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CHAOTIFYING 2-D PIECEWISE-LINEAR MAPS VIA A PIECEWISE-LINEAR CONTROLLER FUNCTION

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A simple method for chaotifying piecewise-linear maps of the plane using a piecewise-linear controller function is given. A domain of chaos in the resulting controlled map was determined exactly and rigorously.

1. Introduction

A large number of physical and engineering systems have been found to be governed by a class of continuous or discontinuous maps [1, 4, 5, 7–10, 13, 14] where the discrete-time state space is divided into two or more compartments with different functional forms of the map separated by borderlines [14–17]. The theory of discontinuous maps is in its infancy, with some progress reported for 1-D and n-D discontinuous maps in [1, 8, 9, 11, 12, 17], but these results are restrictive and cannot be obtained in the general n-dimensional context [12]. On the other hand, there are many works that focus on the chaotic behavior of discrete mappings. For example, they have been studied as control and anticontrol (chaotification) schemes using Lyapunov exponents [2, 3, 18, 20] to prove the existence of chaos in n-dimensional discrete dynamical systems with the goal of making in some way an originally nonchaotic dynamical system chaotic or enhancing the chaos already existing in such a system.

The goal of this work is to present a simple method for chaotifying an arbitrary 2-D piecewise-linear map (continuous or not) using a simple piecewise-linear controller function, which allows one to determine exactly and rigorously which portion of the bifurcation parameter space is characterized by the occurrence of chaos in the resulting controlled map using the standard definition of the Lyapunov exponents as the test for chaos.

Consider an arbitrary piecewise-linear map $f\colon D \to D$, where $D = D_1 \cup D_2 \subset \mathbb{R}^2$, defined by

$$X_{k+1} = f(X_k) = \begin{cases} AX_k + b & \text{if} \quad X_k \in D_1, \\ BX_k + c & \text{if} \quad X_k \in D_2, \end{cases} \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are 2×2 real matrices,

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \text{and} \qquad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

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are 2×1 real vectors, and

$$X_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \in \mathbb{R}^2$$

is the state variable.

The Lyapunov exponents of a 2-D dynamical system are defined as follows:

Consider the system

$$X_{k+1} = f(X_k), \quad X_k \in \mathbb{R}^2, \quad k = 0, 1, 2, \ldots, \tag{2}$$

where the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the vector field associated with system (2). Let $J(X_k)$ be its Jacobian evaluated at $X_k \in \mathbb{R}^2$, $k = 0, 1, 2, \ldots,$ and define the matrix

$$T_n(X_0) = J(X_{n-1})\, J(X_{n-2}) \ldots J(X_1)\, J(X_0). \tag{3}$$

Moreover, let $J_i(X_0, n)$ be the modulus of the ith eigenvalue of the nth matrix $T_n(X_0)$, where $i = 1, 2$ and $n = 0, 1, 2, \ldots$.

The Lyapunov exponents for a two-dimensional discrete-time system are now defined by

$$l_i(X_0) = \lim_{n \to +\infty} \ln\left(J_i(X_0, n)^{\frac{1}{n}}\right), \quad i = 1, 2. \tag{4}$$

Roughly speaking, chaotic behavior implies sensitive dependence on initial conditions, with at least one positive Lyapunov exponent. Based on this definition, we give in the next section a rigorous proof of chaos in the resulting controlled map (7) obtained below via a simple piecewise-linear controller function applied to map (1). While many algorithms for calculating the Lyapunov exponents would give spurious results for piecewise-linear discontinuous maps, the algorithm used here and given in [21] works for such cases. It essentially takes a numerical derivative and gives the correct result provided that is taken to ensure that the perturbed and unperturbed orbits lie on the same side of the discontinuity. This may require an occasional small perturbation into a region that is not strictly accessible to the orbit.

2. Chaotification Method Using a Piecewise-Linear Controller Function

The main idea of the chaotification method presented in this work is to introduce a piecewise-linear controller function in such a way that the two (in both partitions) system matrices of the resulting controlled piecewise-linear system have the same trace and determinant (invariants) and, hence, the same eigenvalues. Indeed, the controlled map is given by

$$g(X_k) = \begin{cases} AX_k + b & \text{if} \quad X_k \in D_1 \\ BX_k + c & \text{if} \quad X_k \in D_2 \end{cases} + U(X_k), \tag{5}$$

where the controller function $U(X_k)$ is defined by

$$U\left(X_k\right)=\begin{cases}\begin{pmatrix}\left(b_{11}+b_{22}-a_{11}-a_{22}\right)x_k\\ \left(\dfrac{-a_{12}a_{21}+b_{11}a_{22}-b_{11}b_{22}+b_{12}b_{21}+a_{22}b_{22}-a_{22}^2}{a_{12}}\right)x_k\end{pmatrix} & \text{if} \quad X_k\in D_1,\\ \begin{pmatrix}0\\0\end{pmatrix} & \text{if} \quad X_k\in D_2.\end{cases} \tag{6}$$

The controlled system (5) is now given by

$$g\left(X_k\right)=\begin{cases}QX_k+b & \text{if} \quad X_k\in D_1,\\ BX_k+c & \text{if} \quad X_k\in D_2,\end{cases} \tag{7}$$

where the matrix Q is given by

$$Q=\begin{pmatrix}b_{11}+b_{22}-a_{22} & a_{12}\\ \dfrac{b_{11}a_{22}-b_{11}b_{22}+b_{12}b_{21}+a_{22}b_{22}-a_{22}^2}{a_{12}} & a_{22}\end{pmatrix}. \tag{8}$$

The Jacobian matrix of the controlled map (7) is given by

$$J\left(X_k\right)=\begin{cases}Q & \text{if} \quad X_k\in D_1,\\ B & \text{if} \quad X_k\in D_2.\end{cases} \tag{9}$$

It is shown in [22] that if we consider a system $x_{k+1}=f\left(x_k\right),\ x_k\in\Omega\subset\mathbb{R}^n,$ such that

$$\left\|f'(x)\right\|=\sqrt{\lambda_{\max}\left(f(x)^T f(x)\right)}\leq N<+\infty, \tag{10}$$

with a smallest eigenvalue of $f(x)^T f(x)$ that satisfies

$$\lambda_{\min}\left(f'(x)^T f'(x)\right)\geq\theta>0, \tag{11}$$

where $N^2\geq\theta,$ then, for any $x_0\in\Omega,$ all the Lyapunov exponents at x_0 are located inside $\left[\frac{\ln\theta}{2},\ln N\right],$ i.e.,

$$\frac{\ln\theta}{2}\leq l_i\left(x_0\right)\leq\ln N,\quad i=1,2,\ldots,n, \tag{12}$$

where $l_i\left(x_0\right)$ are the Lyapunov exponents for the map f and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. We remark that $J\left(X_k\right)$ given in (9) is not well defined due to the discontinuity, but, since B and Q have the same eigenvalues, one has $\|Q\| = \|B\| = \sqrt{\lambda_{\max}\left(B^T B\right)}$. Since

$$B^T B = \begin{pmatrix} b_{11}^2 + b_{21}^2 & b_{11}b_{12} + b_{21}b_{22} \\ b_{11}b_{12} + b_{21}b_{22} & b_{12}^2 + b_{22}^2 \end{pmatrix}$$

is at least a positive semidefinite matrix, all its eigenvalues are real and positive, i.e.,

$$\lambda_{\max}\left(B^T B\right) \geq \lambda_{\min}\left(B^T B\right) \geq 0.$$

Hence, the eigenvalues of $B^T B$ are given by

$$\begin{aligned} \lambda_{\max}\left(B^T B\right) &= \frac{1}{2}b_{11}^2 + \frac{1}{2}b_{12}^2 + \frac{1}{2}b_{21}^2 + \frac{1}{2}b_{22}^2 + \frac{1}{2}\sqrt{d}, \\ \lambda_{\min}\left(B^T B\right) &= \frac{1}{2}b_{11}^2 + \frac{1}{2}b_{12}^2 + \frac{1}{2}b_{21}^2 + \frac{1}{2}b_{22}^2 - \frac{1}{2}\sqrt{d}, \end{aligned} \tag{13}$$

where

$$d = \left((b_{11} + b_{22})^2 + (b_{12} - b_{21})^2\right)\left((b_{12} + b_{21})^2 + (b_{11} - b_{22})^2\right) > 0 \tag{14}$$

for all b_{11}, b_{12}, b_{21}, and b_{22}. Condition (10) gives

$$\|f'(x)\| = \|B\| = \|Q\| = \sqrt{\lambda_{\max}\left(B^T B\right)} = N < +\infty$$

because B and Q have the same eigenvalues. Condition (11) gives the inequality

$$\theta^2 - \left(b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2\right)\theta + (b_{11}b_{22} - b_{12}b_{21})^2 \geq 0 \tag{15}$$

with the condition

$$\theta < \frac{b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2}{2}. \tag{16}$$

Since the discriminant of (15) is equal to $d > 0$, (11) holds if

$$\theta \geq \lambda_{\max}\left(B^T B\right) \quad \text{or} \quad \theta \leq \lambda_{\min}\left(B^T B\right). \tag{17}$$

The condition $\theta \geq \lambda_{\max}\left(B^T B\right)$ is impossible because of condition (16), so that θ must satisfy the condition

$$\theta < \lambda_{\min}\left(B^T B\right) = \frac{b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 - \sqrt{d}}{2}. \tag{18}$$

Now, if

$$\begin{aligned} &2 < b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 < (b_{11}b_{22} - b_{12}b_{21})^2 + 1, \\ &|b_{11}b_{22} - b_{12}b_{21}| > 1, \end{aligned} \tag{19}$$

then $\lambda_{\min}\left(B^T B\right) > 1,$ i.e., $\theta = 1,$ and one has

$$0 < l_i\,(x_0) \leq \ln N, \quad i = 1, 2, \tag{20}$$

i.e., the controlled map (7) converges to a hyperchaotic attractor for all parameters $b_{11}, b_{12}, b_{21},$ and b_{22} satisfying (19).

Finally, the following theorem is proved:

Theorem 1. *The piecewise-linear controller function (6) makes map (1) chaotic in the following case:*

$$\begin{aligned} &2 < b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 < (b_{11}b_{22} - b_{12}b_{21})^2 + 1, \\ &|b_{11}b_{22} - b_{12}b_{21}| > 1. \end{aligned} \tag{21}$$

Finally, we note that Theorem 1 does not guarantee the boundedness of the resulting controlled map (7). This problem is still open for general piecewise-linear maps and flows.

3. Example

In this section, we make chaotic the following piecewise-linear map using the above method:

$$f\,(x_k, y_k) = \begin{cases} \begin{pmatrix} x_k - \alpha y_k + 1, \\ \gamma x_k \end{pmatrix} & \text{if} \quad y_k \geq 0, \\ \begin{pmatrix} x_k + \alpha y_k + 1 \\ -\beta x_k \end{pmatrix} & \text{if} \quad y_k < 0, \end{cases} \tag{22}$$

where $\alpha,\ \beta,$ and γ are bifurcation parameters. Map (22) is a special case of map (1) because one can rewrite it as

$$f\,(X_k) = \begin{cases} AX_k + b & \text{if} \quad y_k \geq 0, \\ BX_k + c & \text{if} \quad y_k < 0, \end{cases}$$

where

$$A = \begin{pmatrix} 1 & -\alpha \\ \gamma & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \alpha \\ -\beta & 0 \end{pmatrix}, \quad b = c = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

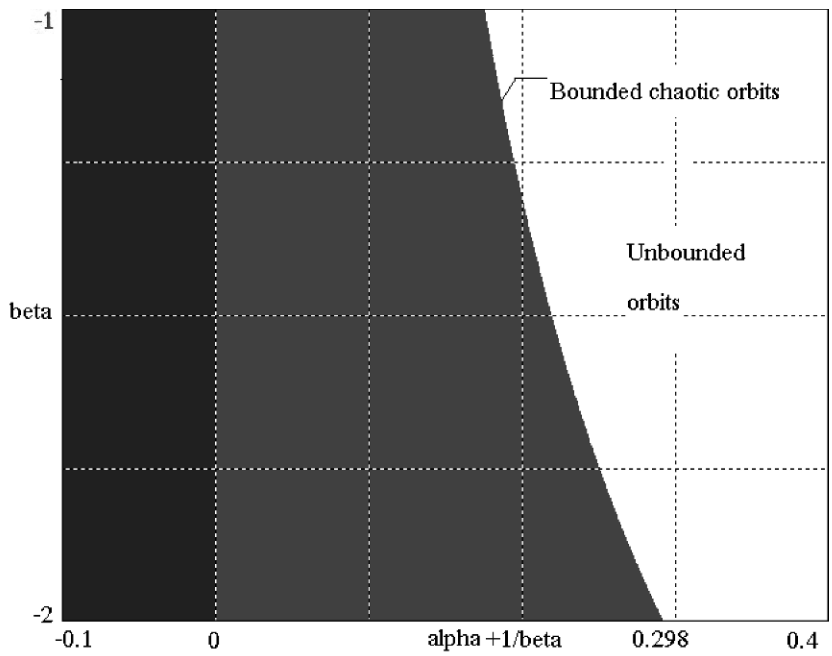

Fig. 1. Regions of dynamical behaviors in the -plane for the controlled map (23) with 2 < 1 and 0.1 1/ < < 1/ + 0.4.

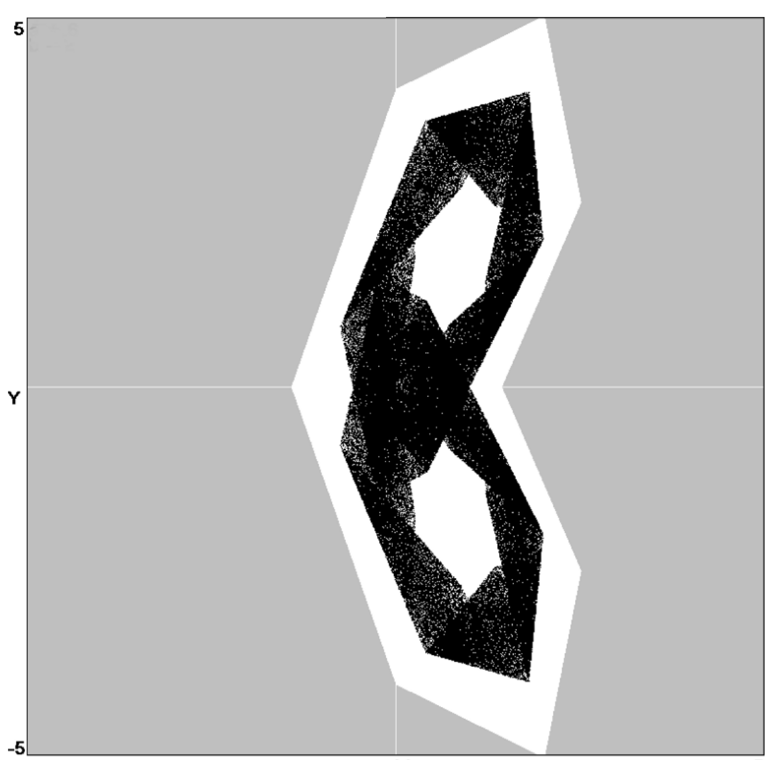

Fig. 2. The new piecewise-linear chaotic attractor obtained from the controlled map (23) with its basin of attraction for = 0.6, = 2, and the initial condition $x_0 = y_0 = 0.01$.

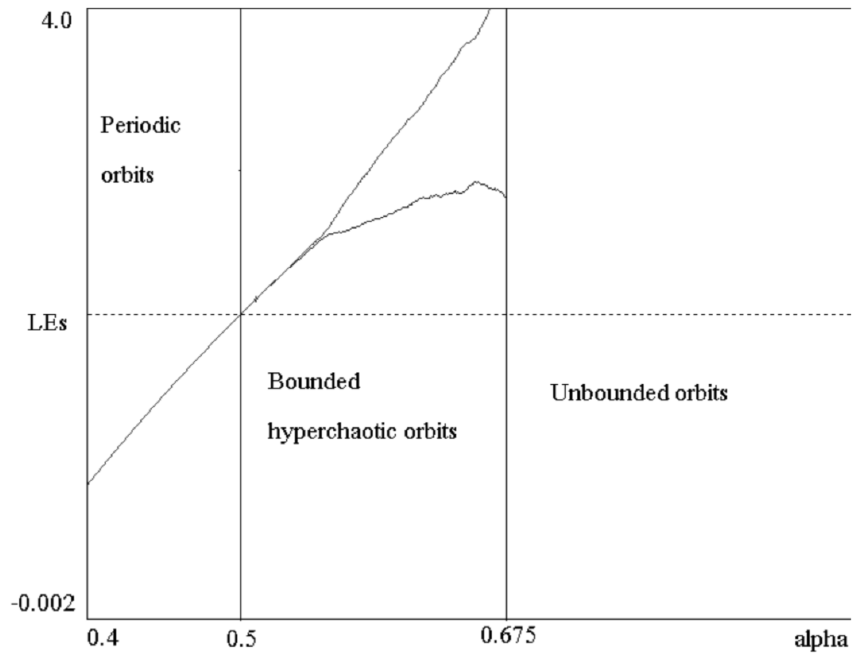

Fig. 3. Variation of the Lyapunov exponents of the controlled map (23) versus the parameter $0.4 \leq \alpha \leq 0.9$, with $\beta = -2$, where the two vertical lines indicate the border between the mentioned dynamical behaviors.

and the two subregions are $D_1 = \left\{(x_k, y_k) \in \mathbb{R}^2 / \ y_k \geq 0\right\}$ and $D_2 = \left\{(x_k, y_k) \in \mathbb{R}^2 / \ y_k < 0\right\}$. Thus, the resulting controlled map is given by

$$g(x_k, y_k) = \begin{cases} \begin{pmatrix} 1 & -\alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if} \quad y_k \geq 0 \\ \\ \begin{pmatrix} 1 & \alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if} \quad y_k < 0 \end{cases} = \begin{pmatrix} 1 - \alpha\,|y_k| + x_k \\ \beta x_k \operatorname{sgn}(y_k) \end{pmatrix}, \tag{23}$$

which is the so-called *discrete butterfly* presented in [19], where $\operatorname{sgn}(\cdot)$ is the standard signum function that gives ± 1 depending on the sign of its argument. For the controlled map (23), condition (21) becomes

$$|\alpha| > \max\left(\frac{|\beta|}{\sqrt{\beta^2 - 1}}, \frac{1}{|\beta|}\right), \quad |\beta| > 1. \tag{24}$$

For illustration, assume that $\beta < -1$. Then we remark that

$$\frac{|\beta|}{\sqrt{\beta^2 - 1}} = \frac{-\beta}{\sqrt{\beta^2 - 1}} < \frac{1}{|\beta|} = \frac{-1}{\beta},$$

and, thus, conditions (24) become

$$|\alpha| > \frac{-1}{\beta}, \quad \beta < -1.$$

Using the obtained analytical results, Fig. 1 shows that, for

$$-2 \leq \beta < -1 \qquad \text{and} \qquad -0.1 - \frac{1}{\beta} < \alpha < \frac{-1}{\beta} + 0.4,$$

the controlled map (23) converges to bounded hyperchaotic attractors or unbounded orbits for $\alpha > -\frac{1}{\beta}$. In this figure, unbounded solutions, periodic solutions, and chaotic solutions are shown in the $\alpha\beta$-plane for the controlled map (23), where we use 5000 different initial conditions and 10^6 iterations for each point. A chaotic attractor for the case with $\alpha = 0.6$ and $\beta = -2$ is shown in Fig. 2.

On the other hand, it is necessary to verify the hyperchaoticity of the attractors by calculating both Lyapunov exponents using the formula $l_1(x_0) + l_2(x_0) = \ln|\det(J)| = \ln|\alpha\beta|$ averaged along the orbit, where $\det(J)$ is the determinant of the Jacobian matrix. The result is shown in Fig. 3 for $0.4 \leq \alpha \leq 0.9$, with $\beta = -2$.

4. Conclusion

A new simple chaotification method for piecewise-linear maps of the plane via a piecewise-linear controller function was presented. A rigorous proof of chaos in the resulting controlled map using the standard definition of the largest Lyapunov exponent was also given.

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