\,(\alpha - 1)\,x^2 + 1.4\alpha x + y + 1, & \text{if } (x, y) \in D_1 \\ 1.4\,(\alpha - 1)\,x^2 - 1.4\alpha x + y + 1, & \text{if } (x, y) \in D_2 \end{cases} \\ 0.3x \end{pmatrix}, \tag{7}$$

where in each of these regions the system (2) is a quadratic map. Notably, the unified system (2) has some special features and advantages as follows:

(1) System (2) is chaotic when $0 \le \alpha \le 1$.

(2) System (2) connects the Hénon and the Lozi maps and realizes the entire transition spectrum from one to the other.

(3) The control parameter α in system (2) reveals the evolution of dynamical behaviors from the Hénon to the Lozi attractors.

(4) System (2) has robust chaotic attractors for $0.493122734 \le \alpha < 1$, while it is absent for $\alpha = 0$ and $\alpha = 1$.

4 Numerical simulations

In this section, the dynamical behaviors of the unified chaotic system (2) will be investigated numerically. For $0 \le \alpha \le 1$, the unified chaotic system has two kinds of chaotic orbits: Hénon-like chaotic attractors over the first portion of the interval $[0, 1[$ and a Lozi-like chaotic attractor over the second portion of the interval $]0, 1]$ as shown in Fig. 3(a) and (c). It seems that this phenomenon is related to the shape of the function f_α, where for values of α close to zero, the function f_α given in (3) behaves like the quadratic term x^2, while the values of α close to unity the function f_α behaves like the absolute value function $|x|$, as shown in Fig. 3(b) and (d). This explains the occurrence of the two kinds of chaotic attractors mentioned above.

5 A rigorous proof of the robustness of the homoclinic chaos

In this section, we begin by studying the existence of the fixed point of the U mapping in order to determine the associated normal form for the unified chaotic map (2), which

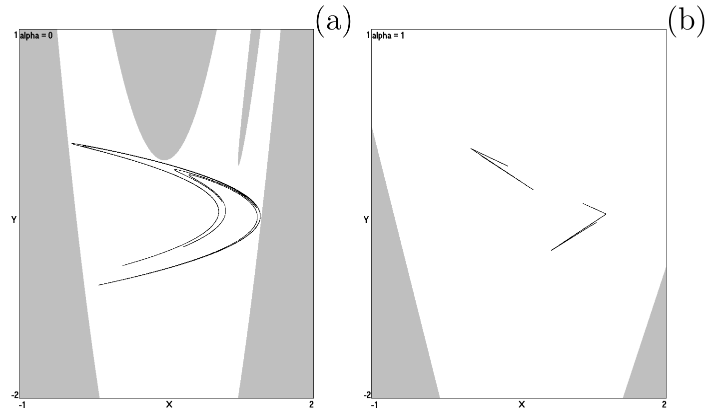

Figure 1: (a) The original Hénon chaotic attractor obtained from the H mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$. (b) The original Lozi chaotic attractor obtained from the L mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$.

permits us to prove rigorously the occurrence of robust homoclinic chaos, where we exclude the values $\alpha = 0$ and $\alpha = 1$ since both the Hénon and Lozi mapping are studied in detail in several works and in the references therein. We will show that if $0 \leq \alpha < 1$, then the unified chaotic map (2) has two fixed points given by:

$$P_1 = (x_1, 0.3x_1) \in D_1 \text{ and } P_2 = (x_2, 0.3x_2) \in D_2, \tag{8}$$

where

$$\begin{cases} x_1 = \frac{-0.7\alpha + 0.35 + \frac{\sqrt{-7.\,56\alpha + 1.\,96\alpha^2 + 6.\,09}}{2}}{1.\,4(\alpha - 1)} \\ \\ x_2 = \frac{0.7\alpha + 0.35 - \frac{\sqrt{-3.\,64\alpha + 1.\,96\alpha^2 + 6.\,09}}{2}}{1.\,4(\alpha - 1)}. \end{cases} \tag{9}$$

Obviously, the fixed points of the unified chaotic map (2) are the real solutions of the system:

$$1 - 1.4 f_\alpha(x) + y = x \text{ and } y = 0.3x. \tag{10}$$

Hence one may easily obtain the two equations:

$$1.4(\alpha - 1)x^2 + (1.4\alpha - 0.7)x + 1 = 0 \text{ for } x < 0 \text{ and } y = 0.3x \tag{11}$$

$$1.4(\alpha - 1)x^2 - (1.4\alpha + 0.7)x + 1 = 0 \text{ for } x > 0 \text{ and } y = 0.3x. \tag{12}$$

If $0 \leq \alpha < 1$, then $1.4(\alpha - 1) < 0$, and the discriminant of the first equation of (11) is $-7.56\alpha + 1.96\alpha^2 + 6.09 > 0$. Thus, one can easily conclude that the only negative solution of the first equation of (11) is:

$$x_1 = \frac{-0.7\alpha + 0.35 + \frac{\sqrt{-7.56\alpha + 1.96\alpha^2 + 6.09}}{2}}{1.4(\alpha - 1)} < 0. \tag{13}$$

On the other hand, the discriminant of the first equation of (12) is $-3.64\alpha + 1.96\alpha^2 + 6.09 > 0$ for all $0 \leq \alpha < 1$. Thus, one can easily conclude that the only positive solution of

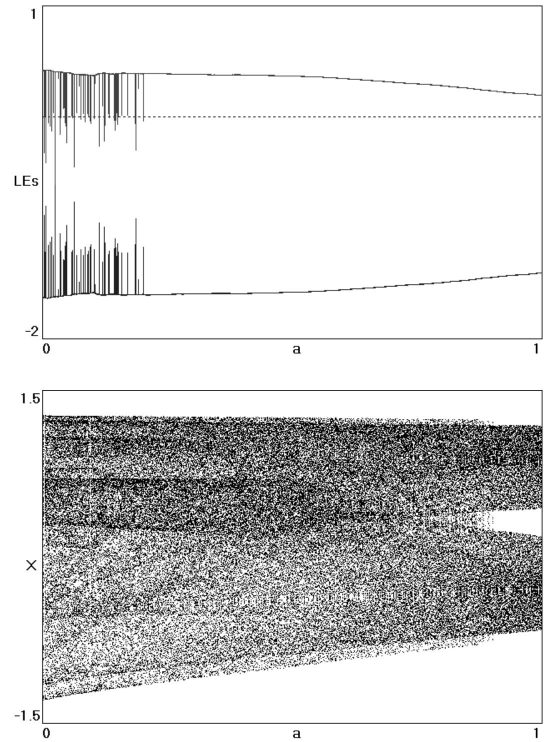

Figure 2: (a) Variation of the Lyapunov exponents of the unified map (2) for $0 \leq \alpha \leq 1$. (b) The bifurcation diagram of the unified chaotic map (2) for $0 \leq \alpha \leq 1$.

the first equation of (12) is:

$$x_2 = \frac{0.7\alpha + 0.35 - \frac{\sqrt{-3.64\alpha + 1.96\alpha^2 + 6.09}}{2}}{1.4\,(\alpha - 1)} > 0. \tag{14}$$

Finally, the unified chaotic map (2) has two simultaneous fixed points defined for $0 < \alpha < 1$ as $P_1 = (x_1, 0.3x_1) \in D_1$ and $P_2 = (x_2, 0.3x_2) \in D_2$.

The Jacobian matrix of the unified chaotic map (2) evaluated at a point (x, y) in the region D_1 is given by:

$$J_1\,(x, y) = \begin{pmatrix} 1.4\alpha - 2.8x + 2.8x\alpha & 1 \\ 0.3 & 0 \end{pmatrix}, \tag{15}$$

and at a point (x, y) in the region D_2 the Jacobian matrix is given by:

$$J_2\,(x, y) = \begin{pmatrix} 2.8x\alpha - 1.4\alpha - 2.8x & 1 \\ 0.3 & 0 \end{pmatrix}. \tag{16}$$

Thus, at P_1 one has:

$$J_1\,(P_1) = \begin{pmatrix} 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} & 1 \\ 0.3 & 0 \end{pmatrix}. \tag{17}$$

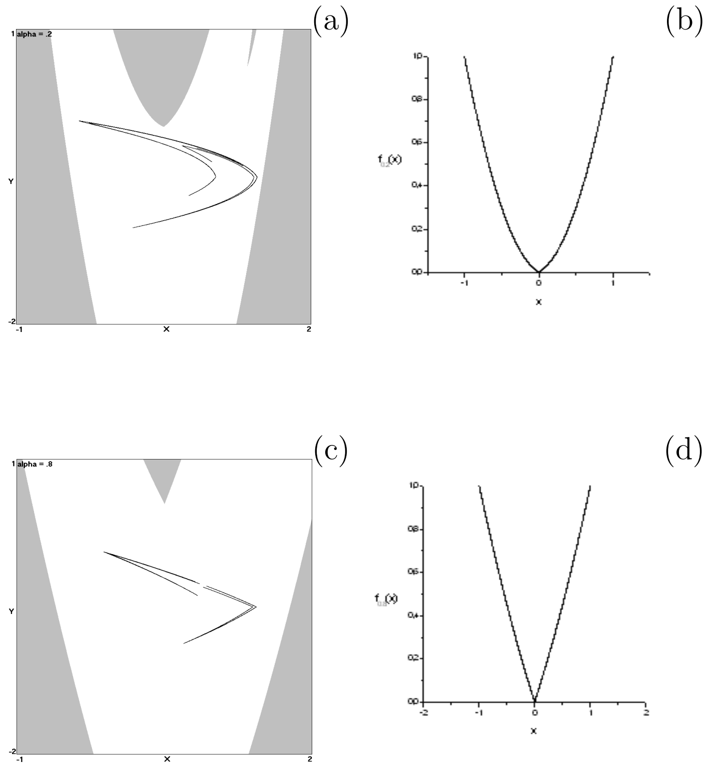

Figure 3: (a) The transition Hénon-like chaotic attractor obtained for the unified chaotic map (2) with its basin of attraction (white) for $\alpha = 0.2$. (b) The graph of the function $f_{0.2}$. (c) The transition Lozi-like chaotic attractor obtained for the unified chaotic map (2) with its basin of attraction (white) for $\alpha = 0.8$. (d) The graph of the function $f_{0.8}$.

The eigenvalues of $J_1(P_1)$ are

$$\begin{cases} \lambda_1 = \frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}+\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35 \\ \lambda_2 = \frac{\sqrt{1.96\alpha^2-7.56\alpha+6.09}-\sqrt{1.96\alpha^2-7.56\alpha+1.4\sqrt{1.96\alpha^2-7.56\alpha+6.09}+7.78}}{2}+0.35, \end{cases} \tag{18}$$

and at P_2 one has:

$$J_2(P_2) = \begin{pmatrix} 0.7-\sqrt{1.96\alpha^2-3.64\alpha+6.09} & 1 \\ 0.3 & 0 \end{pmatrix}. \tag{19}$$

The eigenvalues of $J_2(P_2)$ are:

$$\begin{cases} \omega_1 = \frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}+\sqrt{1.96\alpha^2-3.64\alpha-1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35, \\ \omega_2 = \frac{-\sqrt{1.96\alpha^2-3.64\alpha+6.09}-\sqrt{1.96\alpha^2-3.64\alpha-1.4\sqrt{1.96\alpha^2-3.64\alpha+6.09}+7.78}}{2}+0.35, \end{cases} \tag{20}$$

In the case of two-dimensional piecewise smooth maps, it is possible to choose an appropriate coordinate transformation so that the choice of axis is independent of the parameter.

In so doing, the normal form of map (1) is given by [9]:

$$N(x,y)=\begin{cases}\begin{pmatrix}\tau_1 & 1\\ -\delta_1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}\mu, \text{ if } x<0,\\ \\ \begin{pmatrix}\tau_2 & 1\\ -\delta_2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}\mu, \text{ if } x>0,\end{cases} \tag{21}$$

where μ is a parameter, and $\tau_i, \delta_i, i=1,2$ are the traces and determinants of the corresponding matrices of the linearized map in the two subregion D_1 and D_2 evaluated at P_1 and P_2 respectively, and they are given by:

$$\begin{cases}\tau_1=0.7+\sqrt{1.96\alpha^2-7.56\alpha+6.09},\\ \tau_2=0.7-\sqrt{1.96\alpha^2-3.64\alpha+6.09},\\ \delta_1=\ \delta_2=-0.3,\end{cases} \tag{22}$$

It is shown in [10] that a robust homoclinic chaos (i.e. the existence of an infinity of homoclinic intersections between the two subregions D_1 and D_2) occurs in the piecewise smooth map of the form (21) when:

$$\begin{cases}\tau_1>1+\delta_1, \text{ and } \tau_2<-(1+\delta_2),\\ \delta_1<0, \text{ and } -1<\delta_2<0,\end{cases} \tag{23}$$

and the condition:

$$\frac{\lambda_1-1}{\tau_1-1-\delta_1}>\frac{\omega_2-1}{\tau_2-1-\delta_2}, \tag{24}$$

where the parameter range for boundary crisis is given by:

$$(\lambda_2-\tau_2)\lambda_1-\tau_1+\tau_2+\delta_1>0, \tag{25}$$

because $\delta_1=\delta_2$, where the inequality (25) determine the condition of stability of the chaotic attractor. However, if the first condition (24) is not satisfied, then the condition of existence of the chaotic attractor changes to:

$$\frac{\omega_2-1}{\tau_2-1-\delta_1}<\frac{(\tau_1-\delta_1-\lambda_2)}{(\tau_1-1-\delta_1)(\lambda_2-\tau_2)}, \tag{26}$$

because $\delta_1=\delta_2$. Finally, the formulas (18), (20), and (22), and the inequalities (23), (24), and (25), or the inequalities (23), (25), and (26) if they are satisfied, determine rigorously the region for the parameter α where the unified map (2) has robust homoclinic chaos.

6 Discussion

First, it is clear that the conditions of (23) are satisfied for all $0<\alpha<1$. Second, it is difficult to solve rigorously the conditions for existence of the chaotic attractor (24) or (26) and its condition for stability (25) since these inequalities contain complicated square formulas. Hence, we use numerical estimates of the portion of the range $0\leq\alpha<1$, for

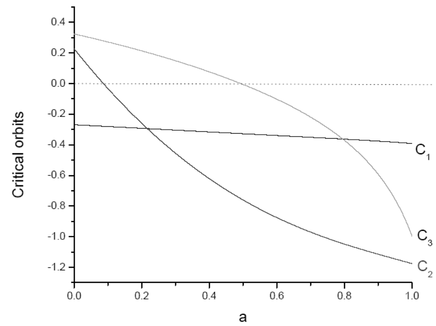

Figure 4: Critical curves corresponding to the conditions (24), (25) and (26).

which robust homoclinic chaos occurs in the unified piecewise smooth map (2). Here we exclude the value $\alpha = 0$, since there is no robust chaos in the Hénon map. We also exclude the value $\alpha = 1$, since both fixed points given in (9) are not defined for this value.

Second, let us consider the critical curves corresponding to the conditions (24), (25), and (26) as follows:

$$\begin{cases} C_1 : \frac{\lambda_1 - 1}{\tau_1 - 1 - \delta_1} - \frac{\omega_2 - 1}{\tau_2 - 1 - \delta_2} = 0, \\ C_2 : (\tau_2 - \lambda_2)\lambda_1 + \tau_1 - \tau_2 - \delta_1 = 0, \\ C_3 : \frac{\omega_2 - 1}{\tau_2 - 1 - \delta_2} - \frac{\delta_1(\tau_1 - \delta_1 - \lambda_2)}{(\tau_1 - 1 - \delta_1)(\delta_2\lambda_2 - \delta_1\tau_2)} = 0, \end{cases} \tag{27}$$

From Fig. 4 we remark that the curve (C_2) has an intersections with the axis $y = 0$, at $\alpha = 0.0866592234$, then conditions (24) holds for $\alpha \in [0, 0.0866592234]$, while the curve (C_1) does not hits the axis $y = 0$, then conditions (25) does not holds for all $0 \le \alpha < 1$, and the curve (C_3) hits the axis $y = 0$ also one time at $\alpha = 0.493122734$, then condition (26) holds when $\alpha \in [0.493122734, 1[$, where the Newton method for finding roots of an algebraic equation was used with an error of 10^{-6}. Thus, the homoclinic chaos presented by the unified chaotic map (2) is robust not stable when $\alpha \in [0.493122734, 1[$, because the condition (25) does not hold in this interval. The chaotic attractor cannot be destroyed by small changes in the parameters, since small changes in the parameters can only cause small changes in the Lyapunov exponents. Hence, the percentage for the parameter $0 \le \alpha < 1$, in which the map (2) converges to a robust chaotic attractor is approximately 50.6 88 percent, this result is also verified numerically by computing Laypunouv exponents and bifurcation diagram as shown in Fig. 2.

For $\alpha < 0.493122734$, the chaos is not robust in some ranges of the variable α, because there are numerous small periodic windows as shown in Figs. 5 (a), 5 (b) for example the period-8 window at $\alpha = 0.025$. Also, for $\alpha = 0.114$, there is some periodic windows. We remark, also the existence of some regions in the $\alpha-$line where the largest Lyapunouv exponent is positive, but this does not guaranty the unicity of the attractor, contrary in the case where $\alpha \in [0.493122734, 1[$, where there is guaranteed that the attractor is unique, due to the analytical expressions (23), (24), (25), and (26). When α approaches 0, there is a break of smoothness and the dynamics is too chaotic and presents some chaotic attractors very similar to the original Hénon attractor shown in Figs. 3 (a). Finally, it is interesting and surprising that the unified system (2) has such a property

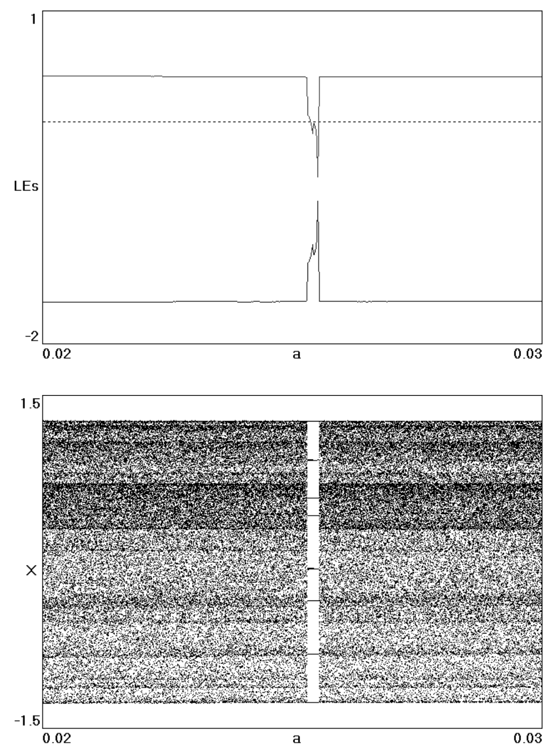

Figure 5: (a) Variation of the Lyapunov exponents of the unified chaotic map (2) for $0.02 \leq \alpha \leq 0.03$. (b) The bifurcation diagram of the unified chaotic map (2) for $0.02 \leq \alpha \leq 0.03$, showing a period-8 attractor obtained for $\alpha = 0.025$.

for an intermediate α while it was absent for $\alpha = 0$ or $\alpha = 1$ since the Hénon map is a *quasiattractor* and the Lozi map is a *Lorenz-type attractor*. These types of chaotic attractors have no robust homoclinic chaos over all portion of their key system parameters.

7 Conclusion

We have reported some results relevant to a new piecewise smooth 2-D discrete chaotic map as a unified chaotic system that contains the original Hénon and the Lozi systems as two extremes and other systems as a transition in between, and which has robust homoclinic chaos over a portion of its key system parameters, while this property is absent for the two systems at its extremes. Dynamics of piecewise continuous (smooth) mappings are a newly emerging area of research, due to the absence of continuity (smoothness), exist theories/methods in dynamical systems are not directly applicable, so new methods are needed for this important area.

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