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The effect of modulating a parameter in the logistic map

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The effect of using the output of one logistic map to modulate the accessible parameter of a second logistic map is examined. Rigorous analytical results provide some predictions on the effect of this type of modulation, and those effects are tested numerically. *© 2008 American Institute of Physics*. [DOI: 10.1063/1.2912729]

Many simple examples are now known of mathematical models that exhibit chaos, and they have been extensively studied. One of the oldest and simplest such systems is the logistic map, which was originally used to model biological population dynamics but has now found application in many other fields. In some ways it is the prototypical example of chaos in discrete-time systems (iterated maps). Chaotic systems are usually characterized by one or more parameters that control the behavior of the system and give rise to bifurcations and other changes in the dynamics. In most studies, these parameters are held constant or allowed to take on a range of unchanging values, whereas in the real world, the parameters governing the dynamics often vary in time. Such cases have been studied very little. Here we consider a situation in which the single parameter of the logistic map is modulated by the output of a second logistic map whose parameter is constant but can be adjusted. Thus we are able to explore periodic and chaotic modulations of the first logistic map. Despite the simplicity of the resulting model, the dynamics exhibit extreme complexity. Some of the features of this model are proved rigorously and tested numerically.

I. INTRODUCTION

The logistic equation is a well-known one-dimensional map with the following form:

$$y_{n+1} = dy_n(1 - y_n). \tag{1}$$

Equation (1) was originally used to model population growth in which the variable y_n represents the fraction of the maximum population that the habitat can support.

In several previous studies, the parameter d was replaced or modulated by some special forms, for example in Ref. 1, the parameter d took two values driven by an external periodic signal, or by a stochastic sequence in Ref. 2 or by a sequence of multiple values in Ref. 3, where the onset of chaos in such a modulated logistic map was studied. In Ref. 4, the effect of applying a periodic perturbation to an accessible parameter of various chaotic systems (including the logistic map) was examined. Numerical results indicated that perturbation frequencies near the natural frequencies of the unstable periodic orbits of the chaotic systems can result in limit cycles for relatively small perturbations. In Ref. 5 the logistic map was modulated with the parameter d varied through a delayed feedback mechanism. This type of modulation gives the possibility of stabilizing the system to periodic dynamics. However, in all the previous works, several nonlinear phenomena are observed by numerical simulations, and a few analytical results were obtained. The lack of smoothness in some of the earlier works makes the study of the resulting modulated map difficult. Our aim in this paper is to find a simple nonlinear modulation of the parameter d making the resulting map smooth and its dynamics dependent on the history. The simplest way to implement this idea is to modulate the forcing parameter d linearly by the signal from another logistic map.

This paper is organized as follows: In the following section, we discuss the model, and then in Sec. III we give several rigorous results that explain the different dynamical regimes resulting from the modulation. In Sec. IV, some numerical simulations confirming the theory are given and discussed. The final section gives some conclusions.

II. THE MODULATED MAP

In this paper the controlling perturbation is taken as

$$d = b + cx_n, \tag{2}$$

where b and c are constant parameters, and x_n is the solution of another logistic map given by

$$x_{n+1} = ax_n(1 - x_n), \tag{3}$$

where $0 \le a \le 4$. The perturbation given by Eq. (2) makes the system two-dimensional,

$$x_{n+1} = ax_n(1 - x_n), \quad y_{n+1} = (b + cx_n)y_n(1 - y_n). \tag{4}$$

However, the resulting structure of the system (4) can be regarded as a combination of two one-dimensional maps for the specific choice of the forcing function (2). Hence, map (4) can be regarded as a 2D map with the property that the first equation does not depend on the second equation. The result is a "master-slave" system.

The goal of this paper is to determine analytically and numerically some predictions for the effect of the modulation (2) on the logistic map (1) for some values of b and c. We

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will prove that the orbits of the modulated map (4) are bounded, and in addition, for some values of b and c, the original and the modulated logistic maps given, respectively, by Eqs. (3) and (4) have identical bifurcation points.

The Jacobian matrix of the modulated map (4) is given by

$$J(x,y) = \begin{bmatrix} a(1-2x) & 0 \\ cy(1-y) & (b+cx)(1-2y) \end{bmatrix}. \quad (5)$$

Because the Jacobian matrix (5) is triangular, the Lyapunov exponents are given by

$$\lambda_1 = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\ln a|2x_n-1|$$
$$\lambda_2 = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|b+cx_n||2y_n-1|. \quad (6)$$

We remark that λ_1 is independent of the choice of the parameters b and c, and is simply the Lyapunov exponent of the logistic map (3).

In the following we will prove some theorems in order to demonstrate rigorously the occurrence of asymptotically stable fixed points, periodic orbits, and chaotic orbits in the modulated map (4).

III. ANALYTICAL RESULTS

Proposition 1: For all $n\in\mathbb{N}$, and

$$0\le a\le 4, \quad 0\le x_0\le \frac{a}{4}, \quad (7)$$

one has

$$0\le x_n\le\frac{a}{4}. \quad (8)$$

Because Proposition 1 is well known, its proof will be omitted here.

Proposition 2: For all $n\in\mathbb{N}$, and

$$0\le a\le 4,$$

$$0\le x_0\le\frac{a}{4}, \quad 0\le y_0\le\frac{4b+ac}{16}, \quad (9)$$

$$0<b\le 4-\frac{ac}{4}, \quad 0<c<\frac{16}{a},$$

one has

$$0\le y_n\le\frac{4b+ac}{16}. \quad (10)$$

Proof: First, for $n=0$, one has

$$0\le y_0\le\frac{4b+ac}{16}.$$

Assume now by induction that

$$0\le y_n\le\frac{4b+ac}{16}.$$

Then one has for $b>0$ and $c>0$, that

$$\frac{16}{4b+ac}(b+cx_n)y_n\left(\frac{4b+ac}{16}-y_n\right)\ge 0.$$

Hence,

$$(b+cx_n)y_n\left(1-\frac{16}{4b+ac}y_n\right)\ge 0.$$

On the other hand, we can rewrite y_{n+1} as follows:

$$y_{n+1} = (b+cx_n)y_n\left(1-\frac{16}{4b+ac}y_n\right) - (b+cx_n)\left(1-\frac{16}{4b+ac}\right)y_n^2\ge 0,$$

because

$$0<b\le 4-\frac{ac}{4}$$

and

$$0<c<\frac{16}{a}.$$

Second, the function $h(y)=(b+cx)y(1-y)$ has a single maximum at $y=\frac{1}{2}$, and it is decreasing in the interval $[\frac{1}{2},\infty[$ because $b+cx\ge b>0$. Thus one has for all $x\in[\frac{1}{2},\infty[$, that

$$h(y)\le h\left(\frac{1}{2}\right)=\frac{b+cx}{4}\le\frac{4b+ac}{16},$$

and increasing in the interval $]-\infty,\frac{1}{2}]$. Thus one has that

$$h(y)\le h\left(\frac{1}{2}\right)\le\frac{4b+ac}{16},$$

i.e.,

$$y_n\le\frac{4b+ac}{16}.$$

As a result, from Propositions 1 and 2, one has that the trajectories of the modulated map (4) are bounded.

Proposition 3: For all $n\in\mathbb{N}$, and

$$0\le a\le 4,$$

$$0\le x_0\le\frac{a}{4}, \quad 0\le y_0\le\frac{4b+ac}{16} \quad (11)$$

$$0<b\le 4-\frac{ac}{4}, \quad 0<c<\frac{16}{a},$$

one has

$$\|(x_n,y_n)\|_2\le\frac{8abc+16a^2+16b^2+a^2c^2}{256}. \quad (12)$$

Proof: We have

$$\|(x_n,y_n)\|_2=\sqrt{x_n^2+y_n^2}\leq\frac{8abc+16a^2+16b^2+a^2c^2}{256}.$$

Theorem 4: *If*

$$0\leq x_0\leq\frac{a}{4},\quad 0\leq y_0\leq\frac{4b+ac}{16}$$
$$0\leq a\leq 4,\quad 0<c<\frac{16}{a},\quad 0<b<4-\frac{ac}{4},\tag{13}$$

then one has

$$\lambda_2\leq\ln\frac{(4b+ac+8)(4b+ac)}{32}.\tag{14}$$

Proof: We have

$$|2y_n-1|\leq 2|y_n|+1\leq\frac{4b+ac+8}{8}$$

and

$$|b+cx_n|\leq b+\frac{ca}{4}.$$

Thus one has

$$\lambda_2\leq\ln\frac{(4b+ac+8)(4b+ac)}{32}.$$

Theorem 5: (1) *The modulated map (4) is asymptotically stable if*

$$0\leq x_0\leq\frac{a}{4},\quad 0\leq y_0\leq\frac{4b+ac}{16}$$
$$0<a<3,\quad 0<b<\sqrt{3}-\frac{1}{4}ac-1,\quad 0<c<\frac{4(\sqrt{3}-1)}{a}.\tag{15}$$

(2) *The modulated map (4) is almost periodic if*

$$0\leq x_0\leq\frac{a}{4},\quad 0\leq y_0\leq\frac{4b+ac}{16}$$
$$3\leq a\leq 3.569\,945\,7,\quad 0<b<\sqrt{3}-\frac{1}{4}ac-1,\tag{16}$$
$$0<c<\frac{4(\sqrt{3}-1)}{a}.$$

(3) *The modulated map (4) is almost chaotic if*

$$0\leq x_0\leq\frac{a}{4},\quad 0\leq y_0\leq\frac{4b+ac}{16}$$
$$3.569\,945\,7<a<4,\quad 0<b<\sqrt{3}-\frac{1}{4}ac-1,\tag{17}$$
$$0<c<\frac{4(\sqrt{3}-1)}{a}.$$

Proof: (1) The modulated map (4) is asymptotically stable if $\lambda_1<0$ and $\lambda_2<0$, and this is possible if $0<a<3$ and

$$\frac{(4b+ac+8)(4b+ac)}{32}<1,$$

i.e., $0<b<\sqrt{3}-1/4ac-1$ and

$$0<c<\frac{4(\sqrt{3}-1)}{a}.$$

(2) The modulated map (4) is periodic if $\lambda_1=0$ and $\lambda_2<0$, and this is possible if $3\leq a<3.569\,945\,7$ and

$$\frac{(4b+ac+8)(4b+ac)}{32}<1,$$

i.e., $0<b<\sqrt{3}-\frac{1}{4}ac-1$ and $0<c<\frac{4(\sqrt{3}-1)}{a}$.

(3) The modulated map (4) is chaotic if $\lambda_1>0$ and $\lambda_2<0$, and this is possible if $3.569\,945\,7<a<4$ and

$$\frac{(4b+ac+8)(4b+ac)}{32}<1,$$

i.e., $0<b<\sqrt{3}-\frac{1}{4}ac-1$ and $0<c<\frac{4(\sqrt{3}-1)}{a}$.■

Theorem 6: *If*

$$(b,c)\in\left]0,\sqrt{3}-\frac{1}{4}ac-1\right[\times\left]0,\frac{4(\sqrt{3}-1)}{a}\right[=J,\tag{18}$$

then the onset of chaos in the modulated logistic map (4) is the same as in the logistic map, i.e., (3) the critical value of a is seen to be 3.569 945 7, *where periodicity just ends* (*the accumulation point*).

Proof: The first Lyapunov exponent of the modulated map (4) is exactly the same as for the logistic map (3) and does not depend on the variable y_n, and the second Lyapunov exponent is negative when $(b,c)\in J$.■

Let us define the following subsets:

$$I_1=\{a\in\mathbb{R}/\lambda_1<0\},\quad I_2=\{a\in\mathbb{R}/\lambda_1=0\},$$
$$I_3=\{a\in\mathbb{R}/\lambda_1>0\},\quad J_1=\{(b,c)\in\mathbb{R}^2/\lambda_2<0\},\tag{19}$$
$$J_2=\{(b,c)\in\mathbb{R}^2/\lambda_2=0\},\quad J_3=\{(b,c)\in\mathbb{R}^2/\lambda_2>0\},$$

and let q and p indicate fixed points for the logistic map (3) and the second component of the modulated map (4), respectively, and let $(x_n)_{n\in\mathbb{N}}^{\text{Periodic}}$ and $(y_n)_{n\in\mathbb{N}}^{\text{Periodic}}$ indicate periodic orbits for the logistic map (3) and the second component of the modulated map (4), respectively, and let $(x_n)_{n\in\mathbb{N}}^{\text{Chaotic}}$ and $(y_n)_{n\in\mathbb{N}}^{\text{Chaotic}}$ indicate chaotic orbits for the logistic map (3) and the second component of the modulated map (4), respectively. Hence, because the first Lyapunov exponent λ_1 depends only on the variable x_n, then we can classify the solutions of the modulated map (4) as follows:

$$
\begin{cases}
(q,p), & \text{if } (a,b,c) \in I_1 \times J_1, \\
[q,(y_n)_{n\in\mathbb{N}}^{\text{Periodic}}], & \text{if } (a,b,c) \in I_1 \times J_2, \\
[q,(y_n)_{n\in\mathbb{N}}^{\text{Chaotic}}], & \text{if } (a,b,c) \in I_1 \times J_3, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Periodic}},p], & \text{if } (a,b,c) \in I_2 \times J_1, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Periodic}},(y_n)_{n\in\mathbb{N}}^{\text{Periodic}}], & \text{if } (a,b,c) \in I_2 \times J_2, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Periodic}},(y_n)_{n\in\mathbb{N}}^{\text{Chaotic}}], & \text{if } (a,b,c) \in I_2 \times J_3, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Chaotic}},p], & \text{if } (a,b,c) \in I_3 \times J_1, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Chaotic}},(y_n)_{n\in\mathbb{N}}^{\text{Periodic}}], & \text{if } (a,b,c) \in I_3 \times J_2, \\
[(x_n)_{n\in\mathbb{N}}^{\text{Chaotic}},(y_n)_{n\in\mathbb{N}}^{\text{Chaotic}}], & \text{if } (a,b,c) \in I_3 \times J_3.
\end{cases}
\tag{20}
$$

(1) If $(a,b,c) \in (I_1 \times J_1) \cup (I_2 \times J_1) \cup (I_3 \times J_1)$, then the effect of the modulation (2) is a stabilization procedure via, respectively, fixed, periodic, and chaotic modulations.

(2) If $(a,b,c) \in (I_1 \times J_2) \cup (I_2 \times J_2) \cup (I_3 \times J_2)$, then the effect of the modulation (2) is a control procedure via respectively fixed point, periodic, and chaotic modulations.

(3) If $(a,b,c) \in (I_1 \times J_3) \cup (I_2 \times J_3) \cup (I_3 \times J_3)$, then the effect of the modulation (2) is a chaotification procedure via, respectively, fixed, periodic, and chaotic modulation.

IV. NUMERICAL SIMULATIONS

Because the basin of attraction depends on the parameters a, b, and c, we must fix these parameters and then calculate the limit sets using appropriate initial conditions. The effect of the modulation given by Eq. (2) can be summarized in three procedures: stabilization, control, and chaotification. Each procedure can be achieved via three types of modulation, namely, fixed-point modulation, periodic modulation, and chaotic modulation. We will give in this section examples for each case, showing their respective effects. As an example, for the stabilization procedure via fixed point modulation when $a=2$, then one has $0 \le x_0 \le 0.5$, $0 \le y_0 \le 0.25b + 0.125c$, $0 < b < 0.732\,05 - 0.5c$, and $0 < c < 1.464\,1$. Assuming that $c=1$, one has $0 < b < 0.232\,05$ and $0 \le y_0 \le 0.25b + 0.125$, and so, for $b=0.2$, one has $0 \le y_0 \le 0.175$. Hence, we choose $x_0 = y_0 = 0.01$. The corresponding orbit is the stable fixed point $(0,0)$. For $a=3.2$, one has $0 \le x_0 \le 0.8$, $0 \le y_0 \le 0.18$, $0 < b < 0.092\,05$, and $0 < c < 0.915\,06$. Assume that $c=0.8$ and $b=0.08$. Hence, we choose $x_0 = y_0 = 0.01$. The corresponding orbit is $\{(0.513\,04,0),(0.799\,46,0)\}$, i.e., it is a period-2 orbit. Generally, for $a=a_m$, where $a_m, m=0,1,2,\ldots$ are the values of a for which the logistic map (3) has periodic orbits, and

$$
0 \le x_0 \le \frac{a_m}{4}, \quad 0 \le y_0 \le \frac{4b + a_m c}{16},
$$

$$
0 < b < \sqrt{3} - \frac{1}{4} a_m c - 1,
$$

and

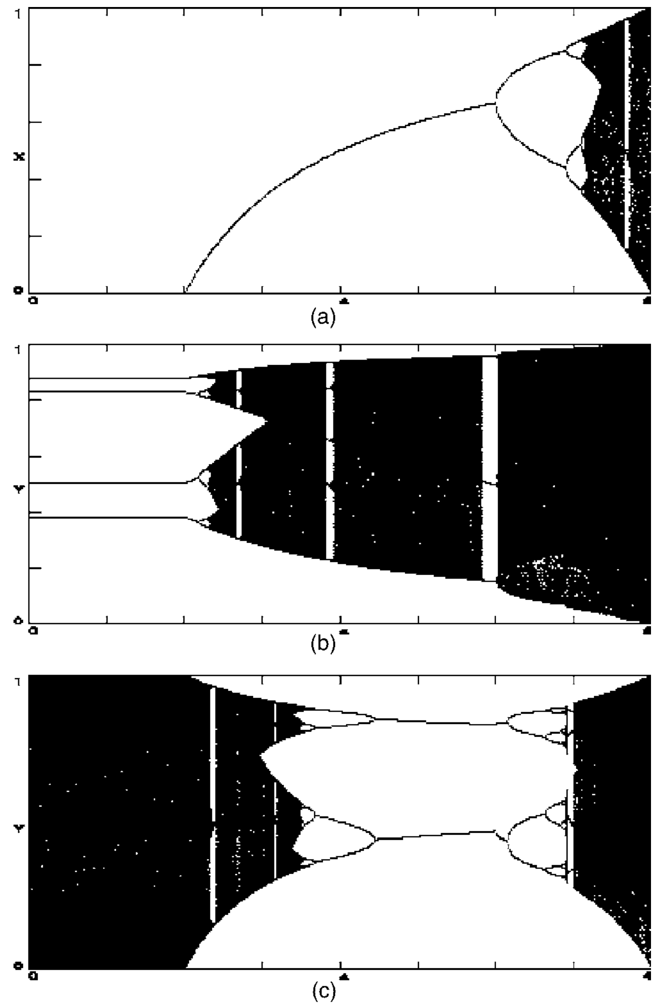

FIG. 1. (a) Bifurcation diagram of the logistic map (3) for $0 \le a \le 4$. (b) Bifurcation diagram of the modulated logistic map (4) for $0 \le a \le 4$ and $b=3.5, c=0.5$. (c) Bifurcation diagram of the modulated logistic map (4) for $0 \le a \le 4$ and $b=4, c=-1$.

$$
0 < c < \frac{4(\sqrt{3} - 1)}{a_m}
$$

for all $m=0,1,2,\ldots$, then the corresponding orbit is the period-m orbit given by $\{(x_i,0)_{1 \le i \le m}\}$. This procedure is the stabilization procedure via periodic modulation. Third, for $a=3.6$, one has $0 \le x_0 \le 0.9$, $0 \le y_0 \le 0.25b + 0.225c$, $0 < b < 0.732\,05 - 0.9c$, and $0 < c < 0.813\,39$. For $c=0.8$, one has $0 < b < 0.012\,05$ and $0 \le y_0 \le 0.25b + 0.18$, and so, for $b=0.01$, one has $0 \le y_0 \le 0.182\,5$. Hence, we choose $x_0 = y_0 = 0.01$. The corresponding orbit is $\{(x_n, y_n = 0)\}_{n\in\mathbb{N}}$, indicating a stabilization procedure via chaotic modulation.

Now Figs. 1(c), 1(b), and 2(c) show, respectively, the following phenomena: If $a=2$, $b=-2$, and $c=2$, then the logistic map (3) converges to a fixed point, and the second component of the modulated map (4) converges to a stable period-4 orbit. If $a=3.2$, $b=3.5$, and $c=0.5$, then the logistic map (3) converges to a period-2 orbit, and the second component of the modulated map (4) converges to a stable period-2 orbit. If $a=3.6$, $b=-2$, and $c=2$, then the logistic map (3) converges to a chaotic orbit, and the second component of the modulated map (4) converges to a stable period-1

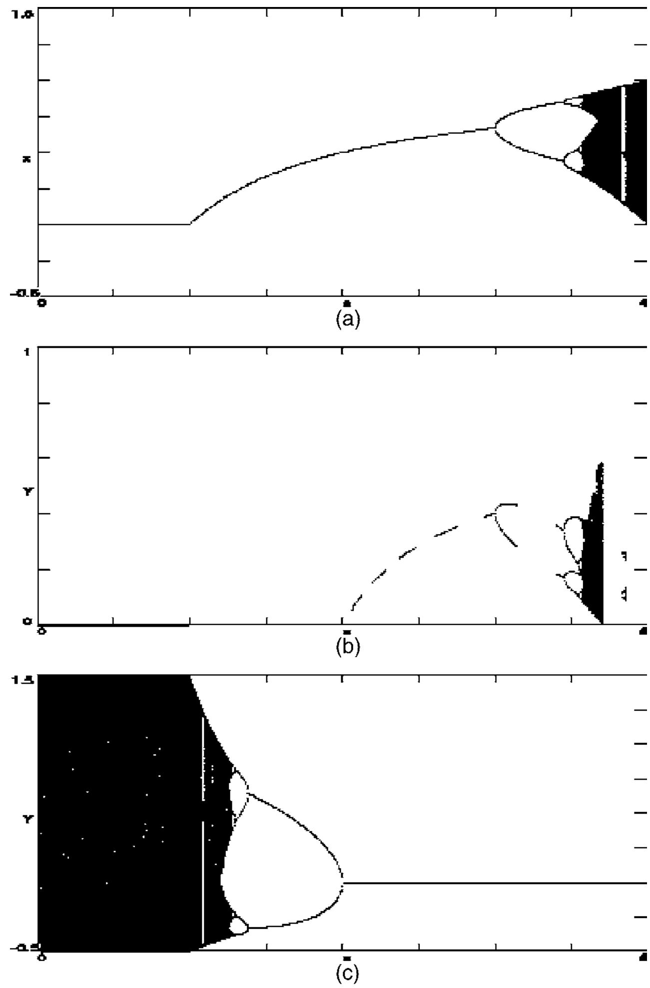

FIG. 2. (a) Bifurcation diagram of the logistic map (3) for $0 \leq a \leq 4$. (b) Bifurcation diagram of the modulated logistic map (4) for $0 \leq a \leq 4$ and $b = -1, c = 4$. (c) Bifurcation diagram of the modulated logistic map (4) for $0 \leq a \leq 4$ and $b = -2, c = 2$.

orbit. These effects are respectively the control procedure via fixed-point modulation, the control procedure via periodic modulation, and the control procedure via chaotic modulation. On the other hand, Figs. 2(c), 3(c), and 1(b) show, respectively, the following phenomena: If $a = 0.5$, $b = -2$, and $c = 2$, then the logistic map (3) converges to a fixed point orbit, and the second component of the modulated map (4) converges to a chaotic orbit. If $a = 3.2$, $b = 1.6$, and $c = 3$, then the logistic map (3) converges to a period-2 orbit, and the second component of the modulated map (4) converges to a chaotic orbit. If $a > 3.569\,945\,7$, $b = 3.5$, and $c = 0.5$, then the logistic map (3) converges to a chaotic orbit, and the second component of the modulated map (4) converges also to a chaotic orbit. These effects are respectively the chaotification procedure via fixed-point modulation, the chaotification procedure via periodic modulation, and the chaotification procedure via chaotic modulation.

Note that some examples of chaotic and hyperchaotic orbits with their basins of attraction (in white) obtained from the modulated map (4) are shown in Fig. 3.

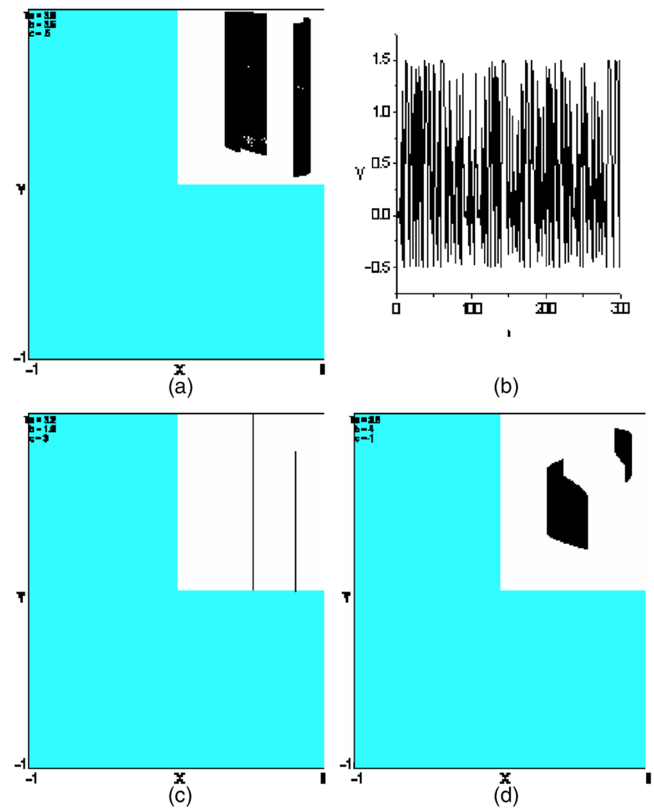

FIG. 3. (Color online) (a) Hyperchaotic orbit with its basin of attraction (in white) obtained from the modulated map (4) for $a = 3.6$, $b = 3.5$, $c = 0.5$. (b) Chaotic orbit obtained from the modulated map (4) for $a = 0.5$, $b = -2$, $c = 2$. (c) Chaotic orbit with its basin of attraction (in white) obtained from the modulated map (4) for $a = 3.2$, $b = 1.6$, $c = 3$. (d) Hyperchaotic orbit with its basin of attraction (in white) obtained from the modulated map (4) for $a = 3.6$, $b = 4$, $c = -1$.

V. DISCUSSION

Most of the results of linearly modulating the parameter of the logistic map by the output of another logistic map can be classified as follows: First, the modulation (2) makes the dynamics of the modulated logistic map similar to the original logistic map, for example if $b = 3.5$ and $c = 0.5$, the dynamics over the range $0 \leq a \leq 4$ of the modulated map (4) is clearly similar to the one for the logistic map (3) as shown in Figs. 1(a) and 1(b). Second, the modulation produces a period-halving bifurcation sequence leading to stable limit cycles after chaos, without the production of period-doubling bifurcations as shown in Fig. 2(c) obtained for $b = -2, c = 2$, and $0 \leq a \leq 4$. Third, the modulation (2) produces a period-halving bifurcation sequence leading to stable limit cycles after chaos and produces a period-doubling bifurcation sequence leading to a chaotic attractor as shown in Fig. 1(c) obtained for $b = 4, c = -1$, and $0 \leq a \leq 4$. The fourth effect is that the modulation makes the dynamics of the modulated logistic map (4) identical to the original logistic map (1). This case is analytically proved for some regions in the *a*-*b*-*c* space. Indeed, the system in Eq. (4) is of a standard master-slave type of the form

$$x_{n+1} = a x_n (1 - x_n) = f(x_n), \tag{21}$$

$$y_{n+1} = (b + c x_n) y_n (1 - y_n) = g(x_n, y_n). \tag{22}$$

The drive $f(x_n)$ is unaffected by the response $g(x_n, y_n)$. Such systems show generalized synchronization in noninvertible maps,[6] because the driving system (21) is noninvertible. The method of Ref. 6, which is based on the auxiliary system approach given in Ref. 7 is applied here, then one has that the generalized synchronization is monotonically stable if for all a, b, and c one has

$$2cx_ny_n - cx_n - 1 < b(1-2y_n) < 2cx_ny_n - cx_n + 1. \tag{23}$$

Assuming $c>0$, then one has $2cx_ny_n - cx_n - 1 \geq -\frac{1}{4}(ac+4)$, and

$$2cx_ny_n - cx_n + 1 \leq \frac{4abc + a^2c^2 + 32}{32}.$$

Thus Eq. (23) becomes

$$-\frac{1}{4}(ac+4) < b(1-2y_n) < \frac{4abc + a^2c^2 + 32}{32}. \tag{24}$$

From Eq. (10) we have

$$-\frac{(4b+ac-8)}{8} \leq 1 - 2y_n \leq 1.$$

Then if

$$0 < b < 2 - \frac{ac}{4},$$

one has

$$0 < -\frac{(4b+ac-8)}{8} \leq 1 - 2y_n \leq 1.$$

Hence, one has

$$-\frac{1}{4}(ac+4) < b < \frac{-(4abc + a^2c^2 + 32)}{4(4b+ac-8)},$$

and this inequality holds for all

$$0 < b < 2 - \frac{ac}{4}.$$

Other conditions on a, b, and c are obtained in which Eq. (9) holds. Finally, the following Theorem has been proved:

Theorem 7: *If*

$$0 \leq a \leq 4,$$

$$0 \leq x_0 \leq \frac{a}{4}, \quad 0 \leq y_0 \leq \min\left(\frac{4b+ac}{16}, \frac{1}{2}\right), \tag{25}$$

$$\max\left(0, 4y_0 - \frac{ac}{4}\right) < b \leq 2 - \frac{ac}{4}, \quad 0 < c < \frac{16}{a},$$

then the generalized synchronization is monotonically stable.

As a test of this result, assume that $b=1, c=0.5, x_0=y_0=0.01$, and $0 \leq a \leq 4$. Figure 4 shows that both the driving and the response systems given by Eqs. (21) and (22) have identical oscillations for all $0 \leq a \leq 4$. This situation includes the case $a=4$, where the orbit occasionally gets stuck in some very high periodic cycle and never reaches zero.

On the one hand, Fig. 5(a) shows regions of unbounded (white), fixed point (gray), periodic (blue), and chaotic (red)

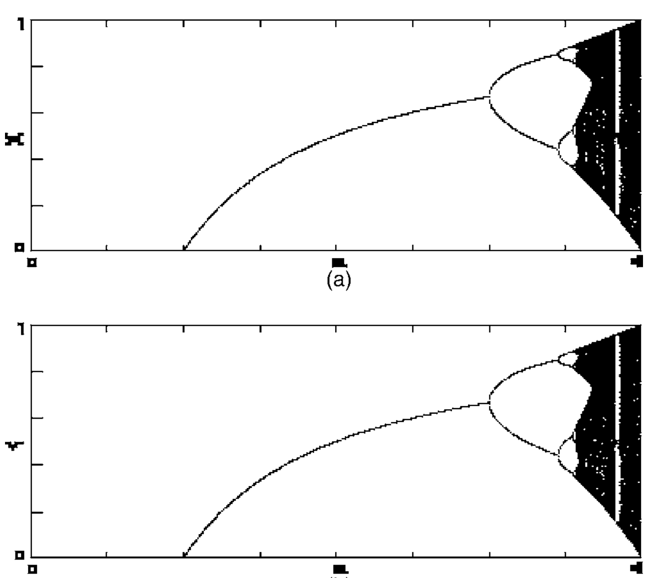

FIG. 4. The driving x_n and the response y_n have identical bifurcations for $b=1$, $c=0.5$, $x_0=y_0=0.01$, and $0 \leq a \leq 4$.

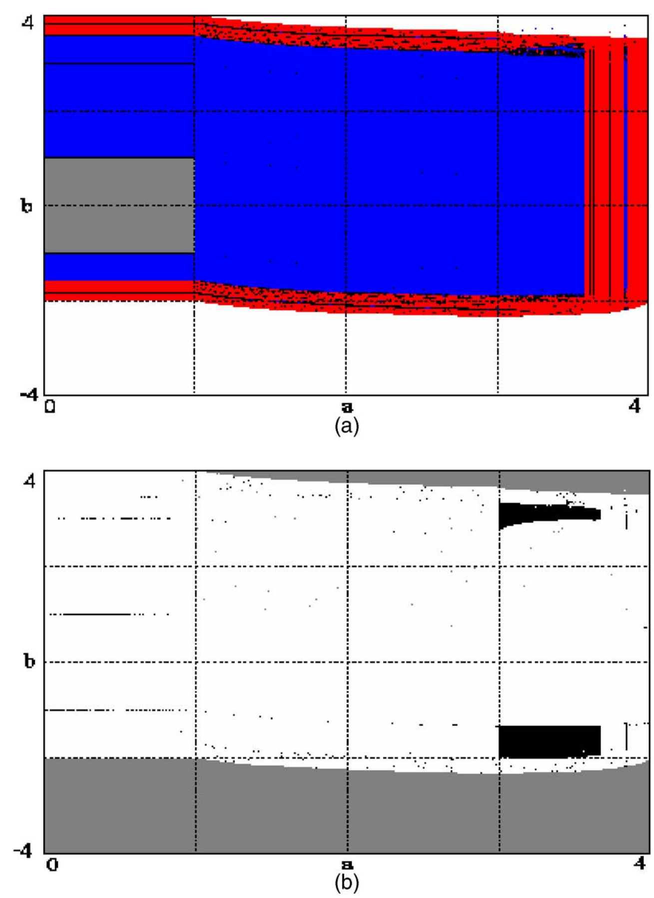

FIG. 5. (Color online) (a) Regions of dynamical behaviors in the ab-plane with $c=0.5$ for the modulated map (4). (b) The regions of ab-space with $c=0.5$ for multiple attractors (in black) and the regions of unbounded attractors (in gray) and the regions of single attractors (in white) for the modulated map (4).

solutions in the ab-plane with $c=0.5$ for the modulated map (4), where we use 10^6 iterations for each point.

On the other hand, an important phenomenon in the modulated map (4) is that the dynamics are multistable, i.e., multiple attractors can coexist. Indeed, Fig. 5(b) shows regions for multiple attractors (in black), regions of unbounded attractors (in gray), and regions of single attractors (in white) for the modulated map (4). A simple comparison between Figs. 5(a) and 5(b) shows that fixed points, periodic, and chaotic orbits of the modulated map (4) can coexist with other attractors, and the coexistence is very clear in the bands $(a,b) \in [3,3.7] \times [-2,-1.6]$ and $(a,b) \in [3,3.7] \times [2.5,3.1]$, where both periodic and chaotic attractors coexist in all three combinations: chaotic-periodic, chaotic-chaotic, and periodic-periodic.

VI. CONCLUSION

The effect of applying a modulation derived from one logistic map to the accessible parameter of a second logistic map was examined. Rigorous analytical and numerical results provided predictions for the effect of this type of modulation.

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