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REVIEW ARTICLE

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Classification of three-dimensional quadratic diffeomorphisms with constant Jacobian

Abstract The 3-D quadratic diffeomorphism is defined as a map with a constant Jacobian. A few such examples are well known. In this paper, all possible forms of the 3-D quadratic diffeomorphisms are determined. Some numerical results are also given and discussed.

Keywords 3-D quadratic diffeomorphism, classification, chaos

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Contents

1 Introduction

The most general 3-D quadratic map is given by

$$f(x,y,z) = \begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_4x^2+a_5y^2+a_6z^2+a_7xy+a_8xz+a_9yz \\ b_0+b_1x+b_2y+b_3z+b_4x^2+b_5y^2+b_6z^2+b_7xy+b_8xz+b_9yz \\ c_0+c_1x+c_2y+c_3z+c_4x^2+c_5y^2+c_6z^2+c_7xy+c_8xz+c_9yz \end{pmatrix} \quad (1)$$

where $(a_i, b_i, c_i)_{0\leqslant i\leqslant 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Polynomial maps of the form (1) are part of the models of storage ring elements in the “thin lens” approximation [21]. In this case the one-turn map is difficult to evaluate with both speed and accuracy, because a modern storage ring consists of 10^3 to 10^4 elements. The occurrence of hyperchaotic and wild-hyperbolic attractors in some 3-D quadratic maps of the form (1) [3, 5, 7–11] makes them useful in potential applications [4, 6]. If the map (1) has constant Jacobian, then it is a 3-D quadratic diffeomorphism. Some possible generalizations of the 2-D Hénon map were introduced in [7–11]. Note that the attractors obtained from these selected generalizations are very similar to the Lorenz and Shimizu-Morioka attractors [1, 2]. Some forms of 3-D quadratic diffeomorphisms are important because they have some relation to Lorenz attractors [1–3, 7], to the study of unfolding 2-D maps to maps of higher dimension, and to the study of homoclinic phenomena in dynamical systems [8, 9], among others.

However, some forms of the 3-D quadratic diffeomorphisms are well known [3–5, 7–11], especially those with a quadratic inverse, where it was shown in Ref. [8] that any 3-D quadratic diffeomorphism with a quadratic inverse and constant Jacobian can be written in a normal form. However, this result does not give any information about the shapes of this type of map. Therefore, in this paper we investigate all the possible forms of the 3-D quadratic diffeomorphisms, especially, those without quadratic inverse, i.e., with an inverse of higher degree.

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The method of analysis is the rigorous computation of the Jacobian of the map (1), and hence we determine sufficient conditions for the 3-D quadratic diffeomorphisms in the sense that their Jacobian is a constant and contains at least one quadratic nonlinearity. Some of the simplest possible cases are presented here and discussed.

2 Sufficient conditions for the 3-D quadratic diffeomorphisms

In this section, we will find sufficient conditions for the 3-D quadratic diffeomorphisms. Indeed, the Jacobian matrix of the map (1) is given by

$$J(x,y) = \begin{pmatrix} a_1+2a_4x+a_7y+a_8z & a_2+2a_5y+a_7x+a_9z & a_3+a_8x+2a_6z+a_9y \\ b_1+2b_4x+b_7y+b_8z & b_2+2b_5y+b_7x+b_9z & b_3+b_8x+2b_6z+b_9y \\ c_1+2c_4x+c_7y+c_8z & c_2+2c_5y+c_7x+c_9z & c_3+c_8x+2c_6z+c_9y \end{pmatrix} \quad (2)$$

The determinant of the Jacobian matrix (2) of the map (1) is given by

$$\begin{aligned} \det J(x,y,z) = \psi_1 + \Phi_1(x,y,z) + \Phi_2(x,y,z) \\ + \Phi_3(x,y,z) \end{aligned} \quad (3)$$

where

$$\begin{aligned} \Phi_1(x,y,z) &= \xi_1x+\xi_2y+\xi_3z+\xi_4xy+\xi_5xz \\ &\quad +\xi_6yz+\xi_7xyz+\xi_8x^2 \\ \Phi_2(x,y,z) &= \xi_9y^2+\xi_{10}z^2+\xi_{11}x^3+\xi_{12}y^3 \\ &\quad +\xi_{13}z^3+(\xi_{14}+\xi_{18})yx^2 \\ \Phi_3(x,y,z) &= \xi_{15}y^2x+(\xi_{16}+\xi_{19})zx^2+\xi_{17}z^2x \\ &\quad +\xi_{20}zy^2+\xi_{21}z^2y \end{aligned} \quad (4)$$

and

$$\begin{aligned} &\xi_1 = {}_{i=2}^{i=4}\psi_i, \quad \xi_2 = {}_{i=5}^{i=7}\psi_i, \quad \xi_3 = -{}_{i=8}^{i=10}\psi_i \\ &\xi_4 = {}_{i=11}^{i=15}\psi_i, \quad \xi_5 = -{}_{i=16}^{i=20}\psi_i, \quad \xi_6 = -{}_{i=21}^{i=26}\psi_i \\ &\xi_7 = -2{}_{i=27}^{i=28}\psi_i, \quad \xi_8 = {}_{i=29}^{i=31}\psi_i, \quad \xi_9 = -{}_{i=32}^{i=34}\psi_i \\ &\xi_{10} = {}_{i=35}^{i=37}\psi_i, \quad \xi_{11} = -2\psi_{38}, \quad \xi_{12} = 2\psi_{39} \\ &\xi_{13} = -2\psi_{40}, \quad \xi_{14} = -4\psi_{41}, \quad \xi_{15} = 2{}_{i=42}^{i=43}\psi_i \\ &\xi_{16} = 4\psi_{44}, \quad \xi_{17} = -2{}_{i=45}^{i=46}\psi_i, \quad \xi_{18} = -2\psi_{47} \\ &\xi_{19} = 2\psi_{48}, \quad \xi_{20} = -2{}_{i=49}^{i=50}\psi_i, \quad \xi_{21} = 2{}_{i=51}^{i=52}\psi_i \end{aligned} \quad (5)$$

where

$$\begin{aligned} \psi_1 &= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2c_1b_3+b_1 \\ &\quad -a_3c_2-a_3b_2c_1 \\ \psi_2 &= 2a_2b_3c_4-2a_2b_4c_3-2a_3b_2c_4+2a_3c_2b_4 \\ &\quad +2b_2a_4c_3-2a_4b_3c_2 \\ \psi_3 &= a_1b_2c_8-a_1b_3c_7-a_1c_2b_8+a_1c_3b_7-a_2b_1c_8 \\ &\quad +a_2c_1b_8+b_1a_3c_7 \\ \psi_4 &= b_1c_2a_8-b_1c_3a_7-a_3c_1b_7-b_2c_1a_8+c_1b_3a_7 \\ \psi_5 &= 2a_1c_3b_5-2a_1b_3c_5+2b_1a_3c_5-2b_1a_5c_3 \\ &\quad -2a_3c_1b_5+2c_1b_3a_5 \\ \psi_6 &= a_1b_2c_9-a_1c_2b_9-a_2b_1c_9+a_2c_1b_9+a_2b_3c_7 \\ &\quad -a_2c_3b_7+b_1c_2a_9 \end{aligned} \quad (6)$$

$$\begin{aligned} \psi_7 &= -a_3b_2c_7+a_3c_2b_7-b_2c_1a_9+b_2c_3a_7-b_3c_2a_7 \\ \psi_8 &= 2a_1c_2b_6-2a_1b_2c_6+2a_2b_1c_6-2a_2c_1b_6 \\ &\quad -2b_1c_2a_6+2b_2c_1a_6 \\ \psi_9 &= a_1b_3c_9-a_1c_3b_9-a_2b_3c_8+a_2c_3b_8-b_1a_3c_9 \\ &\quad +b_1c_3a_9 \\ \psi_{10} &= a_3c_1b_9-a_3c_2b_8-b_2c_3a_8-c_1b_3a_9+b_3c_2a_8+a_3b_2c_8 \\ \psi_{11} &= 4a_3b_4c_5-4a_3b_5c_4-4a_4b_3c_5+4a_4c_3b_5 \\ &\quad +4b_3a_5c_4-4a_5b_4c_3 \\ \psi_{12} &= 2a_1b_5c_8-2a_1c_5b_8-2b_1a_5c_8+2b_1c_5a_8 \\ &\quad +2c_1a_5b_8-2c_1b_5a_8 \\ \psi_{13} &= -2a_2b_4c_9+2a_2c_4b_9+2b_2a_4c_9-2b_2c_4a_9 \\ &\quad -2a_4c_2b_9+2c_2b_4a_9 \\ \psi_{14} &= a_1b_7c_9-a_1c_7b_9-a_2b_7c_8+a_2b_8c_7-b_1a_7c_9 \\ &\quad +b_1a_9c_7+b_2a_7c_8 \\ \psi_{15} &= -b_2a_8c_7+c_1a_7b_9-c_1b_7a_9-c_2a_7b_8+c_2a_8b_7 \\ \psi_{16} &= 4a_2b_4c_6-4a_2c_4b_6-4b_2a_4c_6+4b_2a_6c_4 \\ &\quad +4a_4c_2b_6-4c_2b_4a_6 \\ \psi_{17} &= 2a_1b_6c_7-2a_1b_7c_6-2b_1a_6c_7+2b_1a_7c_6 \\ &\quad +2c_1a_6b_7-2c_1a_7b_6 \\ \psi_{18} &= -2a_3b_4c_9+2a_3c_4b_9+2a_4b_3c_9-2a_4c_3b_9 \\ &\quad +2b_4c_3a_9-2b_3c_4a_9 \\ \psi_{19} &= a_1b_8c_9-a_1b_9c_8-b_1a_8c_9+b_1a_9c_8+a_3b_7c_8 \\ &\quad -a_3b_8c_7 \\ \psi_{20} &= c_1a_8b_9-c_1a_9b_8-b_3a_7c_8+b_3a_8c_7+c_3a_7b_8 \\ &\quad -c_3a_8b_7 \\ \psi_{21} &= 4a_1b_6c_5-4a_1b_5c_6+4b_1a_5c_6-4b_1a_6c_5 \\ &\quad -4c_1a_5b_6 \\ \psi_{22} &= 4c_1a_6b_5-2a_2b_6c_7+2a_2b_7c_6+2b_2a_6c_7 \\ &\quad -2b_2a_7c_6 \\ \psi_{23} &= -2c_2a_6b_7+2a_5c_3b_8-b_2a_8c_9+c_3b_7a_9 \\ \psi_{24} &= 2c_2a_7b_6+2a_3b_5c_8-2a_3c_5b_8-2b_3a_5c_8 \\ &\quad +2b_3c_5a_8 \end{aligned} \quad (7)$$

$$\psi_{25} = -2c_3b_5a_8 + a_2b_8c_9 - a_2b_9c_8 - a_3b_7c_9 + a_3c_7b_9$$
$$\psi_{26} = b_2a_9c_8 + b_3a_7c_9 - b_3a_9c_7 + c_2a_8b_9 - c_2a_9b_8 - c_3a_7b_9$$
$$\psi_{27} = 4a_4b_6c_5 - 4a_4b_5c_6 + 4a_5b_4c_6 - 4a_5c_4b_6 - 4b_4a_6c_5 + 4a_6b_5c_4$$
$$\psi_{28} = a_7b_8c_9 - a_7b_9c_8 - a_8b_7c_9 + a_8c_7b_9 + b_7a_9c_8 - a_9b_8c_7$$
$$\psi_{29} = 2a_2c_4b_8 - 2a_2b_4c_8 + 2a_3b_4c_7 - 2a_3c_4b_7 + 2b_2a_4c_8 - 2b_2c_4a_8$$
$$\psi_{30} = -2a_4b_3c_7 - 2a_4c_2b_8 + 2a_4c_3b_7 + 2b_3c_4a_7 + 2c_2b_4a_8 - 2b_4c_3a_7 \tag{8}$$

$$\psi_{31} = a_1b_7c_8 - a_1b_8c_7 - b_1a_7c_8 + b_1a_8c_7 + c_1a_7b_8 - c_1a_8b_7$$
$$\psi_{32} = 2a_1c_5b_9 - 2a_1b_5c_9 + 2b_1a_5c_9 - 2b_1c_5a_9 + 2a_3b_5c_7$$
$$\psi_{33} = -2a_3c_5b_7 - 2c_1a_5b_9 + 2c_1b_5a_9 - 2b_3a_5c_7 + 2b_3a_7c_5 + 2a_5c_3b_7$$
$$\psi_{34} = -2c_3b_5a_7 + a_2b_7c_9 - a_2c_7b_9 - b_2a_7c_9 + b_2a_9c_7 + c_2a_7b_9 - c_2b_7a_9$$
$$\psi_{35} = 2a_1c_6b_9 - 2a_1b_6c_9 + 2a_2b_6c_8 - 2a_2c_6b_8 + 2b_1a_6c_9 - 2b_1c_6a_9 \tag{9}$$
$$\psi_{36} = -2b_2a_6c_8 + 2b_2a_8c_6 - 2c_1a_6b_9 + 2c_1b_6a_9 + 2c_2a_6b_8 - 2c_2b_6a_8$$
$$\psi_{37} = a_3b_8c_9 - a_3b_9c_8 - b_3a_8c_9 + b_3a_9c_8 + c_3a_8b_9 - c_3a_9b_8$$
$$\psi_{38} = a_4b_8c_7 - a_4b_7c_8 + b_4a_7c_8 - b_4a_8c_7 - c_4a_7b_8 + c_4a_8b_7$$
$$\psi_{39} = a_5c_7b_9 - a_5b_7c_9 + b_5a_7c_9 - b_5a_9c_7 - a_7c_5b_9 + c_5b_7a_9$$
$$\psi_{40} = a_6b_9c_8 - a_6b_8c_9 + b_6a_8c_9 - b_6a_9c_8 - a_8c_6b_9 + c_6a_9b_8$$

$$\psi_{41} = a_4c_5b_8 - a_4b_5c_8 + a_5b_4c_8 - a_5c_4b_8 - b_4c_5a_8 + b_5c_4a_8$$
$$\psi_{42} = 2a_4b_5c_9 - 2a_4c_5b_9 - 2a_5b_4c_9 + 2a_5c_4b_9 + 2b_4c_5a_9 - 2b_5c_4a_9$$
$$\psi_{43} = -a_5b_7c_8 + a_5b_8c_7 + b_5a_7c_8 - b_5a_8c_7 - a_7c_5b_8 + c_5a_8b_7$$
$$\psi_{44} = a_4b_7c_6 - a_4b_6c_7 + b_4a_6c_7 - b_4a_7c_6 - a_6c_4b_7 + c_4a_7b_6$$
$$\psi_{45} = 2a_4b_6c_9 - 2a_4c_6b_9 - 2b_4a_6c_9 + 2b_4c_6a_9 + 2a_6c_4b_9 - 2c_4b_6a_9$$
$$\psi_{46} = a_6b_7c_8 - a_6b_8c_7 - a_7b_6c_8 + a_7c_6b_8 + b_6a_8c_7 - a_8b_7c_6 \tag{10}$$
$$\psi_{47} = a_4c_7b_9 - a_4b_7c_9 + b_4a_7c_9 - b_4a_9c_7 - c_4a_7b_9 + c_4b_7a_9$$
$$\psi_{48} = a_4b_9c_8 - a_4b_8c_9 + b_4a_8c_9 - b_4a_9c_8 - c_4a_8b_9 + c_4a_9b_8$$
$$\psi_{49} = 2a_5b_7c_6 - 2a_5b_6c_7 + 2a_6b_5c_7 - 2a_6c_5b_7 - 2b_5a_7c_6 + 2a_7b_6c_5$$
$$\psi_{50} = a_5b_8c_9 - a_5b_9c_8 - b_5a_8c_9 + b_5a_9c_8 + c_5a_8b_9 - c_5a_9b_8$$
$$\psi_{51} = 2a_5b_6c_8 - 2a_5c_6b_8 - 2a_6b_5c_8 + 2a_6c_5b_8 + 2b_5a_8c_6 - 2b_6c_5a_8$$
$$\psi_{52} = a_6b_7c_9 - a_6c_7b_9 - a_7b_6c_9 + a_7c_6b_9 + b_6a_9c_7 - b_7c_6a_9$$

Hence the map (1) is a 3-D quadratic diffeomorphism if and only if

$$\begin{cases} \psi_1 \neq 0 \\ \Phi_1(x,y,z) = 0 \\ \Phi_2(x,y,z) = 0 \\ \Phi_3(x,y,z) = 0 \end{cases} \tag{11}$$

for all $(x,y,z) \in \mathbb{R}^3$. This is possible if $\xi_i = 0, i = 1, \cdots, 21$, i.e.,

$$\psi_j(a_i,b_i,c_i)_{1\leqslant i\leqslant 9} = 0, \quad j = 2,\cdots,52 \tag{12}$$

Because $\det J(x,y,z)$ is a polynomial function, the only possible case for a non-vanishing constant determinant is when (11) and (12) hold. We define the following subsets of $\mathbb{R}^{27}$:

$$\Omega_1 = \left\{(a_i,b_i,c_i)_{1\leqslant i\leqslant 9} \in \mathbb{R}^{27} : \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right) \neq 0\right\}$$
$$\Omega_j = \left\{(a_i,b_i,c_i)_{1\leqslant i\leqslant 9} \in \mathbb{R}^{27} : \psi_j\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right) = 0\right\}$$
$$j = 2,\cdots,52 \tag{13}$$

Thus if there are vectors $(a_i,b_i,c_i)_{1\leqslant i\leqslant 9} \in \mathbb{R}^{27}$ such that $(a_i,b_i,c_i)_{1\leqslant i\leqslant 9} \neq (0,0,\cdots,0)$, i.e., $\bigcap_{j=1}^{j=52} \Omega_j \neq \varnothing$, then $\det J(x,y,z)$ is a non-zero constant for all $(x,y,z) \in \mathbb{R}^3$. However, the system of equations (13) can be rewritten as

$$\begin{aligned} &\psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right) \neq 0 \\ &AC = O \end{aligned} \tag{14}$$

where $A = A\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)$ is a 51×9 matrix, $C = (c_j)_{1\leqslant j\leqslant 9}$ is a 9×1 vector of unknowns, and O is the null vector of $\mathbb{R}^{51}$. The classical method of Fontené-Rouché can be used to solve this system of equations by introducing the so-called principal determinant. However, this is very hard to do theoretically since it is difficult to determine the set of principal unknowns.

Thus, we have proved the following theorems:

Theorem 1 *The map (1) is a 3-D quadratic diffeomorphism if and only if* $\Omega = \bigcap_{j=1}^{j=52} \Omega_j \neq \varnothing$.

Theorem 2 *The set* $\Omega = \bigcap_{j=1}^{j=52} \Omega_j$ *contains at least one 3-D quadratic diffeomorphism.*

Proof The well known generalized Hénon map given in Ref. [5] satisfies all the above conditions. ■

Note that the existence of an inverse is guaranteed by the so called *real Jacobian conjecture* introduced by O.T. Keller in 1939 [13, 14], and also the upper bound for the degree of the inverse of a quadratic map on $\mathbb{R}^n$ is known to be 2^{n-1} [14], in our case the upper bound is 4. On the other hand, it was shown in Ref. [8] that any 3-D quadratic diffeomorphism with a quadratic inverse and constant Jacobian can be written in the following form:

$$g(x,y,z) = \begin{pmatrix} y \\ z \\ d_0+d_1x+d_2y+d_3z+d_5y^2+d_6z^2+d_9yz \end{pmatrix} \quad (15)$$

where d_i are the bifurcation parameters.

Therefore, we classify all the 3-D quadratic diffeomorphisms into two classes: those with a quadratic inverse, and those with no quadratic inverse, where we determine exactly all the possible forms of these two families. Indeed, as a test of the previous analysis, and for the sake of simplicity and without loss of generality, we can assume that

$$\begin{aligned} &a_0 \neq 0, a_1 = a_3 = 0 \\ &b_0 = b_1 = b_2 = 0 \\ &c_0 = c_2 = c_3 = 0 \\ &c_1 \neq 0, a_2 \neq 0, b_3 \neq 0 \end{aligned} \quad (16)$$

Then one has the following conditions:

$$\begin{aligned} &a_5 = 0, \quad a_7 = 0, \quad a_9 = 0, \quad b_6 = 0, \quad b_8 = 0 \\ &b_8 = 0, \quad c_4 = 0, \quad c_7 = \frac{-c_1 b_9}{b_3}, \quad c_8 = 0 \\ &a_4 b_9 = 0, \quad a_4 c_5 = 0, \quad a_4 c_9 = 0, \quad a_4 b_7 c_6 = 0 \\ &a_8 b_5 = 0, \quad a_8 c_5 = 0, \quad a_8 b_7 = 0, \quad a_8 c_9 = 0, \quad a_8 b_9 = 0 \\ &a_6 b_7 = 0, \quad a_6 b_9 = 0, \quad a_6 b_4 c_9 = 0 \\ &c_6 b_8 = 0, \quad c_6 b_4 = 0 \\ &c_1 a_6 b_5 + a_2 b_7 c_6 = 0, \quad b_4 c_9 = 0 \\ &a_4 b_5 c_6 + b_4 a_6 c_5 = 0 \\ &c_1 b_9^2 + b_3 b_7 c_9 = 0 \end{aligned} \quad (17)$$

Therefore, one of the possible forms of the 3-D quadratic diffeomorphisms is given by

$$f(x,y,z) = \begin{pmatrix} a_0 + a_2 y + a_4 x^2 + a_6 z^2 + a_8 xz \\ b_3 z + b_4 x^2 + b_5 y^2 + b_7 xy + b_9 yz \\ c_1 x + c_5 y^2 - \frac{c_1 b_9}{b_3} xy + c_9 yz \end{pmatrix} \quad (18)$$

with the conditions (16) and (17). More analysis on the conditions (17) show the existence of more than 71 cases. Some of them are listed below, especially those with one and two nonlinearities. On the other hand, we say that the maps f and g given respectively by (15) and (18) are affinely conjugate if there exists an affine transformation h such that

$$g \circ h(x,y,z) = h \circ f(x,y,z), \quad \text{for all } (x,y,z) \in \mathbb{R}^3 \quad (19)$$

As a result if the map (18) is affinely conjugate to the map (15), then the map (18) has quadratic inverse. We use this remark to characterize the 3-D quadratic diffeomorphisms without quadratic inverse. Indeed, the transformation h is defined by

$$h(x,y) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad (20)$$

with the condition of invertibility given by

$$\begin{aligned} d = &(h_{22}h_{33} - h_{23}h_{32})\, h_{11} + (h_{31}h_{23} - h_{21}h_{33})\, h_{12} \\ &+ (h_{21}h_{32} - h_{22}h_{31})\, h_{13} \neq 0 \end{aligned} \quad (21)$$

Such affine transformation h exists if and only if

$$E = \bigcap_{i=0}^{i=30} E_i \neq \varnothing \quad (22)$$

where

$$\begin{aligned} &E_0 : d = -3a_2c_1b_3h_{11}h_{12}h_{13} + a_2^2 b_3 h_{11}^3 + c_1 b_3^2 h_{12}^3 \\ &\qquad + a_2 c_1^2 h_{13}^3 \neq 0 \\ &E_1 : s_1 = s_3 - a_0 h_{11} - a_0 c_1 h_{13} \\ &E_2 : s_2 = s_3 - a_0 c_1 h_{13} \\ &E_3 : a_8 h_{11} = 0 \\ &E_4 : a_6 h_{11} = 0 \\ &E_5 : a_6 h_{13} = 0 \\ &E_6 : a_8 h_{13} = 0 \\ &E_7 : h_{22} = a_2 h_{11} \\ &E_8 : h_{21} = c_1 h_{13} \end{aligned} \quad (23)$$

$$
\begin{aligned}
&E_9 : h_{23} = b_3 h_{12}\\
&E_{10} : h_{32} = c_1 a_2 h_{13}
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{11} : h_{31} = c_1 b_3 h_{12}\\
&E_{12} : h_{33} = b_3 a_2 h_{11}\\
&E_{13} : b_9 h_{12} + c_9 h_{13} = 0\\
&E_{14} : a_4 h_{11} + b_4 h_{12} = 0\\
&E_{15} : b_5 h_{12} + c_5 h_{13} = 0\\
&E_{16} : b_7 h_{12} - \frac{c_1}{b_3} b_9 h_{13} = 0\\
&E_{17} : a_2 b_9 h_{11} + b_3 c_9 h_{12} = 0\\
&E_{18} : a_2 b_5 h_{11} + b_3 c_5 h_{12} = 0\\
&E_{19} : a_2 b_7 h_{11} - c_1 b_9 h_{12} = 0\\
&E_{20} : a_2 b_4 h_{11} + c_1 a_4 h_{13} = 0
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
&E_{21} : \zeta_1 + \zeta_2 = 0\\
&E_{22} : b_9 h_{11} - b_7 h_{13} + b_3 d_9 h_{11} h_{12} + 2 d_5 h_{11} h_{13}\\
&\qquad + 2 c_1 b_3 d_6 h_{12} h_{13} + c_1 d_9 h_{13}^2 = 0\\
&E_{23} : -c_1 a_6 h_{12} + a_2 b_3 d_9 h_{11} h_{12} + a_2^2 b_3 d_6 h_{11}^2\\
&\qquad + b_3 d_5 h_{12}^2 = 0\\
&E_{24} : -a_4 b_3 h_{12} - a_2 b_4 h_{13} + c_1 b_3 d_9 h_{12} h_{13}\\
&\qquad + c_1 b_3^2 d_6 h_{12}^2 + c_1 d_5 h_{13}^2 = 0\\
&E_{25} : -b_3 c_5 h_{11} - c_1 b_5 h_{13} + a_2 c_1 d_9 h_{11} h_{13}\\
&\qquad + a_2 d_5 h_{11}^2 + a_2 c_1^2 d_6 h_{13}^2 = 0\\
&E_{26} : \zeta_3 + \zeta_4 = 0\\
&E_{27} : 2 d_5 h_{12} h_{13} - a_8 h_{12} + a_2 d_9 h_{11} h_{13}\\
&\qquad + 2 a_2 b_3 d_6 h_{11} h_{12} + b_3 d_9 h_{12}^2 = 0\\
&E_{28} : \zeta_5 + \zeta_6 = 0\\
&E_{29} : \zeta_7 + \zeta_8 = 0\\
&E_{30} : \zeta_9 + \zeta_{10} = 0
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
\zeta_1 =& -b_3 c_9 h_{11} - c_1 b_9 h_{13} + 2 b_3 d_5 h_{11} h_{12}\\
\zeta_2 =& 2 a_2 c_1 b_3 d_6 h_{11} h_{13} + c_1 b_3 d_9 h_{12} h_{13}\\
&+ a_2 b_3 d_9 h_{11}^2\\
\zeta_3 =& d_0 + (d_1 + d_2 + d_3 - 1) s_3 - a_0 c_1 b_3 h_{12}\\
&- a_0 d_1 h_{11} - c_1 a_0 (2 d_5 + d_9) h_{13} s_3\\
\zeta_4 =& -c_1 a_0 (d_1 + d_2) h_{13} + (d_5 + d_6 + d_9) s_3^2\\
&+ a_0^2 c_1^2 d_5 h_{13}^2\\
\zeta_5 =& d_1 h_{11} + c_1 d_2 h_{13} - a_2 c_1 b_3 h_{11} + c_1 b_3 d_3 h_{12}\\
&+ c_1 (2 d_5 + d_9) h_{13} s_3\\
\zeta_6 =& b_3 c_1 (2 d_6 + d_9) h_{12} s_3 - a_0 c_1^2 b_3 d_9 h_{12} h_{13}\\
&- 2 a_0 c_1^2 d_5 h_{13}^2\\
\zeta_7 =& -a_0 a_2 c_1^2 d_9 h_{13}^2 + (a_2 c_1 d_3 - 2 a_0 a_2 c_1 d_5 h_{11}\\
&+ s_3 (2 a_2 c_1 d_6 + a_2 c_1 d_9)) h_{13}\\
\zeta_8 =& d_1 h_{12} + a_2 d_2 h_{11} - a_2 c_1 b_3 h_{12}\\
&+ s_3 (2 a_2 d_5 h_{11} + a_2 d_9 h_{11})\\
\zeta_9 =& a_2 b_3 d_3 h_{11} + b_3 d_2 h_{12} + (d_1 - a_2 c_1 b_3) h_{13}\\
&+ b_3 (2 d_5 + d_9) h_{12} s_3\\
\zeta_{10} =& b_3 a_2 (2 d_6 + d_9) h_{11} s_3 - 2 a_0 c_1 b_3 d_5 h_{12} h_{13}\\
&- a_0 a_2 c_1 b_3 d_9 h_{11} h_{13}
\end{aligned}
\tag{26}
$$

Thus the transformation h takes the form:

$$
h(x, y) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ c_1 h_{13} & a_2 h_{11} & b_3 h_{12} \\ c_1 b_3 h_{12} & c_1 a_2 h_{13} & b_3 a_2 h_{11} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} s_3 - a_0 h_{11} - a_0 c_1 h_{13} \\ s_3 - a_0 c_1 h_{13} \\ s_3 \end{pmatrix}
\tag{27}
$$

We remark that If $\left((a_i, b_i, c_i)_{0 \leqslant i \leqslant 9}\right) \in \bar{E} = \bigcup_{i=0}^{i=30} \bar{E}_i$, where $\bar{E}$ is the compliment of E, then maps (15) and (18) are not affinely conjugate. For example if $a_6 \neq 0$ or $a_8 \neq 0$, i.e. $\left((a_i, b_i, c_i)_{0 \leqslant i \leqslant 9}\right) \in \bar{E}_1 \cup \bar{E}_2$ then one has that $h_{11} = 0$ and $h_{13} = 0$, then if $b_4 \neq 0$ or $b_5 \neq 0$ or $b_7 \neq 0$, or $b_9 \neq 0$, or $c_5 \neq 0$ or $c_9 \neq 0$, then $h_{12} = 0$, then the transformation h is not invertible. Thus, a characterization of 3-D quadratic diffeomorphisms without quadratic inverse is given in the following Theorem:

Theorem 3 *If* $\left((a_i, b_i, c_i)_{0 \leqslant i \leqslant 9}\right) \in \bar{E} = \bigcup_{i=0}^{i=30} \bar{E}_i$, *then the map (18) has no quadratic inverse.*

On the other hand, and regarding the conditions (17), it is important to note that the 3-D quadratic diffeomorphism can be written with several nonlinearities, for example all maps with one nonlinearity are given by

$$
f(x, y, z) = \begin{pmatrix} a_0 + a_2 y + a_4 x^2 \\ b_3 z \\ c_1 x \end{pmatrix}
\tag{28}
$$

$$
f(x, y, z) = \begin{pmatrix} a_0 + a_2 y + a_6 z^2 \\ b_3 z \\ c_1 x \end{pmatrix}
\tag{29}
$$

$$
f(x, y, z) = \begin{pmatrix} a_0 + a_2 y + a_8 x z \\ b_3 z \\ c_1 x \end{pmatrix}
\tag{30}
$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_4x^2 \\ c_1x \end{pmatrix} \tag{31}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_5y^2 \\ c_1x \end{pmatrix} \tag{32}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_7xy \\ c_1x \end{pmatrix} \tag{33}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z \\ c_1x+c_5y^2 \end{pmatrix} \tag{34}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z \\ c_1x+c_6z^2 \end{pmatrix} \tag{35}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z \\ c_1x+c_9yz \end{pmatrix} \tag{36}$$

After some tedious calculations, we have proved the following theorem:

Theorem 4 *Any 3-D quadratic diffeomorphism of the family (18) with one nonlinearity has a quadratic inverse.*

Therefore, the dynamics of all the cases from (28) to (36) are conjugate to the dynamics of the map (15). On the other hand, all the 3-D quadratic diffeomorphisms with two nonlinearities are given by

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2+a_6z^2 \\ b_3z \\ c_1x \end{pmatrix} \tag{37}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2+a_8xz \\ b_3z \\ c_1x \end{pmatrix} \tag{38}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z+b_4x^2 \\ c_1x \end{pmatrix} \tag{39}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z+b_5y^2 \\ c_1x \end{pmatrix} \tag{40}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z+b_7xy \\ c_1x \end{pmatrix} \tag{41}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z \\ c_1x+c_5y^2 \end{pmatrix} \tag{42}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z \\ c_1x+c_6z^2 \end{pmatrix} \tag{43}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2 \\ b_3z \\ c_1x+c_9yz \end{pmatrix} \tag{44}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2+a_8xz \\ b_3z+b_4x^2 \\ c_1x \end{pmatrix} \tag{45}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2 \\ b_3z+b_4x^2 \\ c_1x \end{pmatrix} \tag{46}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2 \\ b_3z \\ c_1x+c_5y^2 \end{pmatrix} \tag{47}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2 \\ b_3z \\ c_1x+c_6z^2 \end{pmatrix} \tag{48}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2 \\ b_3z \\ c_1x+c_9yz \end{pmatrix} \tag{49}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_8xz \\ b_3z+b_4x^2 \\ c_1x \end{pmatrix} \tag{50}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_8xz \\ b_3z \\ c_1x+c_6z^2 \end{pmatrix} \tag{51}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_4x^2+b_5y^2 \\ c_1x \end{pmatrix} \tag{52}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_4x^2+b_7xy \\ c_1x \end{pmatrix} \tag{53}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_4x^2 \\ c_1x+c_5y^2 \end{pmatrix} \tag{54}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_5y^2+b_7xy \\ c_1x \end{pmatrix} \tag{55}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_5y^2 \\ c_1x+c_5y^2 \end{pmatrix} \tag{56}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_5y^2 \\ c_1x+c_6z^2 \end{pmatrix} \tag{57}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_5y^2 \\ c_1x+c_9yz \end{pmatrix} \tag{58}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z+b_7xy \\ c_1x+c_5y^2 \end{pmatrix} \tag{59}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z \\ c_1x+c_5y^2+c_6z^2 \end{pmatrix} \tag{60}$$

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y \\ b_3z \\ c_1x+c_5y^2+c_9yz \end{pmatrix} \tag{61}$$

Some tedious calculations show that the maps (37) to (39), (48), (52), (53), (55), (56), (60), and (61) belongs to the map (15) (up to the affine conjugacy). Hence one has the following theorem:

Theorem 5 *The 3-D quadratic diffeomorphisms given by (37) to (39), (48), (52), (53), (55), (56), (60), and (61) are conjugate to the map (15).*

While the remaining maps have no quadratic inverse because the set $E=\bigcap_{i=0}^{i=30} E_i=\varnothing$, and hence the following theorem is proved:

Theorem 6 *The 3-D quadratic diffeomorphisms given by (40) to (47) and by (49) to (51), (54), and (57) to (59) have higher degree inverse.*

For example the inverse of the map (40) is given by:

$$g(u,t,s)=\begin{pmatrix} \dfrac{s}{c_1} \\ \dfrac{-a_0}{a_2}+\dfrac{u}{a_2}+\dfrac{-a_4s^2}{a_2c_1^2} \\ \dfrac{-a_0^2b_5}{a_2^2b_3}+\dfrac{2a_0b_5u}{a_2^2b_3}+\dfrac{t}{b_3}-\dfrac{b_5u^2}{a_2^2b_3}-\dfrac{2a_0a_4b_5s^2}{a_2^2c_1^2b_3}+\dfrac{2a_4b_5s^2u}{a_2^2c_1^2b_3}-\dfrac{a_4^2b_5s^4}{a_2^2c_1^4b_3} \end{pmatrix} \tag{62}$$

An example of a 3-D quadratic diffeomorphism with three nonlinearities is given by:

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_4x^2+a_6z^2+a_8xz \\ b_3z \\ c_1x \end{pmatrix} \tag{63}$$

and it has a quadratic inverse and thus is conjugate to the map (15). Also, an example of those with four nonlinearities is given by:

$$f(x,y,z)=\begin{pmatrix} a_0+a_2y+a_6z^2 \\ b_3z \\ c_1x+c_5y^2+c_6z^2+c_9yz \end{pmatrix} \tag{64}$$

and it has no quadratic inverse, and so on$\cdots$.

Now we restrict our attention to all the possible cases of the 3-D quadratic diffeomorphisms with one nonlinearity with their sufficient conditions, and for each map we give several examples showing the structure of the parameter space for the particular map with selected values of the vectors $(a_i,b_i,c_i)_{0\leqslant i\leqslant 9}\in\mathbb{R}^{30}$, where regions of unbounded behavior are in white, fixed points in gray, periodic orbits in blue, quasi-periodic orbits in green, and chaotic orbits in red. In this case we use $|LE|<0.0001$ as the criterion for quasi-periodic orbits with 10^6 iterations for each point. The choice of one nonlinearity is robust due to the simplicity of the resulting maps. The case with several nonlinearities can be studied using the same logic and the conditions (14). For the class of one nonlinearity, we have 5 different cases:

The first case is given by

$$\begin{cases} f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_4x^2 \\ b_0+b_1x+b_2y+b_3z \\ c_0+c_1x+c_2y+c_3z \end{pmatrix} \\ b_3c_2-b_2c_3=0 \\ \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)\neq 0 \end{cases} \tag{65}$$

The simplest case of the map (65) is

$$f(x,y,z)=\begin{pmatrix} 1+a_3z+a_4x^2 \\ x \\ y \end{pmatrix} \tag{66}$$

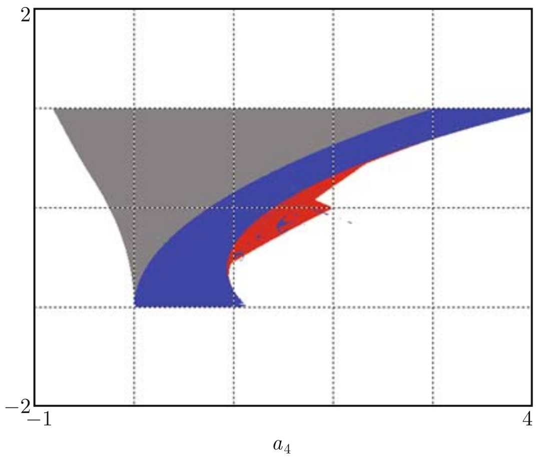

Fig. 1 Regions of dynamical behaviors in the a_4a_3-plane for the map (66).

This is the well known hyperchaotic generalized Hénon map studied in Ref. [5], and its chaotic attractor is very similar to the famous 2-D Hénon map as shown in Fig. 5(a). A generalization of Eq. (66) is studied in Ref. [12] where there are regions with multiple coexisting attractors. We plan to study this property for 3-D diffeomorphisms in a forthcoming paper. Another simple case of the map (65) is given by

$$f(x,y,z)=\begin{pmatrix} 1+a_1z+a_2y-x^2 \\ x \\ y \end{pmatrix} \tag{67}$$

or equivalently

$$f(x,y,z)=\begin{pmatrix} y \\ z \\ 1+a_1x+a_2y-z^2 \end{pmatrix} \tag{68}$$

and it is studied in Ref. [7] where the attractor in this case is very similar to the Lorenz attractor with a lacuna from the Shimizu-Morioka system [1, 2], and it is shown in Fig. 5(b).

The second case is given by

$$\begin{cases} f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_5y^2 \\ b_0+b_1x+b_2y+b_3z \\ c_0+c_1x+c_2y+c_3z \end{pmatrix} \\ c_1b_3-b_1c_3=0 \\ \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)\neq 0 \end{cases} \tag{69}$$

The simplest case of the map (69) is given by

$$f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_3z+a_5y^2 \\ x \\ y \end{pmatrix} \tag{70}$$

This case is also introduced in Ref. [7]. The map (70) is the inverse of a special case of the map (68), and so its attractors are repellers of map (68), and it was shown in Ref. [7] that its attractors are quite different from the attractors of map (68).

The other forms of simple 3-D quadratic diffeomorphisms are given by

$$\begin{cases} f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_6z^2 \\ b_0+b_1x+b_2y+b_3z \\ c_0+c_1x+c_2y+c_3z \end{pmatrix} \\ b_2c_1-b_1c_2=0 \\ \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)\neq 0 \end{cases} \tag{71}$$

$$\begin{cases} f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_7xy \\ b_0+b_1x+b_2y+b_3z \\ c_0+c_1x+c_2y+c_3z \end{pmatrix} \\ c_1b_3-b_1c_3=0 \\ b_3c_2-b_2c_3=0 \\ \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)\neq 0 \end{cases} \tag{72}$$

$$\begin{cases} f(x,y,z)=\begin{pmatrix} a_0+a_1x+a_2y+a_3z+a_9yz \\ b_0+b_1x+b_2y+b_3z \\ c_0+c_1x+c_2y+c_3z \end{pmatrix} \\ b_2c_1-b_1c_2=0 \\ c_1b_3-b_1c_3=0 \\ \psi_1\left((a_i,b_i,c_i)_{1\leqslant i\leqslant 9}\right)\neq 0 \end{cases} \tag{73}$$

The full dynamics of some simple cases of the maps

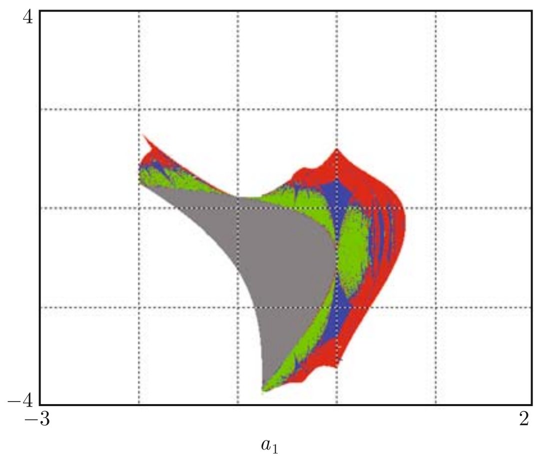

Fig. 2 Regions of dynamical behaviors in the a_1a_2-plane for the map (74).

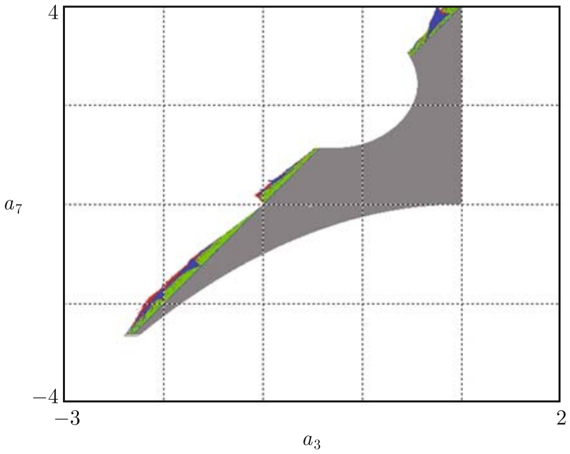

Fig. 3 Regions of dynamical behaviors in the a_3a_7-plane for the map (75).

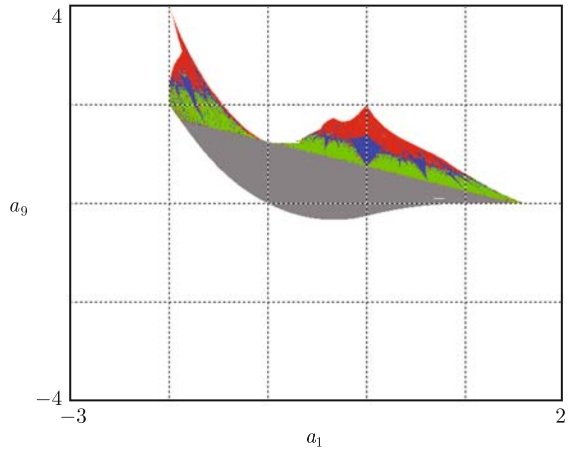

Fig. 4 Regions of dynamical behaviors in the a_1a_9-plane for the map (76).

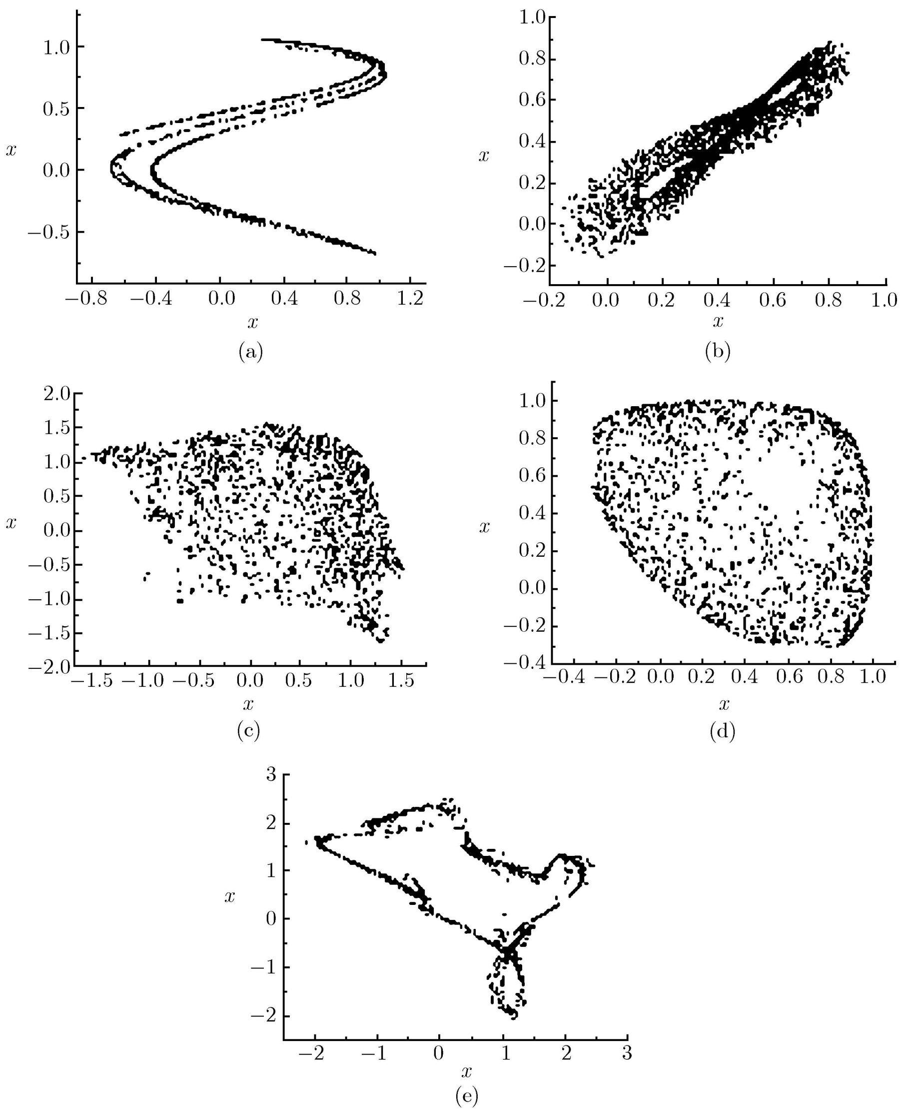

Fig. 5 **(a)** Chaotic attractor obtained from the map (66) with $a_4 = -1.6$, $a_3 = 0.1$; **(b)** Chaotic attractor obtained from the map (68) with $a_1 = 0.7$, $a_2 = 0.85$; **(c)** Chaotic attractor obtained from the map (74) with $a_1 = 0.4$, $a_2 = -0.4$; **(d)** Chaotic attractor obtained from the map (75) with $a_3 = -1$, $a_7 = -1.8$; **(e)** Chaotic attractor obtained from the map (76) with $a_1 = 0.5$, $a_9 = -0.83$.

$$f(x,y,z) = \begin{pmatrix} 1 + a_1x + a_2y - z^2 \\ x \\ y \end{pmatrix} \quad (74)$$

$$f(x,y,z) = \begin{pmatrix} 1 + a_1x + a_9yz \\ x \\ y \end{pmatrix} \quad (75)$$

$$f(x,y,z) = \begin{pmatrix} 1 + a_3z + a_7xy \\ x \\ y \end{pmatrix} \quad (76)$$

are shown in Figs. 2, 3, and 4, respectively. Some corresponding chaotic attractors are also depicted in Fig. 5(c)–(e).

Note that these cases are characterized by a typical quasi-periodic route to chaos, contrary to the situation for the maps given in (66) and (68) [5–7]. This confirms that the maps (66) and (68) are not topologically equivalent to the maps (74), (75), and (76).

3 Applications in physics

Several researchers have defined and studied quadratic 3-D chaotic systems. The first was Lorenz [17] in 1963, where he proposed a simple three linked nonlinear differential equations with complex behavior as a model of a weather system showing rates of change in temperature and wind speed. The behavior of this system was sensitively dependent on the initial conditions of the model, therefore, the prediction of a future state of the system was impossible. Using computer simulation, Ruelle in 1979 show that the first fractal shape identified took the form of a butterfly (the butterfly effect). This justified the usual case where weather prognostication is involved and notoriously wrong (*butterfly in the Amazon might, in principle, ultimately alter the weather in Kansas*).

Recently, chaos has been very useful in many technological disciplines such as in information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications, and so on [18–20]. As a new application of chaos theory the example given in Ref. [20] where the adaptive synchronization with unknown parameters is discussed for a unified chaotic system by using the Lyapunov method and the adaptive control approach. Some communication schemes, including chaotic masking, chaotic modulation, and chaotic shift key strategies, are then proposed based on the modified adaptive method. In these schemes the transmitted signal is masked by chaotic signal or modulated into the system, which effectively blurs the constructed return map and can resist this return map attack. The driving system with unknown parameters and functions is almost completely unknown to the attackers, so it is more secure to apply this method into the communication.

A symplectic map is a diffeomorphism that preserves the area, i.e. the determinant of its Jacobian matrix is one. In the actual work, the 3-D quadratic diffeomorphism is a symplectic map if and only if

$$\psi_1 = 1 \quad (77)$$

On the other hand, Poincaré sections, especially return maps provide tools for the location and stability of resonances and periodic orbits and the existence or nonexistence of invariant tori on the behavior of continuous time systems, such as the Lorenz system which is the best close example connected with physics, the Chen system, Lu system and generalized Lorenz system [18]. It is well known that any periodic (autonomous) Hamiltonian system of degrees of freedom generates a $2n$-dimensional symplectic map by following the flow for one period (parametrized by the value of the Hamiltonian, by considering the first return to a surface of section). Twist maps corresponding to Hamiltonians have the following properties [7–11]:

(1) They are for where the velocity is a monotonic function of the canonical momentum.

(2) They have a Lagrangian variational formulation.

(3) One-parameter families of twist maps typically exhibit all possible types of dynamics, and the properties of the minimizing orbits (the periodic and quasiperiodic orbits) can be found throughout these transitions from simple or integrable motion to complex or chaotic motion. The minimizing orbits are used in the theory of transport that deals with the motion of ensembles of trajectories.

In a virtual viewpoint, every model of physical phenomena is a dynamical system. Almost all fundamental models of physics are Hamiltonian dynamical systems which gives rise to symplectic mappings. For example, the mapping defined by a Hamiltonian flow taking an initial condition to a state some finite time later is a symplectic map. These mappings are included in the study of chemical reactions, or magnetic plasma confinement, especially in magnetic field line mapping, guiding center motion, and plasma wave heating. They are also in charged-particle motion in particle accelerators [15], where a charged particle (either have a positive, negative or no charge) is a particle with an electric charge. It may be either a subatomic particle or an ion. A plasma or the fourth state of matter is a collection of charged particles, or even a gas containing a proportion of charged particles. The simplest accelerator is the cyclotron which consists of a constant magnetic field and a time-dependent voltage drop across a narrow azimuthal gap. The motion of a fluid particle in an incompressible fluid is also Hamiltonian [16].

As an application in the electronic engineering of the 3-D quadratic diffeomorphism given by (15), the following example given in Ref. [6] where a discrete-component electronic implementation of a discrete-time hyperchaotic generalized Hénon map of the form:

$$\begin{cases} x_1\,(k+1) = 1.76 - x_2^2(k) - 0.1x_3(k) \\ x_2\,(k+1) = x_1(k) \\ x_3\,(k+1) = x_2(k) \end{cases} \tag{78}$$

with the initial conditions $x_1\,(0)=1, x_2\,(0)=0.1, x_3\,(0)=0$, the map (78) exhibits a hyperchaotic attractor.

Using analog states, the corresponding circuit designs are relatively simple and uses commonly available parts and is readily constructed. Also, experimental results show that the circuit is functional.

4 Conclusion

In this paper all possible forms of the 3-D quadratic diffeomorphisms are determined. Some numerical results are also given and discussed.

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