Chaotic dynamics on large networks

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Many systems in nature are governed by a large number of agents that interact nonlinearly through complex feedback loops. When the networks are sufficiently large and interconnected, they typically exhibit self-organization and chaos. This paper examines the prevalence and degree of chaos on large unweighted recurrent networks of ordinary differential equations with sigmoidal nonlinearities and unit coupling. The largest Lyapunov exponent is used as the signature and measure of the chaos, and the study includes the effects of damping, asymmetries in the distribution of coupling strengths, network symmetry, and sparseness of connections. Minimum conditions and optimal network architectures are determined for the existence of chaos. The results have implications for the design of social and other networks in the real world in which weak chaos is deemed desirable or as a way of understanding why certain networks might exist on "the edge of chaos." © *2008 American Institute of Physics*. [DOI: 10.1063/1.2945229]

With the advent of readily available fast computers, much of the interest in nonlinear dynamics has turned to the study of high-dimensional networks of interacting agents (or neurons in the case of neural networks). Such networks, with an appropriate nonlinearity and interaction between the neurons can model a wide range of phenomena in the physical, social, and biological sciences. Very large networks often exhibit chaos and other features that may be universal for large classes of such systems. Most studies to date have concerned discrete-time systems (maps) with connections chosen randomly or according to some prescription such as fully connected, scale-free, small-world, or near-neighbor. Here we consider continuous-time, dissipative systems (flows) governed by ordinary differential equations, but we choose a particularly simple example in which all the nonzero couplings are of the same magnitude, although possibly of different signs, so that the effect of network architecture on the dynamics can be unambiguously studied. We are especially interested in the conditions under which such networks are weakly chaotic since such behavior mimics their natural counterparts and provides the conditions for self-organization and pattern formation that are so prevalent in nature.

I. FULLY-CONNECTED RANDOM NETWORKS

As an example of a general, complex, nonlinear, continuous-time, dynamical system, we consider a network[1] of coupled ordinary differential equations with a sigmoidal nonlinearity such as the hyperbolic tangent:

$$\dot{x}_i = -b_i x_i + \tanh \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} x_j, \tag{1}$$

where N is the dimension of the system (the number of variables). This particular nonlinearity is appropriate because it models the common situation in nature where a small stimulus produces a linear response but the response saturates when the stimulus is large, thereby avoiding unbounded and hence unphysical solutions. With an appropriate choice of the vector b_i and the matrix a_{ij}, this system can exhibit a wide range of dynamics including chaos for N as small as 4, and with a sufficiently large N, it can approximate to arbitrary accuracy any dynamical system[2] ensuring, in principle, that the results are relevant to any kind of large network. The details of the system being modeled are contained in these values. In what follows, it will be convenient and sufficient to take $b_i = b$ for all i as the bifurcation parameter. This system is equivalent to an alternate form[2,3] in which the hyperbolic tangent is inside the summation. The $-bx$ term is analogous to frictional damping and guarantees that the solutions are bounded with an attractor for $b > 0$.

Equation (1) can be considered as an artificial neural network,[4] in which the neurons accept weighted inputs from all the other neurons and nonlinearly squash their sum. As a model of the brain, the signals could represent neural firing rates, and the connections would represent synapses. In a food web, the signals could represent the population of the various species, and the connections would represent feeding. In a financial market, the signals could represent wealth of the investors, and the connections would represent trades. In a political system, the signals could represent voters' position along some political spectrum such as Democrat/Republican, and the connections would be the flow of information between individuals that determine their views. In general, the network could represent any collection of nonlinearly interacting agents such as people, firms, animals, cells, molecules, or any number of other entities, with the connections representing the flow of energy, data, goods, or information among them. In each case, there is an implicit external source of energy, money, information, or other resource and usually some loss of that resource from the system.

The system has a static equilibrium with all $x_i=0$ that would represent the state of the system in the absence of external resources, but it can be driven away from that equilibrium by the positive feedback among the neurons, which implies an external source of energy or other resource not explicit in the equations. The damping coefficient (b) describes the rate at which the system decays to its equilibrium state in the absence of external resources. Decreasing b is in some sense equivalent to increasing the interactions among the agents and hence the flow of resources through the system. Similar and more extensive studies have been done on discrete-time neural networks (iterated maps),[5,6] but such models are less common in the physical sciences and lead too easily to chaotic solutions that may not be physically realizable.

A system such as this can exhibit three types of dynamics. (1) Solutions can approach a static equilibrium and thereafter remain forever. Such a system would be "dead," literally in the case of the brain or a food web, but also an unhealthy financial or political state with all individuals locked into their wealth or political views and unresponsive to the other individuals. (2) Solutions can be periodic (or quasi-periodic), cycling repeatedly through the same sequence of values. A person whose brain behaved in such a way would be "stuck in a rut" and not likely to have any creativity or adaptability. (3) Solutions can be chaotic, which is arguably the most healthy state for a natural network, especially if it is only weakly chaotic so that it retains some memory but can explore a vastly greater state space. Weakly chaotic networks exhibit the complex behavior that we normally associate with intelligent living systems.

Our interest here is in the generic behavior of such networks, in particular the likelihood that they will exhibit each of the above dynamics. For that purpose, we take a large but still tractable value of $N=317$ (a prime number approximately equal to $\sqrt{10^5}$) and select the a_{ij} values randomly as $\pm 1/\sqrt{(N-1)}$ with a mean of zero. In contrast to other studies,[7] the matrix is asymmetrical ($a_{ji} \neq a_{ij}$), which complicates theoretical analysis but makes the model more general and enriches the dynamics. Initial conditions are taken as uniformly random in the range $-1 < x_i(0) < 1$, and the equations are iterated using a fourth-order Runge-Kutta integrator with an adaptive step size[8] and an absolute error bound of 10^{-6} at each step.

The Lyapunov exponent is calculated using the method in Ref. 9. An N-dimensional system has N Lyapunov exponents, but this paper will be concerned primarily with the largest Lyapunov exponent since its sign indicates the nature of the dynamics. A practical problem is that the value often converges slowly with large fluctuations, especially in the vicinity of bifurcations. The orbit must be followed for a long enough time to reach and thereafter sample all regions of the attractor, whose dimension can be $\sim N/2$. Convergence of the Lyapunov exponent is ensured by demanding that the amplitude of the fluctuations over the previous thousand time steps be somewhat less than the resolution of the plot. Even so, many of the figures in this paper required several days of computation.

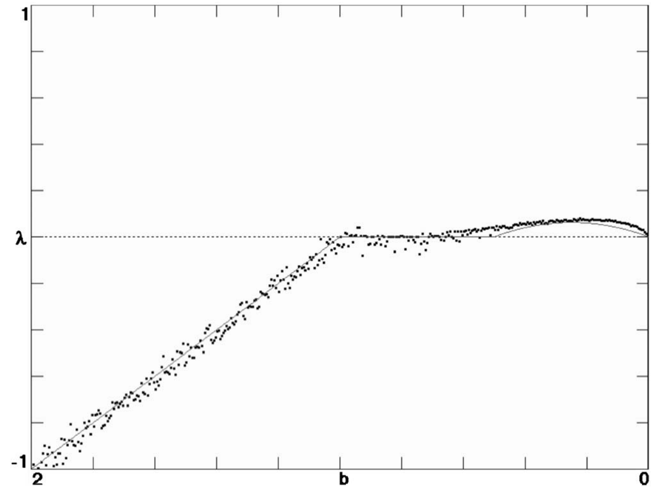

FIG. 1. Largest Lyapunov exponent for a collection of 400 fully connected artificial neural networks in Eq. (1) with $N=317$ and random weights.

Values of b were chosen uniformly in the range $0 < b < 2$, and the Lyapunov exponents for 400 random, fully connected networks are plotted in Fig. 1. Note that the scale on b is backwards so that the network activity increases to the right as the damping is reduced. Three regimes are evident in the figure. For strong damping ($b > 1$), most Lyapunov exponents are negative, implying a stable equilibrium. For intermediate damping, most Lyapunov exponents are zero, implying a periodic (limit cycle) or quasi-periodic (attracting torus) solution. For weak damping, most Lyapunov exponents are positive, implying a chaotic solution and accompanying strange attractor. The smooth curve in Fig. 1 shows a simple piecewise-continuous function

$$\lambda = 1 - b \ (b > 1), \quad \lambda = 0 \ (1 \geqslant b \geqslant 0.5), \quad \lambda = b(0.5 - b) \ (0.5 > b \geqslant 0), \tag{2}$$

which was previously used[10] to fit a similar case with $N=101$ and Gaussian random weights. The case here has a slightly larger maximum Lyapunov exponent ($\lambda \sim 0.07$), which nonetheless occurs near $b=0.25$, and a smaller periodic region (where $\lambda=0$), attributable to the higher N rather than to the different distribution of a_{ij} values. This relatively small Lyapunov exponent is consistent with the appealing but controversial idea that complex adaptive systems evolve at the "edge of chaos."[11] This system exhibits the quasi-periodic route to chaos that is thought to be generic for high-dimensional systems[12,13] with a Lyapunov exponent that depends only weakly on the details of the interactions and hence may be universal in the limit of infinite N.

The results above are for the case in which the interactions a_{ij} are equally positive and negative (mean zero). It is interesting to ask what happens when the random interactions have a bias toward negative or positive values. Figure 2 shows the largest Lyapunov exponent with $b=0.25$ for a collection of 500 networks with $N=317$ and different biases. A bias of -1 corresponds to all a_{ij} negative (all agents competing), and a bias of $+1$ corresponds to all a_{ij} positive (all agents cooperating). Strong competition reduces the Lyapunov exponent, but preserves the chaos, while even

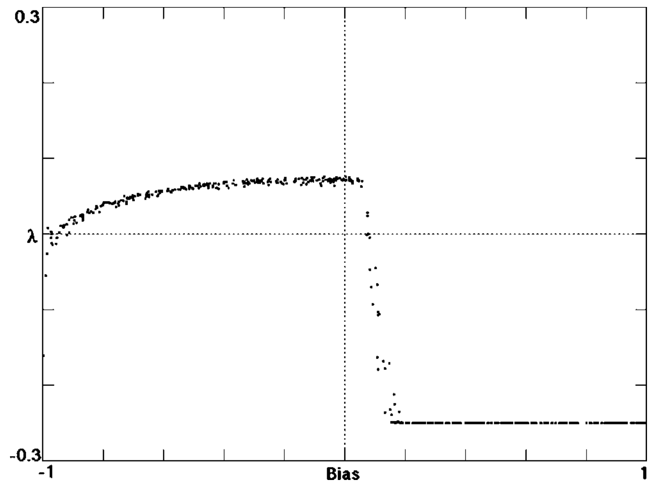

FIG. 2. Largest Lyapunov exponent for a collection of 500 fully connected artificial neural networks in Eq. (1) with $N=317$ and $b=0.25$ with random weights but biased to control the fraction of interactions that are positive.

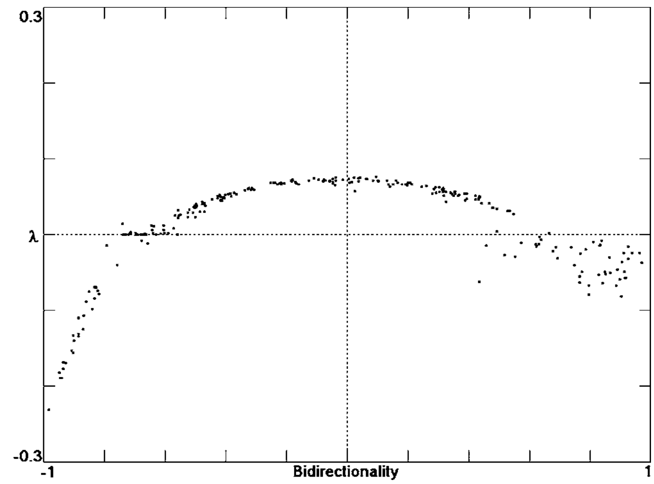

FIG. 3. Largest Lyapunov exponent for a collection of 250 fully connected artificial neural networks in Eq. (1) with $N=317$ and $b=0.25$ with random weights but chosen to control the fraction of interactions that are symmetric ($a_{ji}=a_{ij}$).

modest cooperation drives the system into a stable saturated state, with a rather abrupt phase transition when more than about 55% of the agents are cooperating. The system is near maximally chaotic when there is an equal amount of cooperation and competition, but that condition is very close to the critical state. This result implies that a network of mostly competing agents has more interesting and perhaps more healthy dynamics than one in which the agents are primarily cooperating and the dynamics are stagnant. Perhaps the drive to compete for wealth is the reason financial markets are so inherently chaotic. Note that a network with nearly all agents competing is at "the edge of chaos" (a very small positive Lyapunov exponent).

Equally interesting is to ask what happens when the agents interact with bidirectional symmetry ($a_{ji}=a_{ij}$) or bidirectional antisymmetry ($a_{ji}=-a_{ij}$), but with half competing and half cooperating. Figure 3 shows the largest Lyapunov exponent with $b=0.25$ for a collection of 250 networks with varying degrees of bidirectionality, where $+1$ corresponds to total bidirectional symmetry and -1 corresponds to total bidirectional antisymmetry (or skew symmetry). Both extremes suppress the chaos, but there is a large intermediate region where even a modest asymmetry preserves the chaos. Such a result is perhaps relevant to food webs, where predator/prey interactions have skew symmetry, while other interactions (bees and flowers, for example) are symbiotic.

II. DILUTED NETWORKS

Large networks in nature are not likely to be fully connected, especially if the interactions are of similar magnitude, since that would imply an enormous number of interactions and would be wasteful of resources. This section describes the dynamics of diluted networks[14,15] governed by a sparse matrix of interactions (most a_{ij} values are zero). For this purpose, we define the connectivity as the fraction of connections (a_{ij} values) that are nonzero. Thus, the fully connected networks described above have a connectivity of 1, and a network in which each neuron gets an input from only one other would have a connectivity of $1/(N-1)$ and can

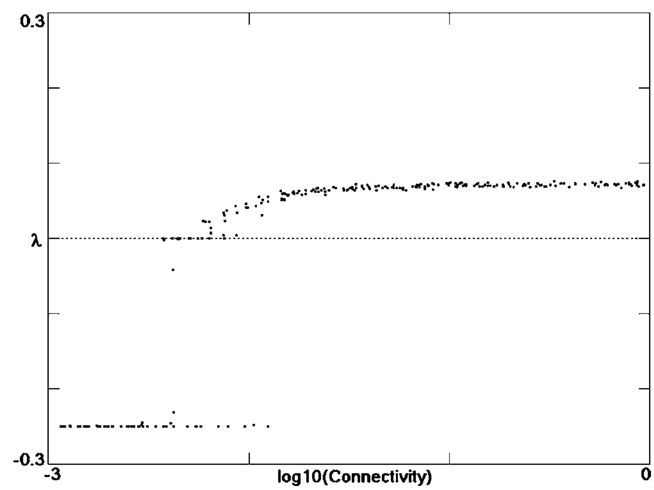

FIG. 4. Largest Lyapunov exponent for a collection of 250 artificial networks in Eq. (1) with $N=317$ and $b=0.25$ with random weights but with varying degrees of connectivity.

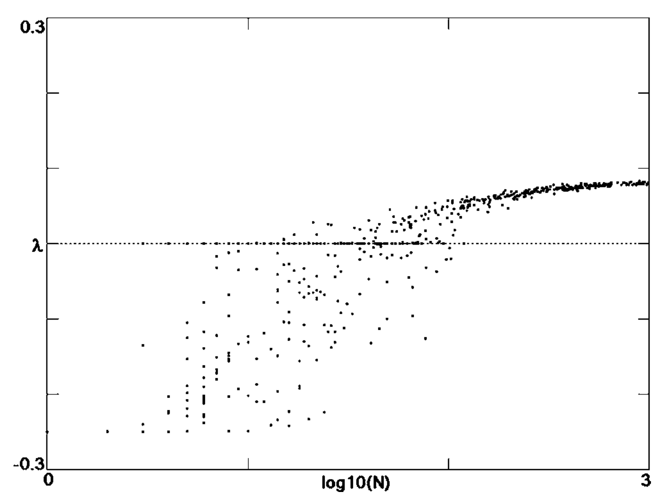

FIG. 5. Largest Lyapunov exponent for a collection of 750 fully connected artificial networks in Eq. (1) with $b=0.25$ and random weights as a function of network size (N).

therefore be arranged into a single continuous ring with only near-neighbor interactions and information propagating unidirectionally around the ring (assuming only those neurons that influence others are assumed to be part of the network). Networks with yet smaller connectivity necessarily have inactive neurons and thus can be reduced to one or more smaller networks, the limiting case of which will be considered in the next section.

Figure 4 shows a collection of 250 networks with $b=0.25$ and different connectivities on a log scale. Clearly the networks can be made quite sparse (only a few inputs per neuron) without affecting the largest Lyapunov exponent until the networks degenerate into a number of smaller networks. To confirm that the loss of chaos when the connectivity is less than about 1% is due to a decrease in network size, Fig. 5 shows the largest Lyapunov exponent for a collection of 750 fully connected networks with $b=0.25$ for values of N from 1 to 1000. Large networks are apparently always chaotic, while small networks are only rarely so, with the transition around $N=50$. Figure 5 also indicates why $N=317$ was chosen, since for that value, nearly all networks for a given b have the same qualitative behavior as even larger networks.

These results imply that a healthy network does not need to be very highly connected, but that some connectivity is crucial. In a food web, it is important that each species has multiple food sources and provides food for several others. Electing the best political leaders requires that each voter is influenced by at least a few others and in turn influences several others.

III. MINIMAL NETWORKS

These observations raise the question of how small a network of this type can be and still exhibit chaos and what are the properties of such a minimal network. Although chaos becomes increasingly rare as N decreases, a brute-force search indicates that the smallest such network has $N=4$. Examination of several million cases with $N=4$ indicated that the case with the greatest Lyapunov exponent is

$$\dot{x}_1 = -bx_1 + \tanh(x_4 - x_2), \quad \dot{x}_2 = -bx_2 + \tanh(x_1 + x_4),$$
$$\dot{x}_3 = -bx_3 + \tanh(x_1 + x_2 - x_4), \quad \dot{x}_4 = -bx_4 + \tanh(x_3 - x_2) \tag{3}$$

with $b=0.043$, for which $\lambda=(0.031\,64, 0, -0.073\,13, -0.130\,51)$ as determined using the Wolf algorithm[16] with a

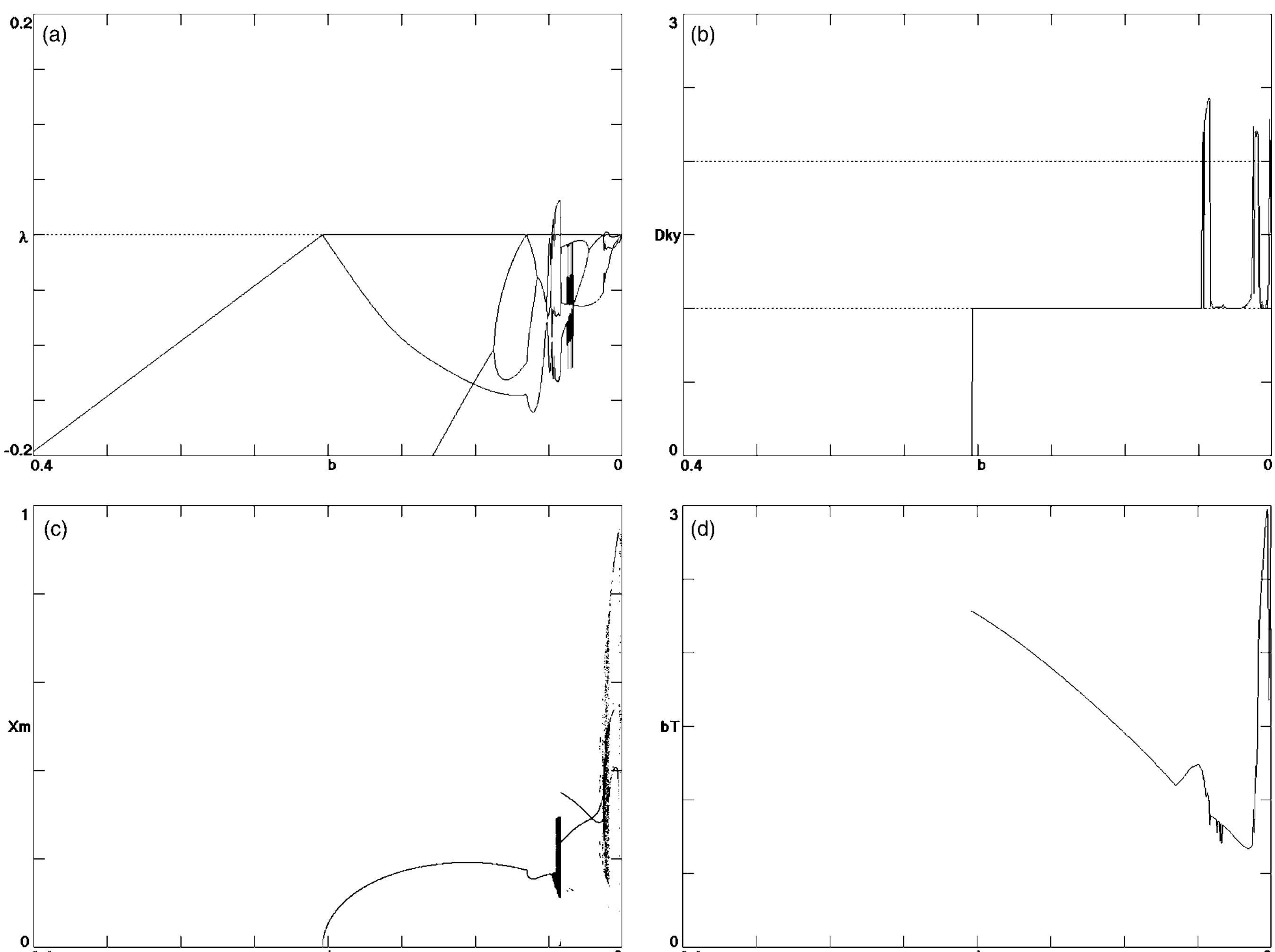

FIG. 6. Route to chaos for the minimal network in Eq. (3) with $N=4$ (a) Lyapunov exponents, (b) Kaplan–Yorke dimension, (c) bifurcation diagram, and (d) cycle time.

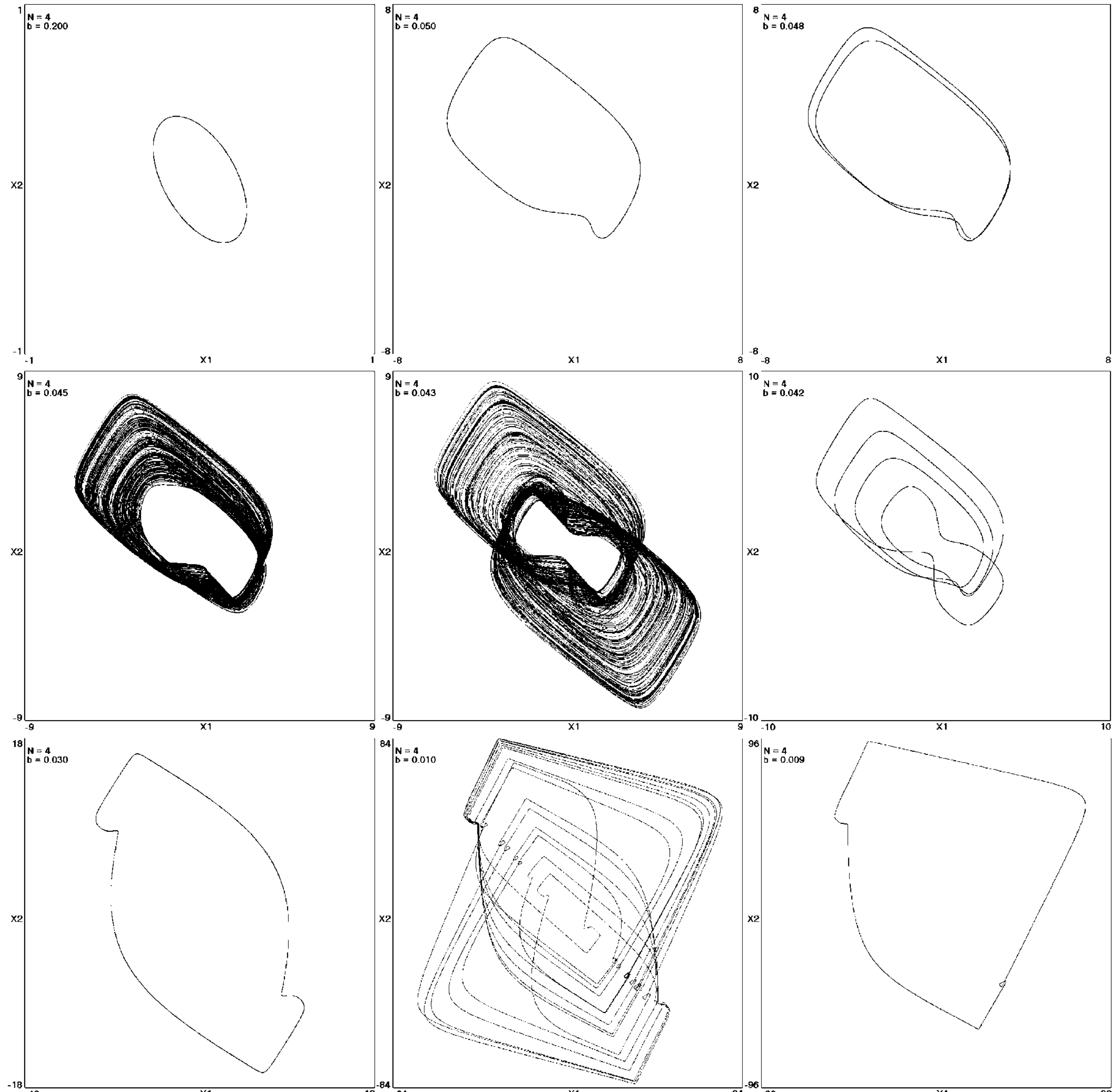

FIG. 7. Attractors at various values of b for the minimal network in Eq. (3) with $N=4$.

Kaplan–Yorke dimension[17] of $D_{\mathrm{KY}}=2.432\,63$. Its route to chaos as shown in Fig. 6 resembles the much more complicated cases in Fig. 1 except that the chaos only exists over two relatively narrow ranges of b. Initial conditions are not critical, but are taken as $x_i(0)=(1.2,0.4,1.2,-1)$, which is near the attractor for $b=0.043$.

This case is sufficiently simple to invite theoretical analysis. For large, positive values of b, the equilibrium at the origin is stable with eigenvalues that satisfy the equation

$$\Delta^4+3\Delta^2+\Delta+1=0, \tag{4}$$

where $\Delta=-b-\lambda$. Equation (4) has solutions $\Delta=0.203\,723\pm1.663\,928i$ and $\Delta=-0.203\,723\pm0.560\,668i$. The largest two Lyapunov exponents for large b are thus equal and given by $\lambda=0.203\,723-b$, and a Hopf bifurcation occurs at $b=0.203\,723$ with angular frequency $\omega=0.560\,668$, in agreement with Fig. 6. For smaller values of b, a limit cycle forms as shown in Fig. 7 and grows in size as b decreases until a pitchfork bifurcation occurs at $b\cong0.065\,12$, whereupon a pair of symmetric limit cycles are born. These limit cycles continue to grow until a period-doubling cascade begins at $b\cong0.049\,29$, culminating in chaos at $b\cong0.0476$ with a pair of symmetric strange attractors that undergo an attractor-merging crisis[18] at $b\cong0.0450$. The chaos persists, except for periodic windows, until an inverse period-doubling cascade begins at $b\cong0.041\,85$, leading eventually to a symmetric limit cycle, whereupon a similar sequence occurs at a larger scale with a new band of

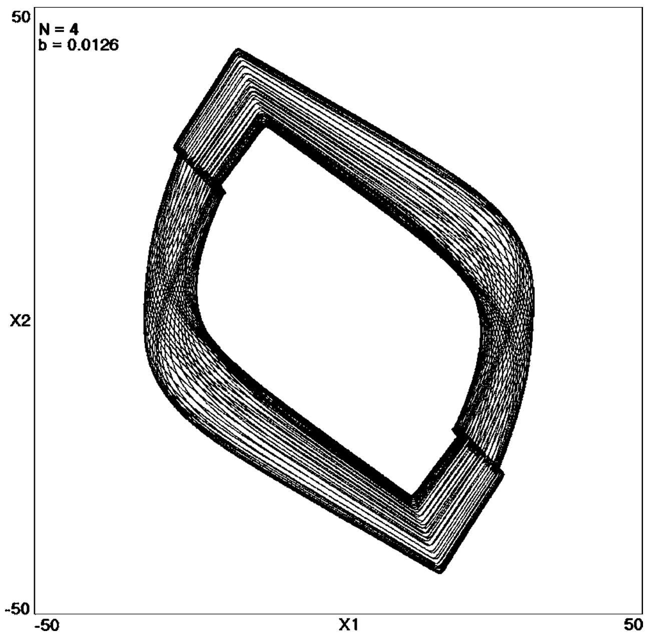

FIG. 8. Attracting 2-torus at $b=0.0126$ for the minimal network in Eq. (3) with $N=4$.

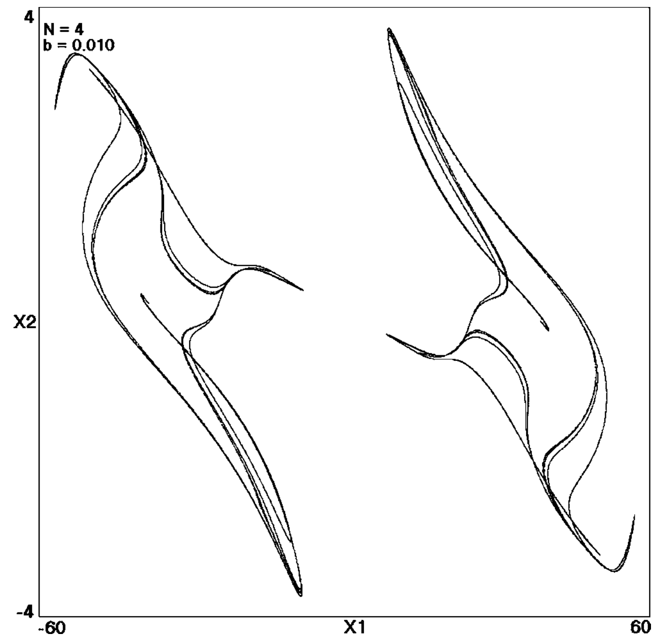

FIG. 10. Poincaré section in the x_1x_2 plane for the minimal network in Eq. (3) with $b=0.010$ and $x_4=0$.

chaos around $b=0.010$, where the Lyapunov exponents are $\lambda=(0.002\,10, 0, -0.011\,76, -0.030\,34)$ with a Kaplan–Yorke dimension of $D_{\mathrm{KY}}=2.178\,45$. There is a narrow band around $b\cong 0.0126$ where an attracting 2-torus exists with Lyapunov exponents $\lambda=(0, 0, -0.000\,444, -0.049\,956)$, as shown in Fig. 8. Figures 9 and 10 show Poincaré sections for the chaotic bands at $b=0.043$ and $b=0.010$, respectively, in the x_1x_2 plane for $x_4=0$.

It would be interesting to implement this system electronically using the saturation properties of operational amplifiers as the sigmoidal nonlinearity, although the narrow range of damping over which it is chaotic would likely make the circuit rather delicate.

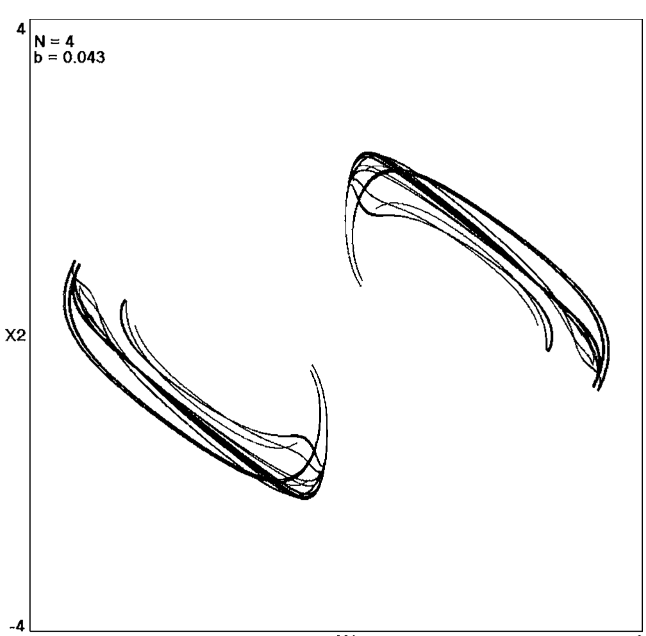

FIG. 9. Poincaré section in the x_1x_2 plane for the minimal network in Eq. (3) with $b=0.043$ and $x_4=0$.

IV. FULLY CONNECTED CIRCULANT NETWORKS

Now consider cases in which the neurons are arranged in a homogeneous ring with each neuron interacting with its neighbors in an identical fashion. The weight matrix then becomes a vector with $N-1$ components a_j, and the network takes the form

$$\dot{x}_i = -bx_i + \tanh \sum_{j=1}^{N-1} a_j x_{i+j}, \tag{5}$$

where $a_j = \pm 1/\sqrt{(N-1)}$ and $x_{i+j}=x_{i+j-N}$ for $i+j>N$ (periodic boundary conditions).

Chaos is very rare in such networks, even with $N=317$, but it does exist, as Fig. 11 shows. The bifurcation sequence is similar to the noncirculant networks previously described. The distribution of a_j values for this case is unremarkable and seemingly random with 156 positive and 160 negative. However, it is remarkable that the dynamics in the periodic and chaotic regimes are asymmetric (the x_i values are not all equal) even though the equations are symmetric. This symmetry-breaking is a common but counter-intuitive feature of such circulant networks. It is necessary to use asymmetric initial conditions to avoid synchronization that would preclude the chaos, but they are otherwise not critical except when there are multiple coexisting attractors. The symmetric solution is unstable, and so any small perturbation of the initial conditions from exact symmetry will suffice.

Circulant networks provide the opportunity to study spatiotemporal chaos. One way to illustrate this behavior is with a spatiotemporal plot in which the values of $x_i(t)$ are plotted in the it plane using a gray scale, as shown in Fig. 12 for the

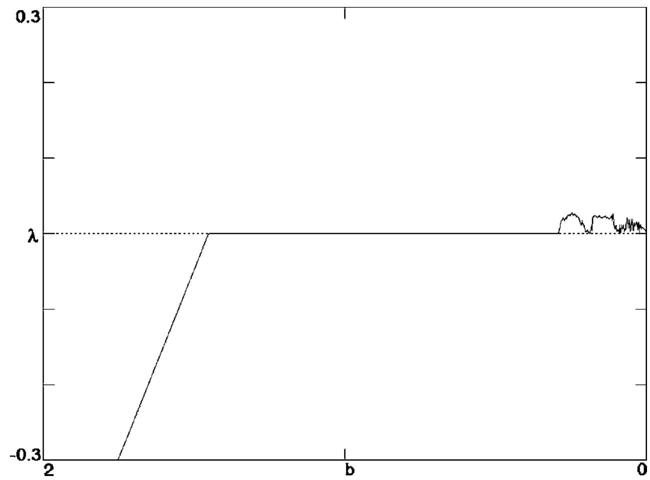

FIG. 11. Route to chaos for a fully connected circulant network in Eq. (5) with $N=317$.

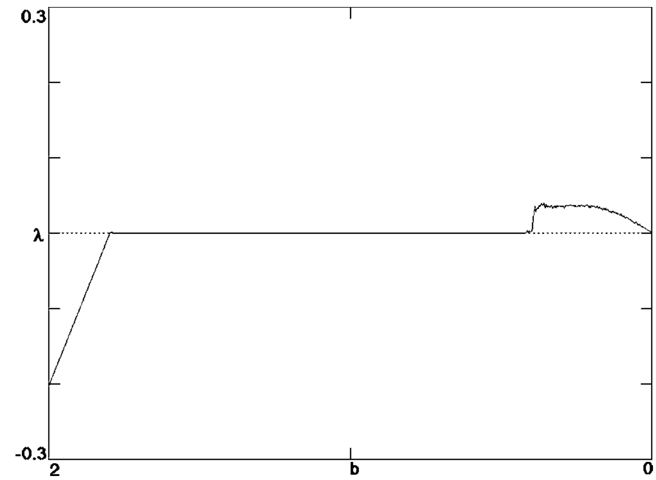

FIG. 13. Route to chaos for the diluted circulant network in Eq. (6) with $N=317$.

case above with $b=0.25$. This case has a largest Lyapunov exponent of $\lambda=0.0248$. It shows some nearly periodic, counter-rotating structures. The initial conditions in Fig. 12 were chosen on the attractor, but random initial conditions would permit one to observe the self-organization that is so characteristic of networks in ecology (niches), finance (pockets of wealth), and politics (red/blue States).

V. DILUTED CIRCULANT NETWORKS

Circulant networks can also be diluted, which even further reduces the number of parameters. Diluted ring networks have been previously studied[19] using both discrete-time and continuous-time models,[20] but typically not with circulant matrices. As with fully connected circulant networks, chaos is rare in diluted circulant networks and is apparently nonexistent in maximally diluted circulant networks (each neuron receiving an input from only one other). However, there are some highly diluted cases with $N=317$ that do exhibit chaos, one example of which is

$$\dot{x}_i = -bx_i + \tanh(x_{i+42} - x_{i+126} + x_{i+254}). \tag{6}$$

Its route to chaos is shown in Fig. 13 and the corresponding spatiotemporal plot for $b=0.36$ is shown in Fig. 14, where the largest Lyapunov exponent is $\lambda=0.0381$. An interesting unanswered question is why those three connections give chaos while most others do not. In fact, changing any of the connections one place to the right or left [replacing 126 with 125 or 127 in Eq. (6), for example] or even changing N from 317 to 316 or 318 destroys the chaos for $b=0.36$. However, the case in Eq. (6) is robustly chaotic in the sense that no periodic windows are evident in Fig. 13 over a wide range of b. One clue might be the fact that 126 is 3×42, while 254 is approximately 6×42.

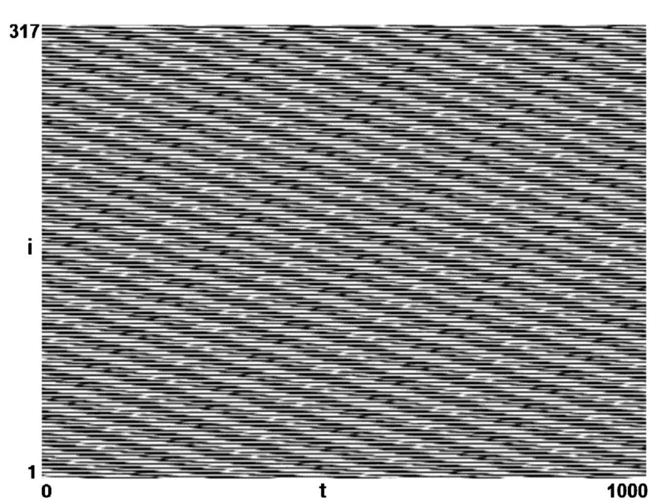

FIG. 12. Spatiotemporal plot of a fully connected circulant network in Eq. (5) with $b=0.25$ and $N=317$.

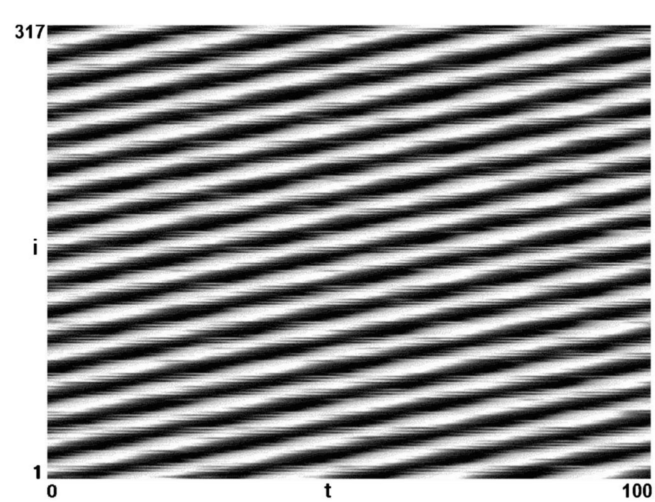

FIG. 14. Spatiotemporal plot of the diluted circulant network in Eq. (6) with $b=0.36$ and $N=317$.

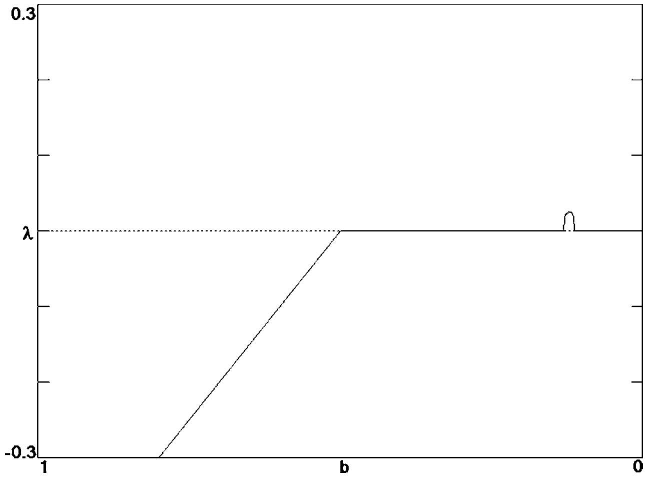

FIG. 15. Route to chaos for the minimal circulant network in Eq. (7) with $N=5$.

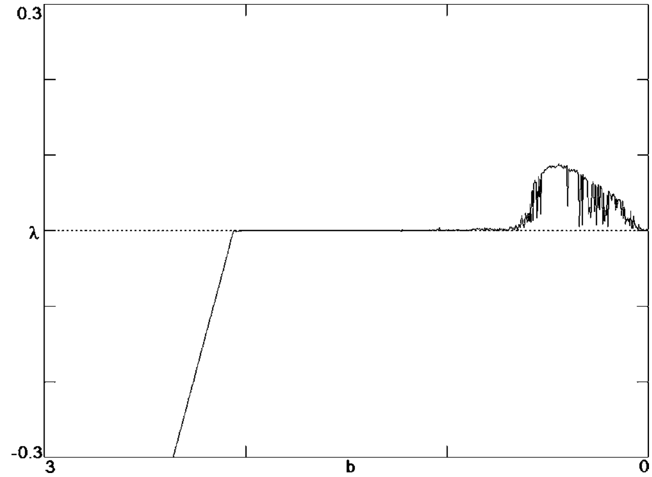

FIG. 17. Route to chaos for the minimal circulant network in Eq. (8) with $N=317$.

VI. MINIMAL CIRCULANT NETWORKS

It is interesting to find the circulant network of the form of Eq. (5) with the smallest N for which chaos occurs. An extensive search indicates that such a network has $N=5$, and the case with the largest Lyapunov exponent appears to be

$$\dot{x}_i = -bx_i - \tanh(x_{i+2} + x_{i+4}), \tag{7}$$

whose largest Lyapunov exponent is $\lambda=0.0248$ at $b=0.12$. Its route to chaos is shown in Fig. 15, and the attractor projected onto the x_1x_2 plane is shown in Fig. 16. This is a case worthy of further study. This is also a case that would be interesting to implement electronically using saturating operational amplifiers, although its chaotic operation is also likely to be rather delicate.

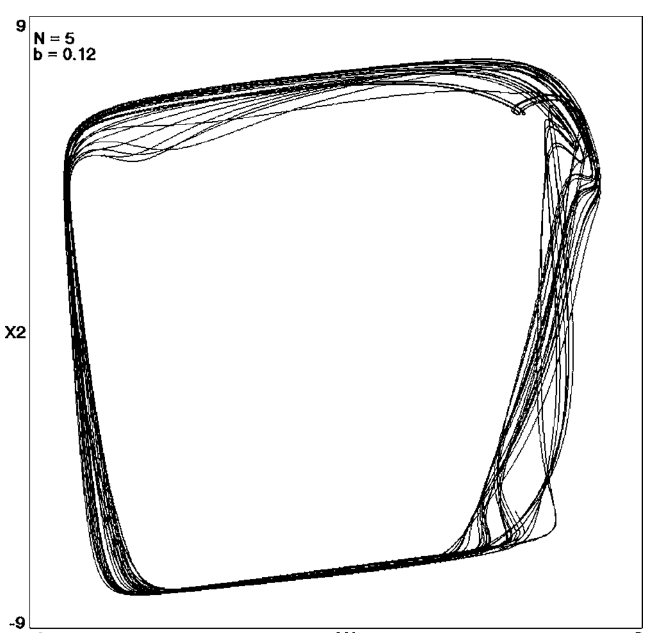

FIG. 16. Strange attractor for the minimal circulant network in Eq. (7) with $b=0.12$ and $N=5$.

VII. NEAR-NEIGHBOR CIRCULANT NETWORKS

Finally, we consider the case in which the network is circulant, but the interactions are only with a small number of near neighbors. Chaos appears to require at least six neighbors for $N=317$, and the simplest example appears to be

$$\dot{x}_i = -bx_i + \tanh(x_{i+1} + x_{i+2} - x_{i+3} - x_{i+4} + x_{i+5} - x_{i+6}), \tag{8}$$

whose largest Lyapunov exponent is $\lambda=0.0831$ at $b=0.5$. Its route to chaos is shown in Fig. 17, and the spatiotemporal plot in the *it* plane is shown in Fig. 18. This case is also worthy of further study.

VIII. CONCLUSIONS

This paper has reported the prevalence and degree of chaos in large unweighted recurrent networks of ordinary differential equations with sigmoidal nonlinearities and unit coupling. Such networks are generally chaotic when there are sufficiently many neurons, even when each neuron has only a few connections to the others. Chaos tends to be sup-

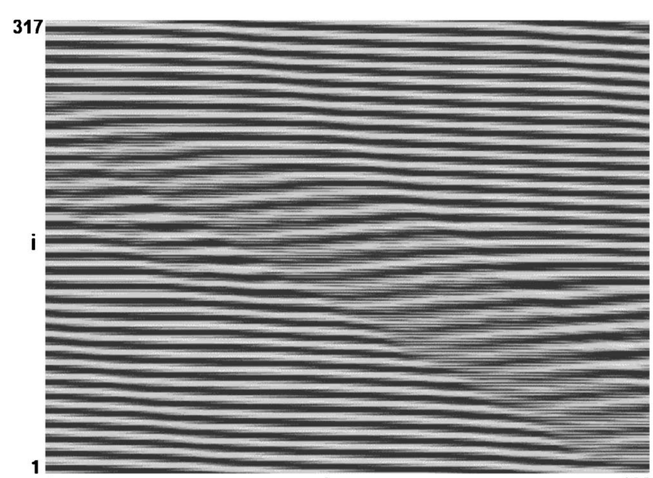

FIG. 18. Strange attractor for the minimal circulant network in Eq. (8) with $b=0.5$ and $N=317$.

pressed when symmetries are imposed on the connections, but there are some highly diluted and highly symmetric situations in which chaos occurs. In particular, circulant networks, in which the neurons are arranged in a homogeneous ring with each neuron interacting identically with its neighbors, provide elegant examples of symmetry breaking, self-organization, and spatiotemporal chaos.

The implication to real-world networks is that they are likely to be chaotic only if they are sufficiently large and have a structure that is not too highly ordered. It is likely that networks evolve (dynamics *of* the network) so that the dynamics *on* the network are weakly chaotic since the networks would otherwise lose their utility and adaptability. Such networks become increasingly delicate as their size and interactivity decrease, showing the importance of biodiversity, a healthy political discourse, and a robust economy.

ACKNOWLEDGMENTS

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