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THE DISCRETE HYPERCHAOTIC DOUBLE SCROLL

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In this paper we present and analyze a new piecewise linear map of the plane capable of generating chaotic attractors with one and two scrolls. Due to the shape of the attractor and its hyperchaoticity, we call it "*the discrete hyperchaotic double scroll*." It has the same nonlinearity as used in the well-known Chua circuit. A rigorous proof of the hyperchaoticity of this attractor is given and numerically justified.

Keywords: Piecewise linear map; border collision bifurcation; discrete hyperchaotic double scroll.

1. Introduction

It is well-known that if two or more Lyapunov exponents of a dynamical system are positive throughout a range of parameter space, then the resulting attractors are hyperchaotic. The importance of these attractors is that they are less regular and are seemingly "almost full" in space, which explains their importance in fluid mixing [Scheizer & Hasler, 1996; Abel *et al.*, 1997; Ottino, 1989; Ottino *et al.*, 1992]. On the other hand, the attractors generated by Chua's circuit [Chua *et al.*, 1986] given by $x' = \alpha(y - h(x)), y' = x - y + z, z' = -\beta y$ are associated with saddle-focus homoclinic loops and are not hyperchaotic, where $h(x) = (2m_1 x + (m_0 - m_1)(|x+1| - |x-1|))/2$. The double scroll attractor for this case is shown in Fig. 1.

The double scroll is more complex than the Lorenz-type and the hyperbolic attractors [Mira, 1997], and thus it is not suitable for some potential applications of chaos such as secure communications and signal masking [Kapitaniak *et al.*, 1994; Thamilmaran *et al.*, 2004]. Hyperchaotic attractors make robust tools for some applications, but this circuit does not exhibit hyperchaos because of its limited dimensionality [Chua *et al.*, 1986]. To resolve this problem, several works have focused on the hyperchaotification of Chua's circuit using several techniques such as coupling many Chua circuits as in [Kapitaniak *et al.*, 1994] where a 15-D dynamical system is obtained. However, the resulting system is complicated and difficult to construct. A simpler method introduces an additional inductor in the canonical Chua circuit as given in [Thamilmaran *et al.*, 2004], where a 4-D dynamical system is obtained that converges to a hyperchaotic attractor by a border collision bifurcation [Banerjee & Grebogi, 1999]. On the other hand, the study of piecewise linear maps [Devaney, 1984; Cao & Liu, 1998; Aharonov *et al.*, 1997; Ashwin & Fu, 2002]

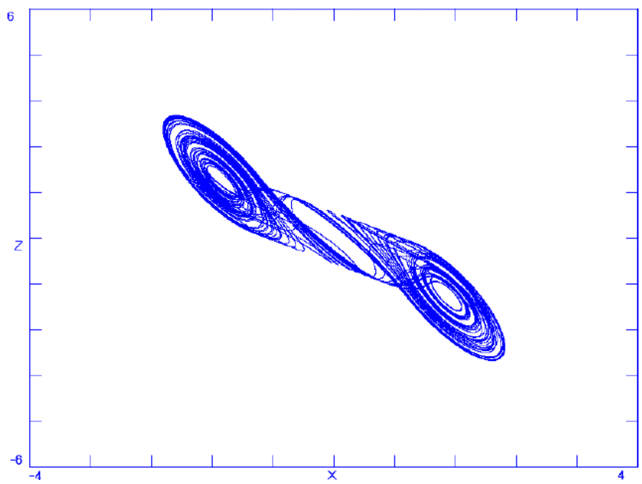

Fig. 1. The classic double scroll attractor obtained for $\alpha = 9.35$, $\beta = 14.79$, $m_0 = -1/7, m_1 = 2/7$ [Chua *et al.*, 1986].

can contribute to the development of the theory of dynamical systems, especially in finding new chaotic attractors with applications in science and engineering [Scheizer & Hasler, 1996; Abel *et al.*, 1997]. Furthermore, the techniques employed in the circuit realization of smooth maps are simple, and the approach can be extended to other systems such as piecewise linear or piecewise smooth maps [Suneel, 2006]. Also, it seems that the circuit realizations of low-dimensional maps is simpler than with high-dimensional continuous systems. For this reason, we present a discrete version of Chua's circuit attractor governed by a simple 2-D piecewise linear map that is capable of producing hyperchaotic attractors with the same shape as the classic double scroll attractor, which is not hyperchaotic. We analytically show the hyperchaoticity of the attractor and numerically show that the proposed map behaves in a similar way to the 4-D dynamical system given in [Thamilmaran *et al.*, 2004], i.e. both hyperchaotic attractors are obtained by a border collision bifurcation.

2. The Discrete Hyperchaotic Double Scroll Map

In this section, we present the new map and show some of its basic properties.

Consider the following 2-D piecewise linear map:

$$f(x, y) = \begin{pmatrix} x - ah(y) \\ bx \end{pmatrix} \tag{1}$$

where a and b are the bifurcation parameters, h is given above by the characteristic function of the so-called double scroll attractor [Chua *et al.*, 1986], and m_0 and m_1 are respectively the slopes of the inner and outer sets of the original Chua circuit. Systems such as the one in Eq. (1) typically have no direct application to particular physical systems, but they serve to exemplify the kinds of dynamical behaviors, such as routes to chaos, that are common in physical chaotic systems. Thus an analytical and numerical study is warranted. Due to the shape of the new attractor and its hyperchaoticity, we call it the "discrete hyperchaotic double scroll" because of its similarity to the well-known Chua circuit [Chua *et al.*, 1986].

One of the advantages of the map (1) is its extreme simplicity and minimality in view of the number of terms and conservation of some important properties of the classic double scroll. Firstly, the associated function $f(x, y)$ is continuous in $\mathbb{R}^2$, but it is not differentiable at the points $(x, -1)$ and $(x, 1)$ for all $x \in \mathbb{R}$. Secondly, the map (1) is a diffeomorphism except at points $(x, -1)$ and $(x, 1)$ when $abm_1m_0 \neq 0$, since the determinant of its Jacobian is nonzero if and only if $abm_1 \neq 0$ or $abm_0 \neq 0$, but it does not preserve area and it is not a reversing twist map for all values of the system parameters. Thirdly, the map (1) is symmetric under the coordinate transformion $(x, y) \to (-x, -y)$, and this transformation persists for all values of the system parameters. Therefore, the chaotic attractor obtained for map (1) is symmetric just like the classic double scroll [Chua *et al.*, 1986]. On the other hand, and due to the shape of the vector field f of the map (1), the plane can be divided into three linear regions denoted by: $R_1 = \{(x, y) \in \mathbb{R}^2/y \geq 1\}$, $R_2 = \{(x, y) \in \mathbb{R}^2/|y| \leq 1\}$, $R_3 = \{(x, y) \in \mathbb{R}^2/y \leq -1\}$, where in each of these regions the map (1) is linear. However, it is easy to verify that for all values of the parameters m_0, m_1 such that $m_0m_1 > 0$, the map (1) has a single fixed point $(0, 0)$, while if $m_0m_1 < 0$, the map (1) has three fixed points, and they are given by $P_1 = ((m_1 - m_0)/bm_1, (m_1 - m_0)/m_1)$, $P_2 = (0, 0)$, $P_3 = ((m_0 - m_1)/bm_1, (m_0 - m_1)/m_1)$. Obviously, the Jacobian matrix of the map (1) evaluated at the fixed points P_1 and P_3 is the same and is given by $J_{1,3} = \begin{pmatrix} 1 & -abm_1 \\ 1 & 0 \end{pmatrix}$. Therefore, the two equilibrium points P_1 and P_3 have the same stability type. The Jacobian matrix of the map (1) evaluated at the fixed point P_2 is given by $J_2 = \begin{pmatrix} 1 & -abm_0 \\ 1 & 0 \end{pmatrix}$, and the characteristic polynomials for $J_{1,3}$ and J_2 are given respectively by $\lambda^2 - \lambda + abm_1 = 0$ and

$\lambda^2 - \lambda + abm_0 = 0$, where the local stability of these equilibria can be studied by evaluating the eigenvalues of the corresponding Jacobian matrices given by the solution of their characteristic polynomials.

3. The Hyperchaoticity of the Attractor

In this section, we give sufficient conditions for the hyperchaoticity of the discrete hyperchaotic double scroll given by the map (1). Note that this property is absent for the classic double scroll [Kapitaniak *et al.*, 1994; Thamilmaran *et al.*, 2004].

It is shown in [Li & Chen, 2004] that if we consider a system $x_{k+1} = f(x_k), x_k \in \Omega \subset \mathbb{R}^n$, such that

$$\|f'(x)\| \leq N < +\infty \tag{2}$$

with a smallest eigenvalue of $f'(x)^T f'(x)$ that satisfies

$$\lambda_{\min}(f'(x)^T(f'(x))) \geq \theta > 0, \tag{3}$$

where $N^2 \geq \theta$, then, for any $x_0 \in \Omega$, all the Lyapunov exponents at x_0 are located inside $[\ln\theta/2, \ln N]$, that is,

$$\frac{\ln\theta}{2} \leq l_i(x_0) \leq \ln N, \quad i = 1, 2, \ldots, n, \tag{4}$$

where $l_i(x_0)$ are the Lyapunov exponents for the map f. For the map (1), one has that

$$\|f'(x,y)\| = \begin{cases} \sqrt{\dfrac{b^2 + a^2m_1^2 + \sqrt{2b^2 + b^4 + 2a^2m_1^2 + a^4m_1^4 - 2a^2b^2m_1^2 + 1} + 1}{2}}, & \text{if } |y| \geq 1 \\ \sqrt{\dfrac{b^2 + a^2m_0^2 + \sqrt{2b^2 + b^4 + 2a^2m_0^2 + a^4m_0^4 - 2a^2b^2m_0^2 + 1} + 1}{2}}, & \text{if } |y| \leq 1 \end{cases} < +\infty \tag{5}$$

and

$$\lambda_{\min}(f'(x)^T(f'(x))) = \begin{cases} \dfrac{b^2 + a^2m_1^2 - \sqrt{2b^2 + b^4 + 2a^2m_1^2 + a^4m_1^4 - 2a^2b^2m_1^2 + 1} + 1}{2}, & \text{if } |y| \geq 1 \\ \dfrac{b^2 + a^2m_0^2 - \sqrt{2b^2 + b^4 + 2a^2m_0^2 + a^4m_0^4 - 2a^2b^2m_0^2 + 1} + 1}{2}, & \text{if } |y| \leq 1. \end{cases} \tag{6}$$

If

$$\begin{cases} |a| > \max\left(\dfrac{1}{|m_1|}, \dfrac{1}{|m_0|}\right) \\ |b| > \max\left(\dfrac{|am_1|}{\sqrt{a^2m_1^2 - 1}}, \dfrac{|am_0|}{\sqrt{a^2m_0^2 - 1}}\right) \end{cases} \tag{7}$$

then both Lyapunov exponents of the map (1) are positive for all initial conditions $(x_0, y_0) \in \mathbb{R}^2$, and hence the corresponding attractor is hyperchaotic. For $m_0 = -0.43$ and $m_1 = 0.41$, one has that $|a| > 2.439$, and for $b = 1.4$, one has that $|a| > 3.323$. As a test of the previous analysis, Fig. 2 shows the Lyapunov exponent spectrum for the map (1) for $m_0 = -0.43$, $m_1 = 0.41, b = 1.4$, and $-3.365 \leq a \leq 3.365$. The regions of hyperchaos are $-3.365 \leq a \leq -3.323$ and $3.323 \leq a \leq 3.365$.

On the other hand, the discrete hyperchaotic double scroll shown in Fig. 3 results from a stable period-3 orbit transitioning to a fully developed chaotic regime. This particular type of bifurcation is called a border-collision bifurcation as shown in Fig. 4, and it is the only observed scenario. If we fix parameters $b = 1.4, m_0 = -0.43$, and $m_1 = 0.41$ and vary $a \in \mathbb{R}$, then the map (1) exhibits the following dynamical behaviors as shown in Fig. 4.

In the interval $a < -3.365$, the map (1) does not converge. For $-3.365 \leq a \leq 3.365$, the map (1) begins with a reverse border-collision bifurcation, leading to a stable period-3 orbit, and then collapses to a point that is reborn as a stable period-3 orbit leading to fully developed chaos. For $a > 3.365$, the map (1) does not converge. However, it seems that the proposed map behaves in a similar way to the 4-D dynamical system given in [Thamilmaran *et al.*, 2004], i.e. both hyperchaotic attractors are obtained by a border-collision bifurcation [Banerjee & Grebogi, 1999].

Figure 5 shows regions in the *ab*-plane given by $(a, b) \in [-3.365, 3.365] \times [-2, 2]$ of unbounded

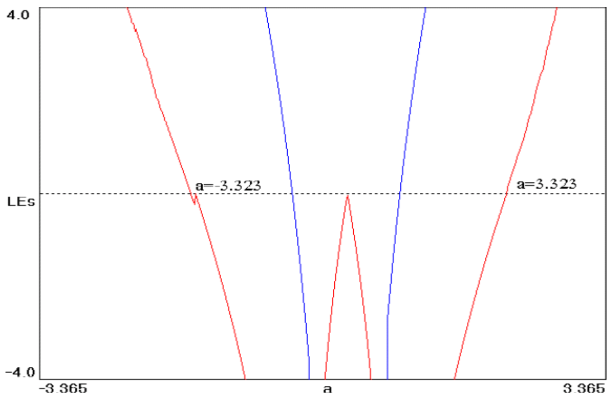

Fig. 2. Variation of the Lyapunov exponents of map (1) versus the parameter $-3.365 \le a \le 3.365$ with $b = 1.4, m_0 = -0.43$, and $m_1 = 0.41$.

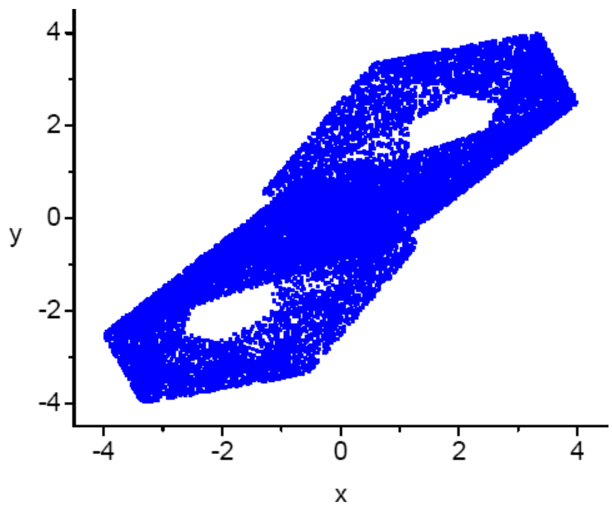

Fig. 3. The discrete hyperchaotic double scroll attractor obtained from the map (1) for $a = 3.36, b = 1.4, m_0 = -0.43$, and $m_1 = 0.41$ with initial conditions $x = y = 0.1$.

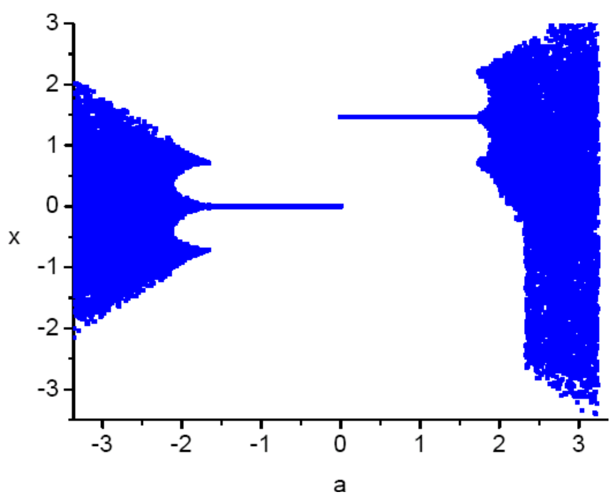

Fig. 4. The border collision bifurcation route to chaos of map (1) versus the parameter $-3.365 \le a \le 3.365$ with $b = 1.4$, $m_0 = -0.43$, and $m_1 = 0.41$.

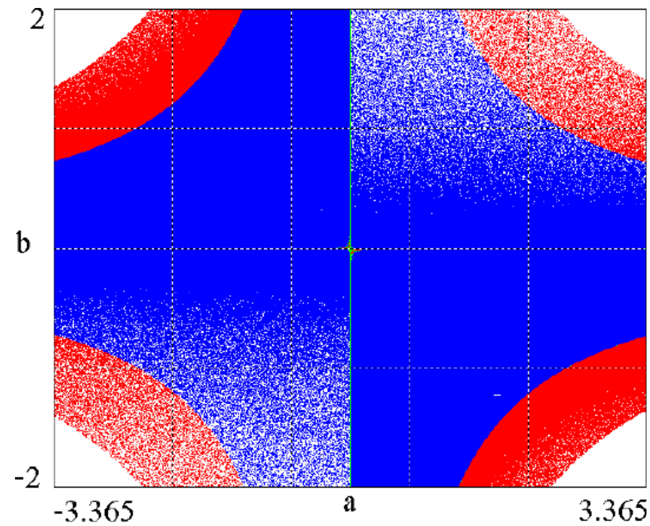

Fig. 5. Regions of dynamical behaviors in the ab-plane for the map (1).

(white), periodic orbits of periods 1 and 3 (blue), and chaotic (including hyperchaotic attractors) (red) solutions in the ab-plane for the map (1), with 10^6 iterations for each point.

4. Conclusion

We have described a new simple 2-D discrete piecewise linear chaotic map that is capable of generating a hyperchaotic double scroll attractor. Some important detailed dynamical behaviors of this map were further investigated.

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