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The Unified Chaotic System Describing the Lorenz and Chua Systems

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Abstract: This paper introduces the dynamical behaviors of a unified nonlinear chaotic system which describes a two-family chaotic system containing the original Lorenz and the original Chua systems as two extremes and some other systems as a transition in between via a new constructed joint function. This system can display two kinds of attractors, those with one scroll and those with two scrolls.

Keywords: Unified system, double-scroll attractor, quasi-attractor, Lorenz-type attractor.

1 Introduction

THE discovery of chaos in three-dimensional smooth autonomous systems is due to Lorenz [1] where he analyzes the following system:

$$\begin{aligned} x' &= \sigma(y-x) \\ y' &= rx - y - xz \\ z' &= -bz + xy \end{aligned} \tag{1}$$

and obtains a typical chaotic attractor for:

$$\sigma = 10, \quad r = 28, \quad b = \frac{8}{3}. \tag{2}$$

Many other systems of three smooth autonomous ordinary differential equations with two quadratic nonlinear terms have been found. These include the Chen's

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system [2] and the unified chaotic system which describes a large family of chaotic systems containing the Lorenz and Chen systems as two extremes and the Lü system as a transition in between [3]. These systems have many similar properties, e.g. they are all dissipative systems and have at most three equilibria and two-scroll chaotic attractors, and they possess Hopf and period-doubling bifurcations.

A new continuous-time, three-dimensional autonomous system is presented in [4]. This new system is capable of realizing the well-known quasi-attractors and Lorenz-type strange attractors in three-dimensional autonomous system as special choices of some real function and eight bifurcation parameters; especially the Lorenz [1], the Chen [2], the Lü [5], and the Chua [6] models are obtained; in addition to other new chaotic attractors. This model is given by the following system of equations:

$$\begin{aligned} x' &= a_1(y-h(x)) \\ y' &= a_2x+by+a_3z-a_4xz \\ z' &= -a_5z+a_6xy-a_7y \end{aligned} \tag{3}$$

where $(a_i)_{1\leq i\leq 7}$ and b are the bifurcation parameters and $h(x)$ is some real function. For example:

(1) For $h(x)=x, a_3=a_7=0, b=-1$, one has the Lorenz model [1].
(2) For $h(x)=x, a_3=a_7=0, a_2=b-a_1$, one has the Chen model [3].
(3) For $h(x)=x, a_2=a_3=a_7=0$, and $a_4=1$, one has the Lü model [4].
(4) For $h(x)=m_1x+\frac{1}{2}(m_0-m_1)(|x+1|-|x-1|)$, and $a_2=a_3=1, a_4=a_5=a_6=0, b=-1$, one has the Chua model.

Observations of chaotic behavior in electrical circuits date back to Van der Pol in 1927 who reported *irregular noise* from his neon bulb circuit. The first proposed real physical dynamical system capable of generating chaotic phenomena in the laboratory similar to those in the Lorenz system was invented by Chua [7] who synthesized a simple third-order autonomous circuit given by:

$$\begin{aligned} x' &= \alpha(y-h(x)) \\ y' &= x-y+z \\ z' &= -\beta y \end{aligned} \tag{4}$$

where

$$h(x)=m_1x+\frac{1}{2}(m_0-m_1)(|x+1|-|x-1|) \tag{5}$$

is the characteristic function. This system exhibits a remarkable variety of dynamics and gives a chaotic attractor called the *double-scroll attractor* [6] obtained for:

$$\alpha=9.35, \quad \beta=14.79, \quad m_0=-\frac{1}{7}, \quad m_1=\frac{2}{7}. \tag{6}$$

There are several studies comparing Chua's system and the Lorenz equation [8], pointing out that Chua's system has several advantages over the Lorenz equation: Chua's system has only one nonlinearity with one variable, whereas the Lorenz system has two nonlinearities, each with two variables. Furthermore, Chua's circuit is easy to build in the laboratory in contrast to the Lorenz system.

Strange attractors can be classified into three principal classes: hyperbolic, Lorenz-type, and quasi-attractors. The hyperbolic attractors are the limit sets for which Smale's "axiom A" is satisfied and are structurally stable. Periodic orbits and homoclinic orbits are dense and are of the same saddle type, which is to say that they have the same index (the same dimension for their stable and unstable manifolds). However, the Lorenz-type attractors are not structurally stable, although their homoclinic and heteroclinic orbits are structurally stable (hyperbolic), and no stable periodic orbits appear under small parameter variations. The quasi-attractors are the limit sets enclosing periodic orbits of different topological types (for example stable and saddle periodic orbits) and structurally unstable orbits. For example, the attractors generated by Chua's circuit [6] (see Fig. 3(d)) associated with saddle-focus homoclinic loops are quasi-attractors. Note that this type is more complex than the above two attractors.

The principal motivation of this work is to develop and analyze a new unified chaotic system which describes the original Lorenz and Chua systems as two extremes with some other systems as a transition in between and to present the first mathematical model that describes with one key parameter both the Lorenz system as a Lorenz-type attractor and the Chua system as a quasi-attractor.

The paper is organized as follows: In the next section, the new three-dimensional system is presented. Some basic properties are given in Section 3. The stability of its equilibrium points are briefly discussed in Section 4. In Section 5, the evolution of the new system with respect to its key parameter is analyzed by means of Lyapunov exponents and bifurcation diagrams for an associated Poincaré map. The final section concludes the paper.

2 The Unified Model

The unified chaotic model is given by the following system of equations:

$$\begin{aligned} x' &= (-0.65\mu + 10)\left(y - h_\mu(x)\right) \\ y' &= (-27\mu + 28)x - y + \mu z - (1-\mu)xz \\ z' &= (1-\mu)xy - 14.79\mu y - \frac{8}{3}(1-\mu)z \end{aligned} \tag{7}$$

where we use the set of parameters (2) for the Lorenz system (1) and the set of parameters (6) for the Chua system (4) with $\mu \in [0,1]$ as a control parameter. The joint function h_μ is given by:

$$h_\mu(x) = \frac{(7-5\mu)x}{7} - \frac{3\mu}{14}\left(|x+1| - |x-1|\right) \tag{8}$$

which is a new proposed generalized characteristic function for the Chua's diode where h_μ is an odd and continuous piecewise linear function that connects the two complex systems. Then the model (7) is a continuous system with three nonlinearities xy, xz, and $h_\mu(x)$, that with $\mu = 0$ reduces to the original Lorenz system and with $\mu = 1$ reduces to the original Chua system.

The system (7) has at least two types of chaotic attractors when $\mu \in [0,1]$: Lorenz-type and quasi-attractors, because it connects the Lorenz and the Chua systems and realizes the transition between them via the proposed joint function h_μ given by (8). The control parameter μ in system (7) allows the evolution of dynamical behaviors from the Lorenz attractor to the Chua attractor. The analytical study of system (7) is difficult because it requires the solution of a third-order algebraic equation at each step.

3 Some Basic Properties

When $\mu \in]0,1[$, the unified system (7) is not symmetric under the natural coordinate transforms $(x,y,z) \longrightarrow (-x,-y,-z)$ and $(x,y,z) \longrightarrow (-x,-y,z)$. Thus it does not preserve the same symmetry properties of the Lorenz and Chua systems. Also it is clear that the z-axis is not invariant. Therefore, the divergence of the flow is given by:

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = \begin{cases} 10.46\mu - 0.46429\mu^2 - \dfrac{41}{3}, & \text{if } x \geq 1 \\ 14.745\mu - 0.74286\mu^2 - \dfrac{41}{3}, & \text{if } |x| \leq 1 \\ 10.46\mu - 0.46429\mu^2 - \dfrac{41}{3}, & \text{if } x \leq -1. \end{cases} \tag{9}$$

Thus if one has:

$$10.46\mu - 0.46429\mu^2 - \frac{41}{3} < 0 \text{ and } 14.745\mu - 0.74286\mu^2 - \frac{41}{3} < 0, \tag{10}$$

i.e.

$$0 \leq \mu < 0.97473, \tag{11}$$

then the unified system (7) has a bounded, globally-attracting ω-limit set. Finally, the unified system (7) is dissipative when $0 \leq \mu < 0.97473$. Thus all trajectories are ultimately confined to a specific subset having zero volume, and the asymptotic motion settles onto an attractor. This result has been confirmed by computer simulations.

4 Equilibrium Points and Their Stability

The equilibria of the unified system (7) are given by:

$$P(x) = \left(x, h_\mu(x), \frac{3(1479\mu - 100x + 100x\mu) h_\mu(x)}{800(\mu - 1)} \right), \tag{12}$$

where x is the solution of the equation:

$$\begin{aligned} \left(440\mu - 224 - 216\mu^2\right) x + s(x) h_\mu(x) = 0 \\ s(x) = 8 - 8\mu + 47.37\left(\left(\mu^2 - \mu\right)x + \mu^2\right) + \left(3\mu^2 - 6\mu + 3\right)x^2. \end{aligned} \tag{13}$$

Note that the origin is an equilibrium point for all the system parameters, and also that the number of equilibrium points depends mainly on equation (13) where the possible number of its roots is between one and nine.

After some tedious calculations, one has that the variable x satisfies the following equation:

$$\begin{aligned} &x^3 + p_1 x^2 + p_2 x + p_3 = 0 \text{ if } x \geq 1, \text{ and } \mu \in [0,1] \\ &x = 0 \text{ and } x^2 + r_1 x + r_2 = 0 \text{ if } |x| \leq 1, \text{ and } \mu \in [0,1] - \{0.875\} \\ &x^3 + q_1 x^2 + q_2 x + q_3 = 0 \text{ if } x \leq -1 \text{ and } \mu \in [0,1] \end{aligned} \tag{14}$$

where

$$\begin{aligned} p_1 &= \frac{83.777\mu^2 - 35.121\mu^3 - 48.656\mu}{3 - \frac{15}{7}\mu^3 + \frac{51}{7}\mu^2 - \frac{57}{7}\mu} \\ q_1 &= \frac{78.634\mu^2 - 32.55\mu^3 - 46.084\mu}{3 - \frac{15}{7}\mu^3 - \frac{57}{7}\mu + \frac{51}{7}\mu^2} \\ p_2 &= \frac{\frac{2984}{7}\mu - 216 - 142.61\mu^2 - 54.137\mu^3}{3 - \frac{15}{7}\mu^3 + \frac{51}{7}\mu^2 - \frac{57}{7}\mu} \\ q_2 &= \frac{\frac{2984}{7}\mu - 216 - 13.534\mu^3 - 183.22\mu^2}{3 - \frac{15}{7}\mu^3 - \frac{57}{7}\mu + \frac{51}{7}\mu^2} \\ p_3 &= -q_3 = \frac{\frac{24}{7}\mu^2 - 20.301\mu^3 - \frac{24}{7}\mu}{3 - \frac{15}{7}\mu^3 + \frac{51}{7}\mu^2 - \frac{57}{7}\mu} \\ r_1 &= \frac{101.51\mu^2 - 54.137\mu^3 - 47.37\mu}{\frac{69}{7}\mu^2 - \frac{24}{7}\mu^3 - \frac{66}{7}\mu + 3} \\ r_2 &= \frac{\frac{2960}{7}\mu - 216 - 159.49\mu^2 - 54.137\mu^3}{\frac{69}{7}\mu^2 - \frac{24}{7}\mu^3 - \frac{66}{7}\mu + 3} \\ d &= 2592 - 13221\mu + 28622\mu^2 - 34618\mu^3 + 25480\mu^4 \\ &\quad - 11044.\mu^5 + 2188.4\mu^6. \end{aligned} \tag{15}$$

For the case where $|x| \leq 1$, the second equation of (14) has one solution $x_1 = 0$ if $\mu \in]0.84176, 1]$. Also, we remark that the first and the third equations of (14) have no zero solutions because $p_3 = -q_3 \neq 0$ for all $\mu \in [0,1]$. Finally, finding the equilibria of the unified system (7) requires in each case the solution of a third-order algebraic equation of the form $x^3 + Ax^2 + Bx + C = 0$, according to the position of the variable x. By setting $x = -A/3 + w$, we have: $w^3 + Pw + Q = 0$, where $P = -A^2/3 + B$ and $Q = 2A^3/27 - AB/3 + C$. If we set $\Delta = 4P^3 + 27Q^2$, then if $\Delta > 0$, there is a unique negative real solution:

$$x = -\frac{A}{3} + \left(-\frac{Q}{2} + \sqrt{\frac{Q^2}{2} + \frac{P^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{Q}{2} - \sqrt{\frac{Q^2}{2} + \frac{P^3}{27}}\right)^{\frac{1}{3}}, \tag{16}$$

and if $\Delta < 0$, there are three real solutions:

$$\begin{aligned} x_1 &= -\frac{A}{3} + 2\sqrt{-\frac{P}{3}} \sin\left(\frac{\theta}{3}\right) \\ x_2 &= -\frac{A}{3} + 2\sqrt{-\frac{P}{3}} \sin\left(\frac{\theta + 2\pi}{3}\right) \\ x_3 &= -\frac{A}{3} + 2\sqrt{-\frac{P}{3}} \sin\left(\frac{\theta + 4\pi}{3}\right) \end{aligned} \tag{17}$$

where $\theta = \arcsin\left(\sqrt{\frac{-27Q^2}{4P^3}}\right) \in [0, \pi]$.

The case $\Delta = 0$ corresponds to a measure-zero set of parameters. Therefore, by a slight perturbation of parameters, without changing the behavior of the system, a system belonging to one of the two cases is obtained.

Let us now examine the stability of the equilibria of the unified system (7). For this purpose, the Jacobian matrix at an equilibrium point $P(x)$ given in (12) is expressed by:

$$J(P(x)) = \begin{bmatrix} (0.65\mu - 10)\, h'_\mu(x) & -0.65\mu + 10 & 0 \\ j(x) & -1 & (\mu - 1)x + \mu \\ (1 - \mu)\, h_\mu(x) & (1 - \mu)x - 14.79\mu & \frac{8}{3}(\mu - 1) \end{bmatrix} \tag{18}$$

where

$$h'_\mu(x) = \begin{cases} \left(1 - \frac{5}{7}\mu\right)x, & \text{if } |x| \geq 1 \\ \frac{-(8\mu - 7)}{7}, & \text{if } |x| \leq 1 \end{cases} \tag{19}$$
$$j(x) = \frac{\left(144.37x\mu - 3x^2 + 3x^2\mu\right) h_\mu(x) - 216\mu + 224}{8}$$

For studying the stability of these equilibria, the exact value of the eigenvalues are obtained by using the Cardan method's given above for solving cubic equations $\lambda^3+A\lambda^2+B\lambda+C=0$, where:

$$\begin{aligned} A &= (10-0.65\mu)\,\acute{h}_\mu(x)-\frac{8}{3}\mu+\frac{11}{3} \\ B &= \xi_1\acute{h}_\mu(x)+\xi_2 h_\mu(x)+\xi_3 \\ C &= \xi_4\acute{h}_\mu(x)+\xi_5 h_\mu(x)+\xi_6 \end{aligned} \tag{20}$$

and

$$\begin{aligned} \xi_1 &= 1.7333\mu^2-29.05\mu+\frac{110}{3} \\ \xi_2 &= \left(0.24375\mu^2-3.9938\mu+\frac{15}{4}\right)x^2+\left(11.73\mu^2-180.46\mu\right)x \\ \xi_3 &= (\mu-1)^2x^2+\frac{1579}{100}\mu(\mu-1)x-2.76\mu^2+285.53\mu-\frac{832}{3} \\ \xi_4 &= b_1x^2+b_2x+b_3 \\ \xi_5 &= b_1x^2+b_4x+b_5 \\ \xi_6 &= 46.8\mu^3-815.33\mu^2+1515.2\mu-\frac{2240}{3} \end{aligned} \tag{21}$$

and

$$\begin{aligned} b_1 &= \frac{-(13\mu-200)(\mu-1)^2}{20} \\ b_2 &= \frac{-\mu\left(2566\mu^2-42040\mu+39475\right)}{250} \\ b_3 &= -9.6135\mu^3+149.63\mu^2-28.4\mu+\frac{80}{3} \\ b_4 &= \frac{-(\mu-1)\left(3193\mu^2-49188\mu+1000\right)}{100} \\ b_5 &= \frac{-\mu(\mu-1)(13\mu-200)}{20}. \end{aligned} \tag{22}$$

Note that if $\Delta>0$, there is one real eigenvalue given by equation (16) and two complex conjugate eigenvalues:

$$(\lambda_C)^{\pm}=-\frac{A}{3}-\frac{w_R}{2}\pm\frac{i}{2}\sqrt{4P+3(w_R)^2} \tag{23}$$

where

$$w_R=\left(-\frac{Q}{2}+\sqrt{\frac{Q^2}{2}+\frac{P^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{Q}{2}-\sqrt{\frac{Q^2}{2}+\frac{P^3}{27}}\right)^{\frac{1}{3}}. \tag{24}$$

5 Numerical Simulation

In this section, the dynamical behaviors of the unified system (7) with respect to the variable $\mu \in [0,1]$ are investigated numerically where the bifurcation diagram is obtained via an appropriate Poincaré section Σ defined by:

$$\Sigma = \left\{ (x,z) \in \mathbb{R}^2 \;/\; y = 0, \right\} \tag{25}$$

with the resulting points $\{x_n\}_{n \in \mathbb{N}}$ computed using the Hénon method, and a set of one of them is recorded after transients have decayed and plotted versus the desired parameter as shown in Fig. 1(b). The calculations of limit sets of the unified system

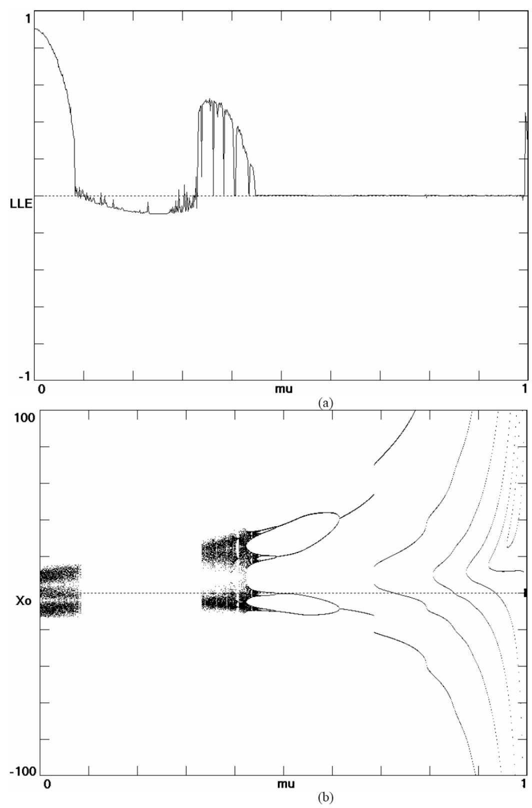

Fig. 1. (a) Variation of the largest Lyapunov exponent of the unified system (7) versus the parameter $\mu \in [0,1]$. (b) Bifurcation diagram of the variable x_n plotted versus control parameter $\mu \in [0,1]$.

(7) were performed using a fourth-order Runge-Kutta algorithm with a constant step size $\Delta t = 10^{-3}$ and initial conditions $(-0.1, -0.1, -0.1)$. Then, to determine

the long-time behavior and chaotic regions, we numerically computed the largest Lyapunov exponent. To see some chaotic behavior of the unified system (7), we present various numerical results. We present here some chaotic attractors with all the phase portraits shown in the *x*-*z* plane.

Fig. 1(a) shows the largest Lyapunov exponent of the unified system (7) with respect to the parameter $\mu \in [0,1]$. There are apparently several equilibrium zones, several periodic zones, and at least four chaotic zones corresponding to a positive Lyapunov exponent. The Lorenz attractor obtained for $\mu = 0$, a stable equilibrium for $\mu = 0.1$, a strange attractor for $\mu = 0.4$, and a limit cycle obtained for $\mu = 0.5$ are shown in Figs. 2(a)-(d), respectively. Some additional periodic orbits are shown in Figs. 3(a)-(c) for larger values of μ along with the Chua chaotic attractor obtained for $\mu = 1$ in Fig. 3(d).

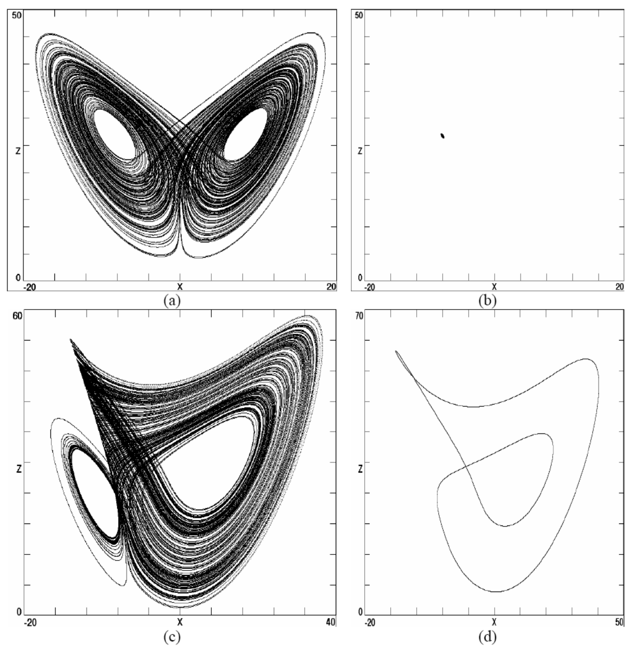

Fig. 2. Phase portraits of the unified system (7). (a) The original Lorenz chaotic attractor obtained for $\mu = 0$. (b) A stable equilibrium obtained for $\mu = 0.1$. (c) Another chaotic attractor obtained for $\mu = 0.4$. (d) A periodic orbit obtained for $\mu = 0.5$.

6 Remarks

Here are some concluding remarks about the dynamics of the unified system (7) concerning homoclinic and periodic orbits:

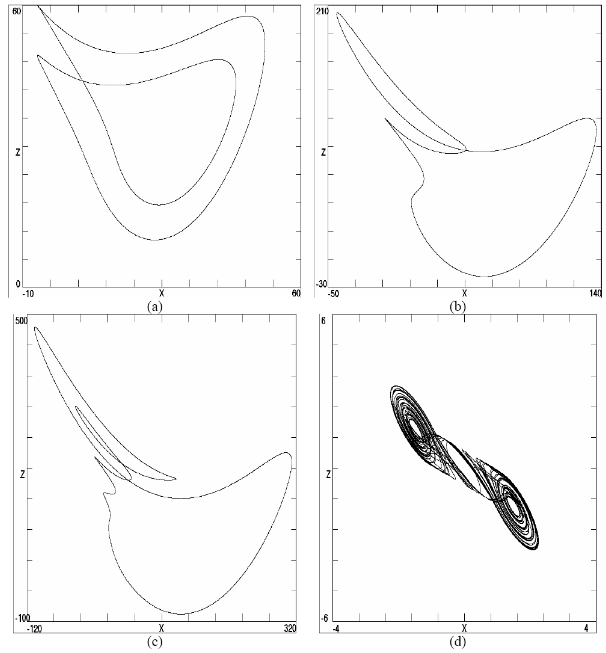

Fig. 3. Phase portraits of system (7), where (a) and (b) and (c) are periodic orbits obtained for $\mu = 0.6, \mu = 0.8$, and $\mu = 0.9$ respectively. (d) The original Chua chaotic attractor obtained for $\mu = 1$.

1. There are no homoclinic orbits of the Shilnikov type for system (7) containing the equilibrium point $(0,0,0)$ when $0 < \mu < 1$, because it is easy to show numerically that $\Delta < 0$ if $0 \leq \mu < 0.97526$, and $\Delta \geq 0$ if $0.97526 \leq \mu \leq 1$. Thus $(0,0,0)$ is of a saddle-focus type when $0.97526 \leq \mu \leq 1$, and the calculation of the largest Lyapunouv exponent shows that the unified system (7) converges to a chaotic orbit for $0.97526 \leq \mu < 1$ as shown in Fig 1(a). Hence there is no homoclinic orbit containing the equilibrium point $(0,0,0)$ that emerges from the Lorenz system to the Chua system.
2. Some periodic orbits do not result from a Hopf bifurcation. For example, for $\mu = 0.6$, the unified system (7) has a periodic orbit as shown in Fig. 3(b). Its equilibrium points are $P_1 = (27.343, 15.367, 29.724)$ with the eigenvalues $\{-171.09, -1.0417, 164.57\}$ and $P_2 = (-4.1214, -2.0980, 20.695)$ with the eigenvalues $\{-35.641, -0.94866, 29.031\}$ and $P_0 = (0,0,0)$ with the eigenvalues $\{-13.969, -0.84169, 7.2526\}$.
3. For $\mu = 0$ and $\mu = 1$, it is well known that there are regions of multiple attractors, but for $0 < \mu < 1$, there are no observed regions of multiple attractors.

7 Conclusion

This paper introduces the dynamical behaviors of a unified chaotic system that can describe a two-family chaotic system containing the original Lorenz and the original Chua systems as two extremes and some other systems as a transition in between via a new constructed joint function using a simple variable constant controller. The unified system (7) has contributed to a better understanding of the relationship between the Lorenz and Chua systems and therefore deserves further investigation.

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