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SIMPLE CONSERVATIVE, AUTONOMOUS, SECOND-ORDER CHAOTIC COMPLEX VARIABLE SYSTEMS

DELMAR MARSHALL
Physics Department, Amrita Vishwa Vidyapeetham,
Clappana P.O., Kollam, Kerala 690-525, India
delmarshall@amritapuri.amrita.edu

J. C. SPROTT
Physics Department, University of Wisconsin,
1150 University Avenue, Madison, WI 53706, USA

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It is shown that, for analytic functions f, systems of the form $\ddot{z} = f(z, \dot{z})$ and $\ddot{z} = f(z)$ cannot produce chaos; and that systems of the form $\ddot{z} = f(z^*, \dot{z}^*)$ and $\ddot{z} = f(z, z^*)$ are conservative. Eight simple chaotic systems of the form $\ddot{z} = f(z, z^*)$ with quadratic and cubic polynomial $f(z, z^*)$ are given. Lyapunov spectra are calculated, and the systems' phase space trajectories are displayed. For each system, a Hamiltonian is given, if one exists.

Keywords: Chaos; complex variables; Duffing equation; Hénon–Heiles system; conservative system; Hamiltonian system.

1. Introduction

Since Edward Lorenz [1993] coined his famous "butterfly effect" metaphor in 1972, chaotic systems with real variables have been studied and written about extensively [Strogatz, 1994; Guckenheimer & Holmes, 1983; Hilborn, 1994; Nayfeh & Balachandran, 1995; Sprott, 2003]. In the last two decades, these investigations have included systems with complex variables, with applications in many areas, including rotor dynamics [Cveticanin, 1995], loading of beams and plates [Nayfeh & Mook, 1979], plasma physics [Rozhanskiĭ & Tsendin, 2001], optical systems [Newell & Moloney, 1992] and high-energy accelerators [Dilão & Alves-Pires, 1996]. Theoretical studies have focused on finding approximate solutions to various classes of complex valued equations, and on the control of chaos [Cveticanin, 2001; Mahmoud & Bountis, 2004; Mahmoud, 1998; Mahmoud *et al.*, 2001].

Following a number of previous studies [Sprott, 1994, 1997a, 1997b; Malasoma, 2000; Marshall & Sprott, 2009], a search was conducted for simple examples of autonomous second-order chaotic complex systems.

Section 2 discusses theoretical considerations and numerical methods, including proscription of chaos in systems $\ddot{z} = f(z)$ and $\ddot{z} = f(z, \dot{z})$; and systems $\ddot{z} = f(z, z^*)$ and $\ddot{z} = f(z^*, \dot{z}^*)$ being conservative. In Secs. 3 and 4, eight simple chaotic quadratic and cubic systems of the form $\ddot{z} = f(z, z^*)$, with polynomial f, are presented. Section 5 is a brief summary.

2. Theoretical Considerations and Numerical Techniques

2.1. *Systems involving $\dot{z}$ and $\dot{z}^*$*

This search for simple autonomous second-order chaotic complex systems began with a search for dissipative systems of the form $\ddot{z} = f(z, \dot{z})$. This form is equivalent to the two-dimensional, first-order complex variable system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z_1, z_2) \end{aligned} \tag{1}$$

and with $z = x + iy$ and $f(z, \dot{z}) = u(x, y, \dot{x}, \dot{y}) + iv(x, y, \dot{x}, \dot{y})$, to the four-dimensional real system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{y}_1 &= y_2 \\ \dot{x}_2 &= u(x_1, y_1, x_2, y_2) \\ \dot{y}_2 &= v(x_1, y_1, x_2, y_2) \end{aligned} \tag{2}$$

If f is analytic, the Cauchy–Riemann equations require that $\partial u/\partial x_1 = \partial v/\partial y_1, \partial u/\partial y_1 = -\partial v/\partial x_1, \partial u/\partial x_2 = \partial v/\partial y_2$ and $\partial u/\partial y_2 = -\partial v/\partial x_2$. Thus the Jacobian of (2) is

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial y_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial y_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \\ -b & a & -d & c \end{bmatrix}, \tag{3}$$

which has complex conjugate pairs of eigenvalues

$$\begin{aligned} &\frac{1}{2}[(c + id) \pm \sqrt{c^2 - d^2 + 4a + 2i(cd + 2b)}] \\ &\frac{1}{2}[(c - id) \pm \sqrt{c^2 - d^2 + 4a - 2i(cd + 2b)}]. \end{aligned} \tag{4}$$

These can in turn be written as

$$\begin{aligned} &\frac{1}{2}[(c + id) \pm (p + iq)] \\ &\frac{1}{2}[(c - id) \pm (p - iq)]. \end{aligned} \tag{5}$$

A theorem of Haken [1983] requires that, given a bounded solution with a positive Lyapunov exponent and not containing a fixed point, there must also be a zero Lyapunov exponent. Thus, since the Lyapunov exponents are the time averages of the real parts of (4), or equivalently (5), $\langle c \rangle = \pm \langle p \rangle$, where the angle brackets denote time average.

At the same time, for a bounded solution, trace $(J) = 2c$, the sum of the Lyapunov exponents, must average to something less than or equal to zero, i.e. $\langle c \rangle \le 0$. If $\langle c \rangle = -\langle p \rangle$, then the nonzero Lyapunov exponents are both $(\langle c \rangle - \langle p \rangle)/2 = \langle c \rangle \le 0$, which contradicts our assumption of a positive Lyapunov exponent. Without at least one positive Lyapunov exponent, no chaos is possible. Similarly, if we suppose that $\langle c \rangle = \langle p \rangle$, then the nonzero exponents are both $(\langle c \rangle + \langle p \rangle)/2 = \langle c \rangle \le 0$, and again no chaos is possible.

If no chaos is possible for $\ddot{z} = f(z, \dot{z})$, is chaos possible for the complex conjugate twin $\ddot{z} = f(z^*, \dot{z}^*)$? For this system, with analytic f, the Jacobian is

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial y_1} & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial y_1} & \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \\ b & -a & d & -c \end{bmatrix}, \tag{6}$$

because the Cauchy–Riemann equations change sign under complex conjugation. The eigenvalues for this Jacobian are complicated, precluding easy analysis, but it is clear by inspection that trace $(J) = 0$, so this system is conservative, or phase-space volume preserving, and may be Hamiltonian. A search for chaotic cubic and quadratic polynomial systems of this form was conducted in the manner described below in Sec. 2.2, but all cases diverged.

One could search for chaos in systems of the form $\ddot{z} = f(z, \dot{z}, z^*, \dot{z}^*)$, but it was judged that they would not be simple systems.

2.2. *Systems* $\ddot{z} = f(z), \ddot{z} = f(z^*)$ *and* $\ddot{z} = f(z, z^*)$

The form $\ddot{z} = f(z, z^*)$ is equivalent to the two-dimensional, first-order complex variable system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z_1, z_1^*) \end{aligned} \tag{7}$$

and with $z = x + iy$ and $f(z, z^*) = u(x, y) + iv(x, y)$, to the four-dimensional real system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{y}_1 &= y_2 \\ \dot{x}_2 &= u(x_1, y_1) \\ \dot{y}_2 &= v(x_1, y_1) \end{aligned} \tag{8}$$

The Jacobian of (8) is

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial u}{\partial x_1} & \dfrac{\partial u}{\partial y_1} & 0 & 0 \\ \dfrac{\partial v}{\partial x_1} & \dfrac{\partial v}{\partial y_1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix}, \quad (9)$$

which has zero trace, so these systems are conservative, or phase-space volume preserving, and may be Hamiltonian.

The eigenvalues of the Jacobian (9) are

$$\pm\frac{1}{\sqrt{2}}[(a+d) \pm \sqrt{(a-d)^2 + 4bc}]^{\frac{1}{2}}. \quad (10)$$

For systems of the form $\ddot{z} = f(z)$, if f is an analytic function, the Cauchy–Riemann equations require that $a = d$ and $b = -c$. The eigenvalues (10) then become

$$\pm\sqrt{a \pm ib} \quad (11)$$

Again Haken [1983] requires a zero Lyapunov exponent, and the Lyapunov exponents are the time averages of the real parts of the eigenvalues (11) over a trajectory. The real part of (11) is $\pm[\sqrt{a^2+b^2} + a/2]^{\frac{1}{2}}$. If it averages to zero with the + sign, it also will average to zero with the −sign, and vice versa, so all four Lyapunov exponents must be zero. Chaos requires at least one positive Lyapunov exponent, so there can be no chaos for systems of the form $\ddot{z} = f(z)$.

On the other hand, for systems of the form $\ddot{z} = f(z^*)$, for analytic f, as mentioned above, the Cauchy–Riemann equations change sign: $a = -d$ and $b = c$. The eigenvalues (10) become

$$\pm\sqrt[4]{(a^2+b^2)}, \quad \pm i\sqrt[4]{(a^2+b^2)} \quad (12)$$

The second pair satisfies Haken [1983], allowing no easy conclusions about the first pair. A search for chaotic cubic and quadratic polynomial systems of this form was conducted as described below, but all cases diverged.

At this point, a search for chaos with systems of the more general form $\ddot{z} = f(z, z^*)$ was conducted. For simplicity, the search was limited to cubic and quadratic polynomials f, with real coefficients, as follows:

$$f(z,z^*) = a_0 + a_1 z + a_2 z^2 + a_3 z z^* + a_4 z^* + a_5 (z^*)^2 + a_6 z^3 + a_7 z^2 z^* + a_8 z (z^*)^2 + a_9 (z^*)^3 \quad (13)$$

The coefficients a_i were randomly chosen, as were the initial conditions. The random values were taken from the squared values of a Gaussian normal distribution with mean zero and variance 1, with the original signs of the random values restored after squaring.

During the search, trajectories were followed using both fixed-step and adaptive-step fourth-order Runge–Kutta integrators [Press *et al.*, 1992]. When a chaotic system was found, it was simplified further, while preserving the chaotic behavior, by scaling or reducing the coefficients a_i to simple one-digit values, or to one, if possible.

The largest Lyapunov exponent was calculated using an adaptive-step fourth-order Runge–Kutta integrator [Press *et al.*, 1992], and the method detailed in [Sprott, 2003]. From (10), the Lyapunov exponents occur in equal pairs, with opposite signs. Thus if the largest (positive) exponent is known, and there is one zero exponent [Haken, 1983], the other two exponents follow.

3. Quadratic Systems

The simplest three quadratic polynomial $\ddot{z} = f(z, z^*)$ systems found were, with Lyapunov spectra in braces, and starting from initial conditions $(z_0, \dot{z}_0)$,

$$\ddot{z} = (z^*)^2 - z \quad \{0.0435, 0, 0, -0.0435\} \quad (0.5i, 0) \quad (14)$$

$$\ddot{z} = z^2 - z^* \quad \{0.0053, 0, 0, -0.0053\} \quad (0.3 + 0.1i, 0.6i) \quad (15)$$

$$\ddot{z} = z^* - zz^* \quad \{0.0039, 0, 0, -0.0039\} \quad (0.85 + 0.11i, 0.03 - 0.6i) \quad (16)$$

The three phase space trajectories are displayed in Fig. 1. System (14), in the form

$$\begin{aligned} \ddot{x} &= x^2 - y^2 - x \\ \ddot{y} &= -2xy - y \end{aligned} \quad (17)$$

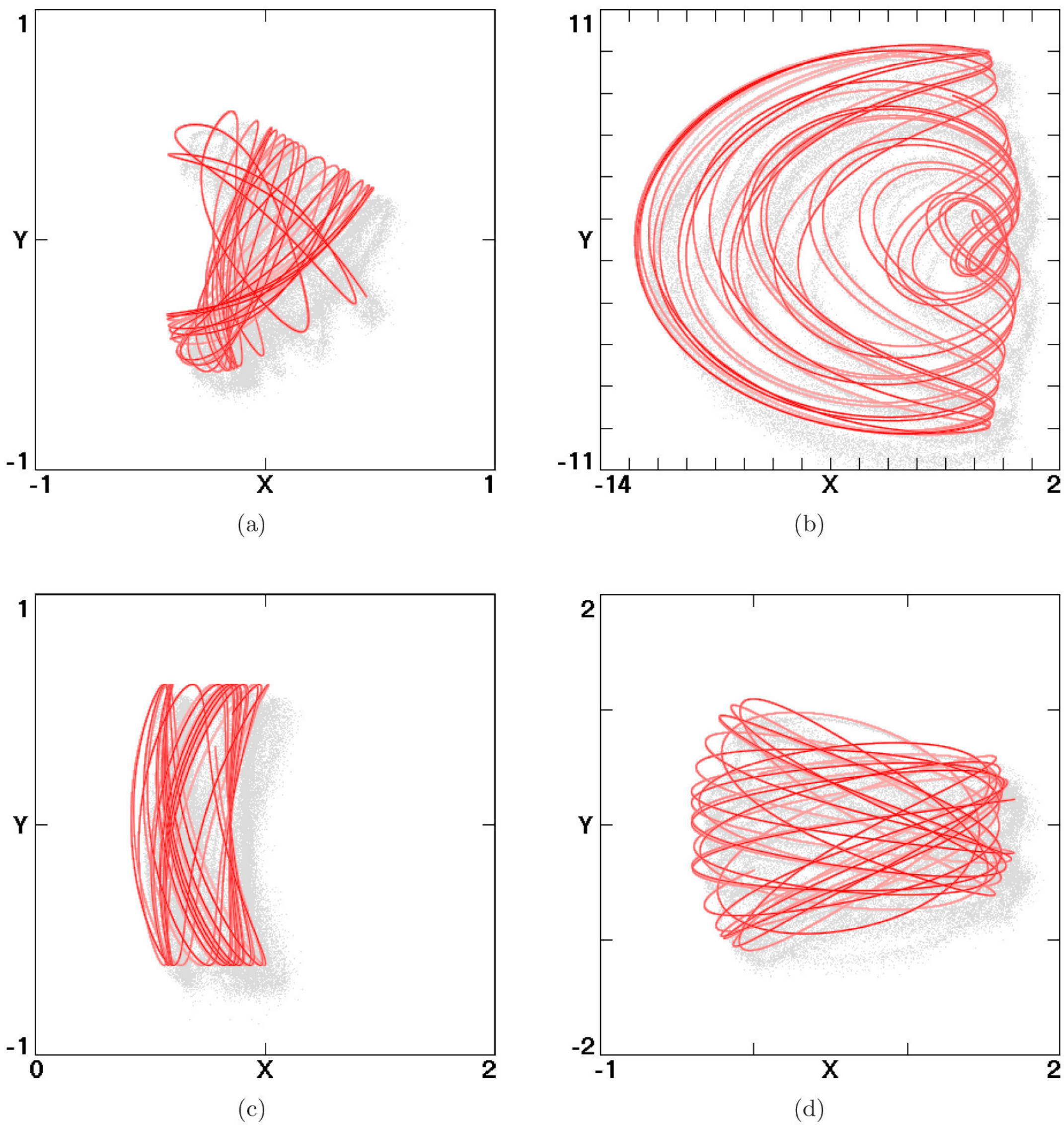

Fig. 1. State space plots: (a) System (14); (b) system (15); (c) system (16); (d) system (20).

may look familiar, because it is the well-known Hénon–Heiles [1964] system, but with x and y reversed. As such, it has a Hamiltonian,

$$H = \frac{(\dot{x}^2 + \dot{y}^2 + x^2 + y^2)}{2} + xy^2 - \frac{x^3}{3}. \tag{18}$$

System (15) can be written

$$\begin{aligned} \ddot{x} &= x^2 - y^2 - x \\ \ddot{y} &= 2xy + y \end{aligned} \tag{19}$$

which is similar to (17), but the sign change in the second equation destroys the possibility of a Hamiltonian. System (16) also lacks a Hamiltonian.

4. Cubic Systems

The five simplest cubic polynomial $\ddot{z} = f(z, z^*)$ systems found were, with Lyapunov spectra in braces, and starting from initial conditions $(z_0, \dot{z}_0)$,

$$\ddot{z} = 1 - z^2 z^* \qquad \{0.0359, 0, 0, -0.0359\} \qquad (-0.4 + 0.2i, -0.3) \tag{20}$$

$$\ddot{z} = z^3 - z^* \qquad \{0.0285, 0, 0, -0.0285\} \qquad (-0.11 - 0.6i, -0.24 + 1.07i) \tag{21}$$

$$\ddot{z} = z^2 - z^2 z^* \qquad \{0.0804, 0, 0, -0.0804\} \qquad (0.5 + 0.5i, 0) \tag{22}$$

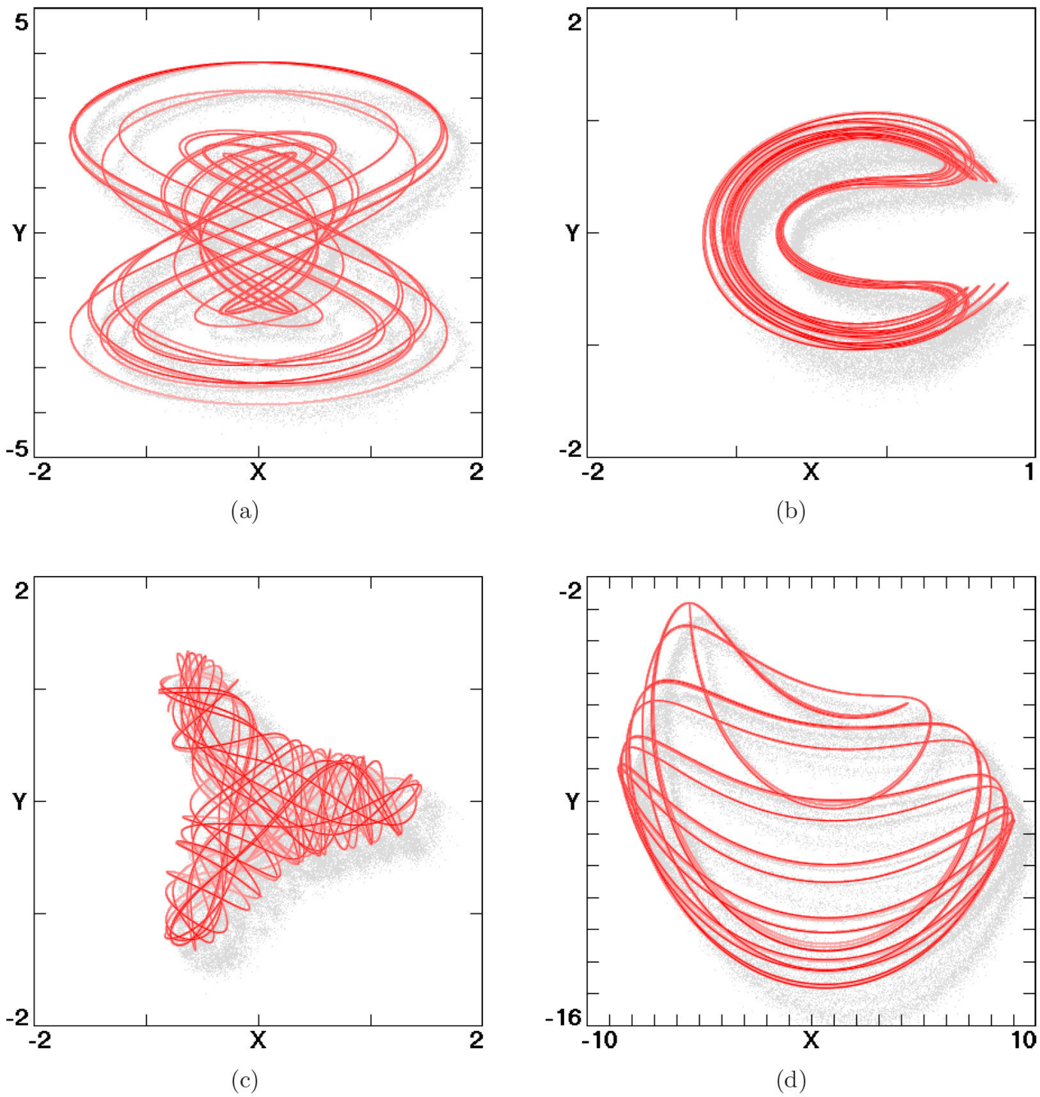

Fig. 2. State space plots: (a) System (21); (b) system (22); (c) system (23); (d) system (24).

$$\ddot{z} = (z^*)^2 - z^2 z^* \qquad \{0.2479, 0, 0, -0.2479\} \qquad (0.5 + 0.5i, 0) \tag{23}$$

$$\ddot{z} = z^3 + zz^* \qquad \{0.0070, 0, 0, -0.0070\} \qquad (-3.79 - 1.72i, 0.89 + 0.37i) \tag{24}$$

The phase space trajectory for system (20) is shown in Fig. 1; those for systems (21)–(24) are shown in Fig. 2. System (20) shares the z^2z^* term with systems (22) and (23). This term is also a constituent of various complex Duffing oscillators studied in [Cveticanin, 2001; Mahmoud & Bountis, 2004; Mahmoud *et al.*, 2001]. System (20), when written in terms of real and imaginary parts x and y, has the Hamiltonian

$$H = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + \frac{(x^4 + y^4)}{4} + \frac{x^2y^2}{2} - x, \tag{25}$$

while systems (21) and (22) have no Hamiltonian.

System (23) has the Hamiltonian

$$H = \frac{(\dot{x}^2 + \dot{y}^2)}{2} + \frac{(x^4 + y^4)}{4} + \frac{x^2y^2}{2} - \frac{x^3}{3} + xy^2, \tag{26}$$

but system (24) does not have a Hamiltonian.

5. Summary

It was shown that, if f is an analytic function, chaos cannot occur in systems of forms $\ddot{z} = f(z)$ and $\ddot{z} = f(z, \dot{z})$, and systems $\ddot{z} = f(z, z^*)$ and $\ddot{z} = f(z^*, \dot{z}^*)$ are conservative. Search for chaotic systems of form $\ddot{z} = f(z^*)$ and $\ddot{z} = f(z^*, \dot{z}^*)$, with quadratic and cubic polynomials f, yielded only systems that diverged. A search for simple chaotic systems of the form $\ddot{z} = f(z, z^*)$, with quadratic and cubic polynomials f, found eight such systems. For each chaotic system, the Lyapunov spectrum was calculated, and the phase space trajectory was displayed. For each chaotic system that has a Hamiltonian, the Hamiltonian was given. Similarities to previously studied systems were noted.

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