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Contact Bifurcations in Two-Dimensional Endomorphisms Related with Homoclinic or Heteroclinc Orbits

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Abstract: In this paper we show the homoclinic bifurcations which are involved in some contact bifurcations of basins of attraction in noninvertible two-dimensional map. That is, we are interested in the link between contact bifurcations of a chaotic area and homoclinic bifurcations of a saddle point or of an expanding fixed point located on the boundary of the basin of attraction of the chaotic area. We shall analyze the particular case of a map having up to three distinct preimages, and the basins's bifurcations are investigated by use of the technique of critical curves.

Keywords: homoclinic points; critical curves; bifurcations in endomorphisms

1 Introduction

Since the celebrated works by Smale [28],[29] it is well known that the existence of homoclinic orbits of a saddle fixed point or cycle is related with the existence of complex dynamics of a system (see also the book by Devaney [5], or [13] or any other text on Dynamical Systems). Really this complex dynamic behavior had been already understood several years before by Poincaré, and the first rigorous proofs are already given by Birkhoff in [3]. The existence of an infinity of periodic points in any neighborhood of an homoclinic orbit was proved by [3] and [27], and supplementary properties of invariant chaotic sets may be found in [24]. Thus we recall that in any neighborhood of a transverse homoclinic orbit of a saddle fixed point of a diffeomorphism T (as for a $k-$cycle we refer to the fixed points of the $k-th$ iterate of the map T) there exists an invariant Cantor set Λ on which the map, or a suitable iterated T^m, is topologically conjugated to the shift map σ on the set Σ_2 of the one-sided infinite sequences on two symbols ($\Sigma_2 = \left\{a = \{.a_i\}_{i=-\infty}^{+\infty};\ a_i = 0\ or\ 1\right\}$ and $\sigma(.a_0a_1a_2...) = (.a_1a_2...)$).

In the case of two-dimensional endomorphisms, that is the object of this work, the existence of homoclinic orbit also plays a fundamental role in the complex dynamic behavior. A first remark is that homoclinic orbits may be associated not only to saddle fixed points or cycles, but also to expanding fixed points (as unstable nodes or foci). This was proved for the first time in

Marotto [18], where it was shown that chaotic dynamics in the sense of Li &Yorke [16] occur in any neighborhood of an expanding fixed point having an homoclinic orbit (a so called *snap-back repellor*). This results was extended in Gardini [10], where it was shown that the first bifurcation value leading an expanding fixed point to become a *snap-back repellor* is associated with the critical curves of the endomorphism. The results have been also reconsidered by the same authors (see [19], [26], [12]).

In the case of transverse homoclinic orbits a saddle fixed point in noninvertible maps, [15] proved that results of Smale recalled above, are valid also for an endomorphism defined by continuously differentiable functions. And the existence of Cantor sets and complex dynamics associated with saddle cycles having transverse homoclinic orbits has been shown to exist also in piecewise smooth systems in examples considered in [11] and [17].

In this work we shall consider another example, showing how relevant is the connection between the contact bifurcations involving the basins of attraction and the invariant attracting chaotic sets. We shall consider in particular the

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homoclinic bifurcation of a saddle cycle belonging to the boundary of the basin of attraction of a chaotic area in a two-dimensional endomorphisms of so-called type $Z_1 - Z_3 - Z_1$ (i.e. whose points have either one or three different rank-1 preimages, depending on some regions of the phase space). We will show, via a numerical study, that at bifurcation value the points of contact between the boundaries of the attractor and its basin converge toward the saddle point. These points of contact are also intersection points between the stable and unstable manifolds of the saddle point, and after the bifurcation value the stable and unstable sets of the saddle have transverse homoclinic points.

Another important mathematical tool used to study the bifurcations which take place in invariant areas of two-dimensional endomorphisms, either for invariant absorbing areas or chaotic areas, is the notion of critical curves, first introduced by Mira in 1964, see [14] and references therein. This tool provides an excellent instrument for the comprehension of many bifurcations occurring in noninvertible maps. This notion is the natural generalization to $\mathbb{R}^2$ (and more generally to $\mathbb{R}^n$ for $n \geq 2$) of the notion of critical points of one-dimensional endomorphisms. Following GM 80 we define the critical curve LC of an endomorphism T in the plane $\mathbb{R}^2$ as the geometrical locus of points x having at least two coincident rank-1 preimages in a set of points denoted as LC_{-1} (so that $T(LC_{-1}) = LC$). When T is differentiable, the locus LC_{-1} is also associated with the Jacobian of T as it belongs to the set of points in which the Jacobian of T vanishes: let $J = \det(DT(x, y)) = 0$ then $LC_{-1} \subseteq J$. A critical curve LC may be constituted of one or several branches, which separate the plane in open regions Z_i. All the points of a region Z_i have the same number i of distinct rank-1 preimages (see also [21], [23]).

Since the pioneering works by Gumowsky and Mira [14], several papers have shown the importance of the critical curves in the bifurcations associated with invariant chaotic areas in noninvertible maps. In [2], [1], [9], [22], [7], it is shown that such contact bifurcations lead either to the chaotic area destruction, or to a sudden modification of the area. Moreover, the critical curves are involved in the bifurcations of the structure of the basins of attraction (of attracting sets of any type), we refer to the classification performed in [21] (see also [23]), and examples will be also shown also in the map considered in this work.

The content of the paper, besides this introduction, consists in a survey of some definitions and properties performed in Section 2, then in Section 3 we introduce the map which will be used to illustrate several kind of bifurcations. In particular, we put in evidence the transition of the basin from a smooth boundary to a fractal boundary with a repelling node on the frontier, or the transition from a smooth boundary to a fractal boundary with a repelling focus on the frontier of the basin, and the last situation will show the transition from a chaotic attractor to a chaotic repellor, via a homoclinic bifurcation of a saddle fixed point on the frontier of the basin.

2 Definitions and properties

In this section, we recall the main definitions and properties associated with some technical tools as absorbing area, chaotic area, as well as contact and homoclinic bifurcations. We consider an endomorphism T which defines a discrete dynamical system in $\mathbb{R}^2$:

$$(x_{n+1},\ y_{n+1}) = T(x_n, y_n) = (f(x_n, y_n; \lambda), g(x_n, y_n; \lambda))$$

where $f(x, y, \lambda)$ et $g(x, y, \lambda)$ are continuous and differentiable functions with respect to the real variables x, y and continuous with respect to the real parameter λ.
Let $J(x, y) = \det(DT(x, y))$ and let us consider the set $J(x, y) = 0$, then the critical set LC is the geometrical locus of points x having at least two coincident rank-1 preimages in points of $LC_{-1} \subseteq J$ $(T(LC_{-1}) = LC)$.
Definition 2.1: An absorbing area E is a closed and bounded set of the phase space such that:

(i) $T(E) \subseteq E$;

(ii) its frontier, ∂E is made up of a finite number of critical arcs of LC, LC_1, LC_2,...., LC_k, such that $LC = T(LC_{-1})$, $LC_i = T^i(LC)$ for $i \geq 1$;

(iii) A neighborhood $U(E)$ exists whose points have an images of finite rank in the interior of E (i.e. all the points of U are mapped into E after a finite number of iterations and cannot escape).

Notice that in the definition given above the number of critical arcs on the boundary of E is assumed finite in number because the case in which the number of arcs becomes infinite the map is at a bifurcation value (an example will be given also in the next section), and at the bifurcation value in the neighborhood $U(E)$ there exist also points which are not mapped in the interior of E, while they are convergent to the frontier of E.
Remark 1. An absorbing area may contain one or several attracting sets.

An absorbing area E is called invariant iff $T(E) = E$. When an absorbing area E is not invariant (but $T(E) \subset E$) then the intersection $\cap_{n>0} T^n(E)$ (*n* finite or infinite) is an invariant absorbing area. When an absorbing area E is invariant

$(T(E) = E)$ then the frontier ∂E of E is detected by a finite number of critical arcs which are all belonging to the images of a particular segment of critical curve belonging to E (see [23]). That is, let $\gamma = E \cap LC_{-1}$, then there exists a finite integer *m* such that:

$$\partial E \subset \cup_{k=1}^{m} T^k(\gamma)$$

When a finite number *m* such that the boundary ∂E of E is included in $\cup_{k=1}^{m} T^k(\gamma)$ does not exist then the invariant absorbing area E is at a bifurcation.

Only invariant absorbing areas are useful considering contact bifurcations between the absorbing area boundaries and the boundaries of their attraction basins (as non invariant absorbing areas do not undergo bifurcations when involved in some contact, and are of no consequence for the invariant attracting set which they include).

Definition 2.2: A chaotic area A is an invariant absorbing area $(T(A) = A)$), the points of which give rise to iterated sequences having the property of sensitivity to initial conditions.

About chaotic areas, it is important to emphasize that the study of such area may be related with the transition to fractal basin boundaries.

Remark 2. In [23] it is also introduced the notions of mixed absorbing areas and mixed chaotic areas. The difference between these last notions and the one given above is that in the boundaries of a mixed absorbing area there exist also segments of the unstable set of some saddle cycles (for more details see also [25]). However, the contact bifurcations which are of interest in this work are related with absorbing areas of not mixed type.

Definition 2.3: We say that at $\lambda = \lambda^*$ a contact bifurcation of E occurs if there is a contact between the boundary of E and its basin.

Proposition 2.1: *When a contact bifurcation of a chaotic area A occurs at $\lambda = \lambda^*$, then crossing this value leads either to the destruction of A, or to a qualitative modification of properties of A (i.e. a sudden modification of the size of such an area).*

The destruction of A, after the crossing of a contact bifurcation value has been shown by [14], and these bifurcations have been considered later in 96. The disappearance of a chaotic area leads to the existence of a strange repellor constituted of an infinity of unstable cycles, their stable sets and the arborescent sequences of their images: all invariant sets which, before the bifurcation, were included in the area A. The existence of such an invariant set, chaotic repellor, gives rise to chaotic transient in the trajectories of initial conditions belonging to the old invariant area, which are then converging towards some different attractor, at finite or infinite distance.

Among the qualitative change of properties of a chaotic area A after the crossing of a value of contact bifurcation we also mention the transformation of an annular chaotic area A (connected area with a hole), either in a simply connected area or in a non connected cyclic area of period k, which occurs when the value λ^* corresponds to a contact between the boundary of A and an isolated unstable focus belonging to the boundary of its basin of attraction, as described in [2], [10], [23].

Now let us turn to the homoclinic orbits, both of saddle cycles and of expanding cycles. For the shake of simplicity we shall consider only a fixed point of T (as for a $k-$cycle we refer to the fixed points of the map T^k).

Definition 2.4: Let S be a saddle fixed point of T, $W^s(S)$ and $W^u(S)$ denoting its stable and unstable sets. A point q is called homoclinic to S if $q \in W^s(S) \cap W^u(S)$ and $q \neq S$. q is a transversal homoclinic point if $W^s(S)$ intersects transversely $W^u(S)$.

An homoclinic orbit $O_o(q)$ is the set of points given by the forward images of the point q (belonging to $W^s(S)$) and the sequence of its preimages which belong to $W^u(S)$ (which necessarily exist): $O_o(q) = \{....q_{-2}, q_{-1}, q, T^n(q); n > 0\} = \{...., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$, where $q_n = T^n(q) \to S$, and $q_{-n} \in T^{-n}(q) \to S$.

It is worth noticing that when the map is noninvertible, the stable set of the saddle fixed point is not necessarily a connected set. In fact, when the fixed point has a rank-1 preimage different from itself, it may occur that the stable set of a saddle is a disconnected set.

When a map is invertible, a fixed point of type repelling node or focus cannot have homoclinic points, while this is not the case when the map is noninvertible, due to the possible existence of a stable set different from the point itself. When such an expanding fixed point has homoclinic points it is called a snap-back repellor by Marotto [18]. A repelling node or focus P is such that a neighborhood $U(P)$ of P exists in which the Jacobian matrix has both the eigenvalues higher than 1 in absolute value, and we say that T is expanding in U, and we denote by T_l^{-1} the local inverse map of T in $U(P)$, such that $T_l^{-1}(P) = P$.

Definition 2.5: A repelling node or focus P is called snap-back repellor if a point q exists in a neighborhood $U(P)$ in which T is expanding such that $T^m(q) = P$.

As clearly also a sequence of preimages of q exists which is convergent to P, we have realized an homoclinic orbit. That is: an homoclinic orbit $O_o(q)$ associated with an homoclinic point q of an expanding fixed point P is the set constituted of successive iterates of q, and an infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(P)$: $O_o(q) = \{T_l^{-n}(q), q, T^n(q); n > 0\} = \{...., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$, where $q_n = T^n(q) \to S$, and $q_{-n} = T_l^{-n}(q) \to S$.

Similarly we can define heteroclinic orbits connecting unstable fixed points or cycles of T, which are points whose images are convergent to one cycle, say Q, and a sequence of preimages exists converging to another cycle, say P. Without loss of generality we can consider a point of this heteroclinic trajectory belonging to a neighborhood $U(P)$ and we denote by T_l^{-1} the local inverse map of T in $U(P)$ (satisfying $T_l^{-1}(P) = P$). Then

Definition 2.6: An heteroclinic orbit $O_e(q)$ connecting a cycle P to a cycle Q associated with q is the set of points given by q together with its images (finite or infinite in number) convergent to Q and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(P)$. $O_e(q) = \{T_l^{-n}(q), q, T^n(q); n > 0\} = \{...., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$ where $q_n = T^n(q) \to Q$, and $q_{-n} = T_l^{-n}(q) \to P$.

Definition 2.7: Let T be an endomorphism of $\mathbb{R}^2$ depending on a parameter λ and let S (resp. P) be a saddle (resp. expanding) fixed point of T. We say that a homoclinic bifurcation occurs at λ^* if crossing $\lambda = \lambda^*$ there is the appearance/disappearance of an infinitely many homoclinic orbits.

As already remarked, the relevance of the notions given above is that whenever an homoclinic orbits exists, it can be proved the existence of invariant sets with chaotic dynamics. For expanding cycles this was proved for the first time by Marotto. In [18] it is proved that if q is a non degenerate homoclinic point of an unstable node or focus P then in any neighborhood of the homoclinic orbit of P there is chaos in the sense of Li and Yorke (which include an infinity of unstable cycles of T).

This role is also associated with heteroclinic orbits when they are in pair, that is, when an heteroclinic orbit $O_e(q)$ exists connecting a cycle P to a cycle Q and another heteroclinic orbit $O_e(r)$ exists connecting the cycle Q to the cycle R. Thus the intersections of stable and unstable manifolds of cycles are the simplest tool to detect homoclinic orbits and thus chaotic behavior.

When a fixed point or cycle of T is homoclinic then there exist infinitely many homoclinic orbits associated to it. When dealing with homoclinic and heteroclinic orbits in noninvertible maps their existence is associated with points belonging to both sides of the set LC_{-1}. A homoclinic or heteroclinic orbit is called *degenerate,* or *critical,* if it contains one point of LC_{-1} and nondegenerate otherwise. The *first homoclinic bifurcation* is the one leading a fixed point from non homoclinic to homoclinic (and vice versa). In general, after the first homoclinic bifurcation, several other homoclinic bifurcations (or explosions) may occur, leading to more and more homoclinic trajectories. The result of Marotto has been extended by Gardini [10] showing that at the first homoclinic bifurcation of an expanding fixed point all the homoclinic orbits are critical. Moreover, any homoclinic bifurcation is associated with critical homoclinic orbits, which are followed by an explosion of new noncritical homoclinic orbits after the bifurcation. This is because critical points in the priemages of a fixed point are those related with news branches in the sequences of preimages, and thus we can say that a homoclinic bifurcation represents an explosion of the global stable set $W^s(S)$. An explosion of homoclinic orbits of a cycle is always associated with an explosion of unstable cycles in the chaotic sets, and these infinity of cycles must have been created via sequences of flip and fold bifurcations. Thus these homoclinic bifurcation values, are probably also accumulation points of simple (fold and flip) bifurcation values.

A relevant homoclinic bifurcation of an expanding fixed point is that associated with annular chaotic areas, as shown in [10]. In the case of a saddle fixed point S on the boundary of a basin of attraction D of chaotic area d, a contact bifurcation such that $T^n(h_0) \longrightarrow S$, when $h_0 \in \partial d \cap \partial D(d)$ at the bifurcation value, can correspond to the birth of homoclinic orbits of S. It is a conjecture, that has been verified in [9], for dynamical systems defined by piecewise linear maps, and also in [17]. Other examples of the appearance of saddle homoclinic orbits which occurs due to contact bifurcations of chaotic areas are given in [7]. Before the contact bifurcation we are sure that homoclinic orbits of a saddle fixed point S belonging to the boundary of the basin of a chaotic area do not exist, at least inside the basin (while outside can exist). Indeed, this occurs whenever the branch of the unstable set of S^* which enters the basin is converging to the chaotic area, so that before the bifurcation the invariant chaotic area has no intersection with the boundary of the basin, which means that such a branch of the unstable set $W^u(S)$ has no intersection with the stable set $W^s(S)$ (belonging to the basin's boundary). While a contact of the invariant chaotic area with the basin's boundary in the stable set of the saddle leads to a contact between $W^u(S)$ and $W^s(S)$, and thus to homoclinic points.

2.1 Properties of $W^s(S)$ and $W^u(S)$

In the case of an invertible map, the stable invariant manifold $W^s(S)$ of a saddle point S is always connected. This is not necessarily true for a noninvertible map. For a saddle cycle S belonging to the boundary of some basin of attraction of some attractor A, its stable set $W^s(S)$ belongs entirely to the frontier of the basin. When $W^s(S)$ has a contact with the critical line LC, $W^s(S)$ becomes non connected and its connected components may be finite in number or infinitely many. These contact bifurcations are those associated with the changes in the structure of the basins of attraction, for example changing a connected basin into a multi connected one or into a disconnected one, as shown in [21] and [23].

When an unstable invariant manifold $W^u(S)$ has a contact with the critical curve LC, a bifurcation leading to the appearance of self-intersections of this manifold may occur. For example, These self-intersections are responsible of the transformation of a closed invariant curve Γ into a chaotic attractor, as shown in [20] and [23]. Indeed, if an arc is mapped into an arc with a point of self-intersection, this point is a point of non differentiability, it is a point having at least two distinct rank-one preimages, and thus it must belong to the critical set. By successive iterations of T the invariant set must include an infinity of points of self-intersections, and thus of points of non differentiability, which means that Γ becomes a fractal set.

3 Example of a cubic recurrence having chaotic attractors and fractal basin

In this paper we consider a dynamical system generated by a family of two-dimensional continuous noninvertible maps T defined by

$$\begin{cases} x_{n+1} = x_n^3 + ax_n + b + y_n \\ y_{n+1} = cx_n + dy_n \end{cases}$$

where a, b, c, d are real parameters. The critical curves are here obtained as locus $J(x, y) = \det(DT(x, y)) = 0$ and are two straight lines in the phase space, LC_{-1} and LC'_{-1}, given by:

$$\left(LC_{-1}:\ x = +\sqrt{\frac{c}{3d} - \frac{a+1}{3}}\ ,\ LC'_{-1}:\ x = -\sqrt{\frac{c}{3d} - \frac{a+1}{3}} \right)$$

and their images LC and $LC^{'}$ are also straight lines given by:

$$\begin{array}{l} LC:\ \ y = dx - dx_0^3 - (ad + d - c)x_0 - db \\ LC^{'}:\ \ y = dx + dx_0^3 + (ad + d - c)x_0 - db \end{array} \quad \text{where} \quad x_0 = \sqrt{\frac{c}{3d} - \frac{a+1}{3}}$$

this endomorphism is of type $Z_1 - Z_3 - Z_1$ as the critical curves LC and LC' separate the phase plane in three regions whose points have 1-3-1 different rank-1 preimages. Depending on the values of the parameters, it is possible to observe several kinds of routes to complex dynamics (via sequences of local bifurcations and global bifurcations, typically homoclinic bifurcations). Here we are interested in showing some dynamic behaviors which cannot be studied by local methods (based on linear approximations around the attractors) but only through a global study of the map, often requiring an interplay among analytical, geometric and numerical methods.

Our map is not invertible and we shall see that the use of the method of critical curves LC_i in order to explain the contact bifurcations which cause the formation of non simply connected basins, and the homoclinic bifurcations leading to fractal boundaries. For maps of dimension higher than one, the methods followed in the determination of the contact bifurcations are based on a systematic computer assisted study, carried out through a continuous dialog between analytic, geometric, and numerical methods, which often require a careful use of computer graphics. Clearly, the detection of contacts among these objects (critical curves, basin boundaries, attractors) as their shapes change may become a very difficult task in maps in dimensions higher than 2. However, as we shall see also in this example, it is a powerful tool in the case of two-dimensional maps.

First we shall show the transition of the basin from a smooth boundary to a fractal boundary with a repelling node on the frontier, then the transition from a smooth boundary to a fractal boundary with a repelling focus on the frontier of the basin, and the third situation will show the transition from a chaotic attractor to a chaotic repellor, via a homoclinic bifurcation of a saddle fixed point on the frontier of the basin.

3.1 Fractal boundary of the immediate basin of attraction

Let us fix the values of the parameters a, b, c and we vary the parameter d. For the values $a = -0.89$, $b = 1$, $c = -1$ and d decreasing from -1.15 to -1.5, we shall see a route from simply connected basic to mult connected and then simply connected again but with fractal boundary.

For $d = -1.15$, as shown in Fig.1, we have a cyclic chaotic attractor of period 3. This 3-pieces chaotic attractor A is the results of a cascade of flip bifurcations of cycles of order 3.2^i, $i = 0, 1, 2...$ (i.e. the standard Feigenbaum route starting from a 3-cycle saddle). Its immediate basin $D_0(A)$ is simply connected, while the total basin $D(A)$ is non connected.

The immediate basin $D_0(A)$ is simply connected as long as the set $D_0 \cap LC$ is connected. The contact bifurcation of the boundary of the immediate basin with the critical curve LC, which occurs as d decreases, will lead to a multiply connected immediate basin (as described in [21] and [23]). This can be seen in Fig.2 at $d = -1.25$. The attracting set is still a 3-pieces cyclic chaotic attractor, and its immediate basin of attraction D_0 is multiply connected, i.e. connected with holes (or lakes) H_i, as $D_0 \cap LC$ is non connected. The lakes H_i are preimages of H_0, which is numerically visible from the value $d = -1.151$. This means that we have a bifurcation value “simply connected $\leftrightarrow$ multiply connected” of the immediate basin $D_0(A)$ between the values $d = -1.150$ and $d = -1.151$. As the parameter d is further decreased, the

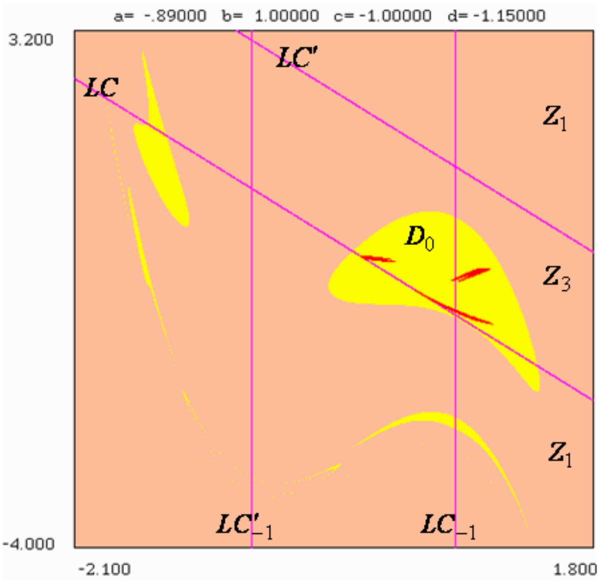

Figure 1: The immediate basin D_0 simply connected

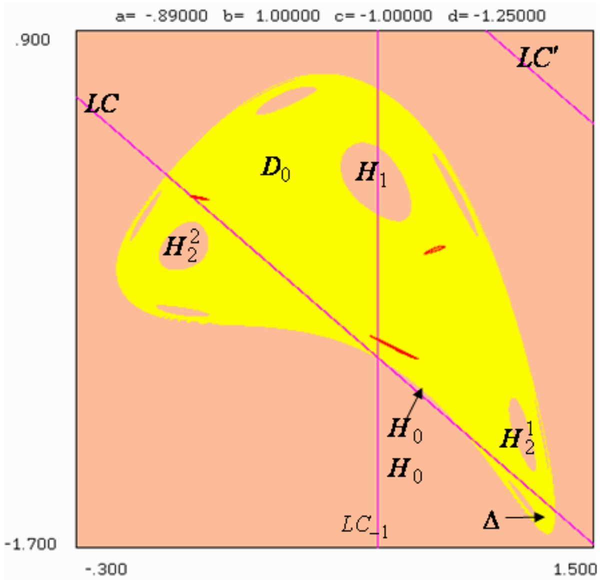

Figure 2: D_0 multiply connected

lakes become closer to the external boundary of D_0 (see Fig.3). The dynamic behavior of these lakes is associated with the structure of the set $D_0 \cap LC$: starting from the two-segments existing in the case shown in Fig.2 this set tends to become again of one unique piece, as it is shown in Fig.4, so that the immediate basin $D_0(A)$ from multiply connected will return again simply connected (as in Fig.4). But now with a boundary which is not smooth (as it was in Fig.1), as infinitely many points of non smoothness exist in the boundary. In Fig.4 it is possible to see that the lakes of the immediate basin changed in bays. This bifurcation is characterized from the fact that the cape Δ crosses LC (at a value of d between the values $d = -1.335$ and $d = -1.336$), and then $D_0 \cap LC$ becomes connected again. The end points of the bays have now a particular shape (see the enlargement shown in Fig.5), and are associated with the arborescent sequence of preimages of a $2-$cycle unstable node belonging to the boundary of D_0. The transition of the boundary from smooth to non smooth occurs soon after this last contact bifurcation and it is associated with the local nature on this 2-cycle on the boundary. In fact let us remark that in invertible maps an unstable cycle *node* cannot belong to basin boundary, which this is possible in the case of noninvertible maps, and this occurrence leads to points of non smoothness on the boundary. In our case, the eigenvalues S_1 and S_2 of this 2-cycle are of opposite signs, $S_1 < 0$, $S_2 > 1$ and $|S_1| < S_2$ for $d > -1.36522$. For $d = -1.36522$ we get $|S_1| = S_2$ and for $d < -1.36522$ we obtain $|S_1| > S_2$. This leads to a repelling node of second kind, which belongs to the frontier, together with all its preimages of any rank, and as the frontier has a cusp point (point of non differentiability) in this 2-cycle, it follows that the frontier includes infinitely many points of non smoothness. The restriction of the map to the frontier of the basin may however be chaotic or not. This depends on the existence of homoclinic orbits for the cycles of this restriction.

3.2 Heteroclinic orbits

Let us fix the values $a = -1$, $c = -1.088$, $d = -0.46$ and decrease b. In Fig.6 we show the situation when $b = 1.004$, we have a 4-pieces cyclic chaotic attractor A around an unstable focus point O. The immediate basin of attraction $D_0(A)$ of the chaotic attractor is multiply connected and its boundary is given by the stable set $W^s(S)$ of a saddle fixed point S. Notice that due to the existence of ”holes” the stable set $W^s(S)$ is non connected and its connected component containing S is the external boundary of D_0. The other connected components are the boundaries of holes (or lakes) H_{-i},

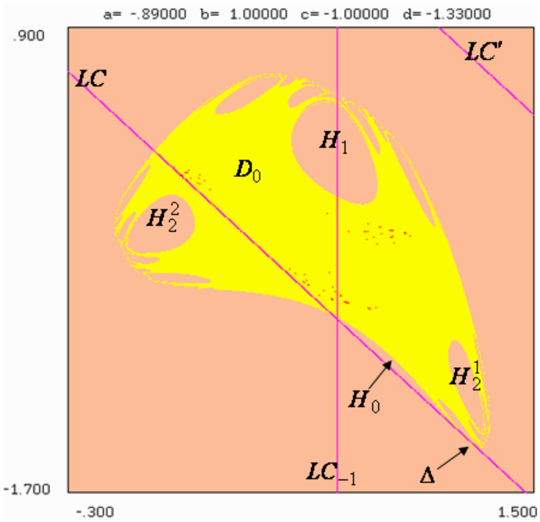

Figure 3: D_0 multiply connected

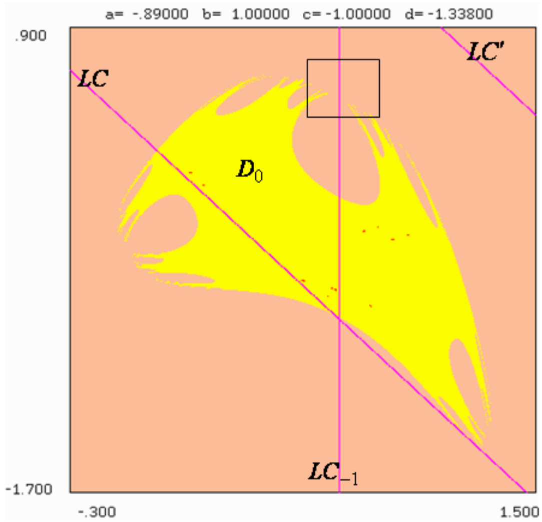

Figure 4: D_0 simply connected after bifurcation

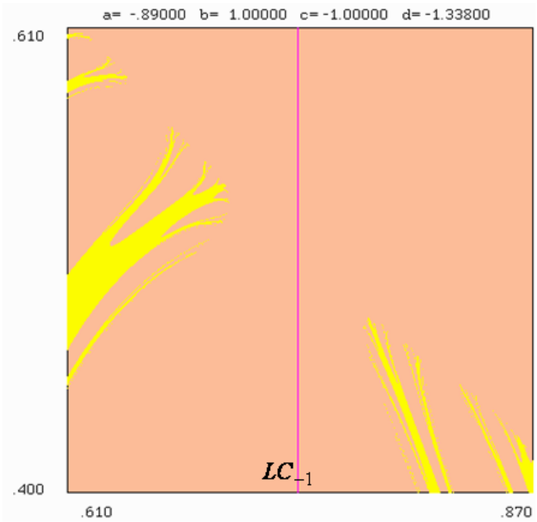

Figure 5: Blow up of the little box chosen in the Fig.4

$i \geq 1$, which are preimages of the bay H_0 ($H_{-1} = T_1^{-1}(H_0) \cup T_2^{-1}(H_0)$, $H_{-i} = T_2^{-i}(H_0)$, $i = 2, 3$). The critical lines $LC_i = T^{i+1}(LC_{-1})$, $i = 0, 1, 2$, can also be used to detect an absorbing area Δ containing the chaotic attractor.

Notice that in Fig.6 the hole H_{-3} is close to, but below, the critical curve LC. As we decrease the value of the parameter b, we shall determine an explosion of the preimages giving the holes, and the bifurcation occurs when the hole H_{-3} has a contact with the critical curve LC (the bifurcation value is $b^* \simeq 1.00305$) and in the case shown in Fig.7, at $b = 1.0024$, and in its enlargement in Fig.8, there are infinitely many holes which are approaching the unstable focus O. At the bifurcation value is $b^* \simeq 1.00305$ a contact occurs between H_{-3} and LC in a point $a_0 \in LC \equiv LC_0$. In this bifurcation value the stable manifold $W^s(S)$ is constitute of the boundaries $\partial_e D_0$, ∂H_{-i}, $i \geq 1$, and the preimages of the critical point a_0. The preimage $a_{-1} = T_2^{-1}(a_0) = T_3^{-1}(a_0) \in LC_{-1}$ is located in the absorbing region Δ, this will give rise to an infinite sequence of preimages of $a_{-1} : a_{-n} = T_2^{-(n+1)}(a_{-1})$, $n > 1$, which converge toward the unstable focus O. In other words, at the bifurcation value b^* there is the appearance of the first heteroclinic orbit connecting O to S. This heteroclinic orbit, constitute of a_{-1}, its images $T^n(a_{-1})$, $n > 0$, converging toward S, and its preimages a_{-n}, $n > 1$, converging toward O, is degenerate (or critical) since $a_{-1} \in LC_{-1}$. The birth of this critical heteroclinc orbit at the bifurcation value is followed by an explosion of (non critical) heteroclinic orbits for $b < b^*$. Indeed, for $b < b^*$, the hole H_{-3} crosses the critical curve LC and this crossing generates a new hole crossing LC_{-1} in a region having infinitely many preimages converging to O, giving rise to an infinity of holes U_{-n} which converge toward O. In Fig.9 we show the

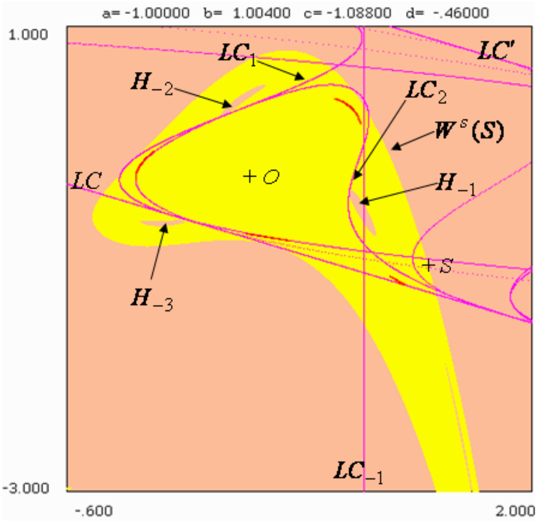

Figure 6: Before the heteroclinic bifurcation

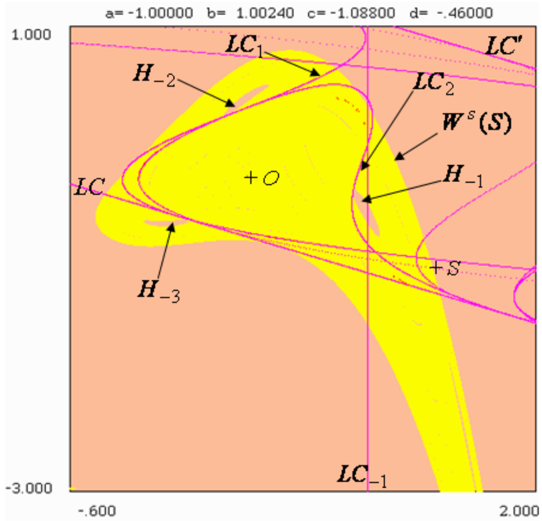

Figure 7: After the heteroclinic bifurcation

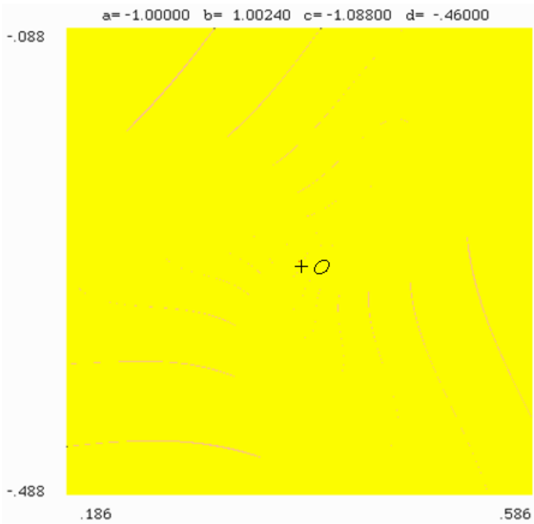

Figure 8: Enlargement in the neighborhood of the focus O

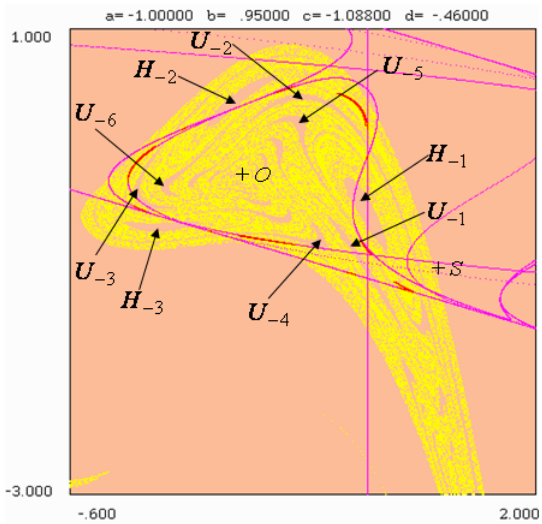

Figure 9: Infinity of heteroclinic orbits

situation far from the contact bifurcation, when the number of holes is increased very much. The stable set $W^s(S)$ is then constituted of frontiers $\partial_e D_0$, ∂H_{-i}, $i \geq 1$ and ∂U_{-n}, $n > 0$, and also new holes may be created by contact with LC and crossing (as we have shown above at $b = b^*$). Thus the boundary of the basin clearly includes infinitely many heteroclinic points belonging to different heteroclinic orbits connecting O to S; each orbit being formed of a point q belonging to some boundary in the region of the unstable focus O, together with its images and of its preimages by T_2^{-1}. We notice however that no chaotic behavior can be associated to such heteroclinic behavior up to now. This is possible only if there are also heteroclinic orbits connecting S to O, and on its turn this is possible only if the preimages of the rank-1 preimage of the focus point O belongs to the frontier.

3.3 Contact bifurcation and homoclinic bifurcation

Differently from the previous case, let us now fix the values $a = -1$, $b = 1$, $d = -0.9$ and decrease the parameter c: we shall describe the first homoclinic bifurcation of a saddle leading to the transition of a chaotic attractor into a chaotic repellor.

At $c = -0.92$ we have a chaotic attractor, shown in Fig.10, resulting from a cascade of flip bifurcations of cycles of period 3.2^i, $i = 0, 1, 2...$ (from a starting $3-$cycle). As c decreases there are more and more tongues of the chaotic area which are approaching the immediate basin boundary, as shown in Fig.11. Some extremity of such areas are denoted by h_i, $i = 1, 2....$ Such points belong to the chaotic area and converge toward the invariant sets, but more and more of them are approaching the boundary, as shown in Fig.12. The boundary of the basin includes a saddle S and its stable set $W^s(S)$, and the regions are alternated on both sides of the unstable set of S. The branch of $W^u(S)$ entering inside basin tends to the attractor so that the closure of this unstable set includes the whole chaotic area. As the tongues in the chaotic area increase, they also approach more and more this branch of unstable set $W^u(S)$, as shown in the enlargement in Fig.13, so that such arcs of unstable set of S approach the stable set of S.

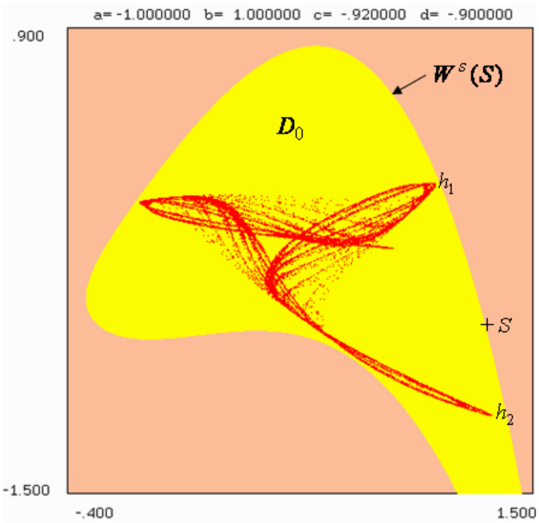

Figure 10: Chaotic attractor

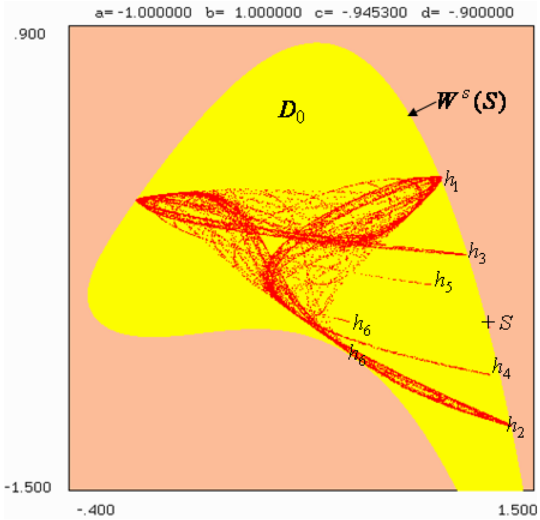

Figure 11: Tongues of the attractor which converge toward $W_s(S)$

It is clear that when the contact bifurcation of the chaotic area with its basin boundary occurs, we shall have the point h_1 on the stable set of S together with all its images, which are extrema of infinitely many tongues of the chaotic area approaching S and its unstable set. For $c = c_1 = -0.94643$ in Fig.12 the tongues' extremities h_i, $i = 1, 2...$, are close to the contact with the boundary of the basin (i.e. $W^s(S)$), and converge in the direction of $W^s(S)$ toward S alternating on each side (due to a negative stable eigenvalue). This values of c_1 is very close to the value of contact bifurcation since for $c_2 = -0.94644 < c_1$ the chaotic attractor no longer exists, it disappears as attractor and a chaotic repellor exists in its place. At this contact bifurcation the saddle S has its first homoclinic points, and we have a tangential contact between one branch of the unstable set (the one entering in the chaotic area) and both the branches of the stable set $W^s_{loc}(S)$ (due to the negative eigenvalue). After the contact bifurcation there will be infinitely many transverse intersections between the sets $W^u(S)$ and $W^s(S)$ and thus infinitely many homoclinic orbits of the saddle S, as shown in Fig.14 for $c = c_2$ Thus the chaotic repellor also includes the saddle S.

4 Conclusion

Numerical simulations are used to illustrate some contact bifurcation occurring in a noninvertible map of the plane, leading to strong changes in the shape of the basins of attraction or in the structure of the invariant attracting sets. The main object of our work has been that to emphasize the connection between contact bifurcations and homoclinic or heteroclinic orbits of cycles belonging to the basin boundary.

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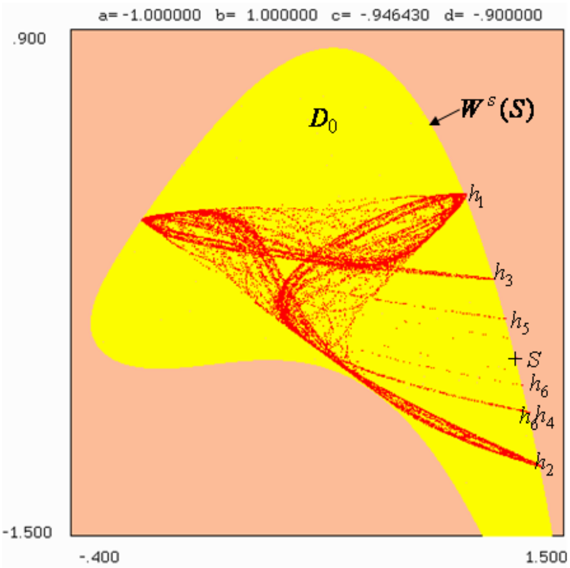

Figure 12: The h_i converge toward S

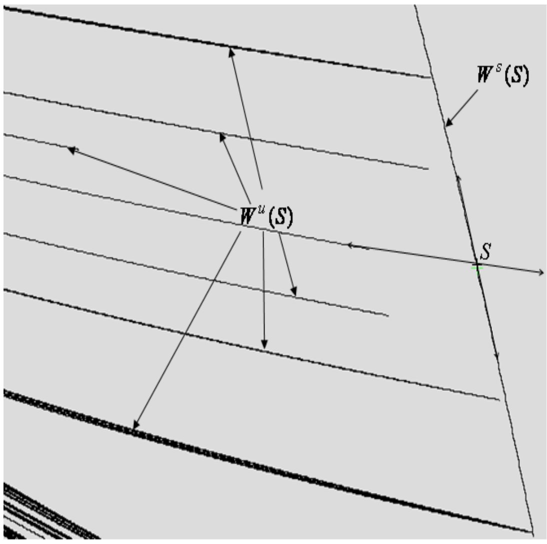

Figure 13: Before the homoclinic bifurcation

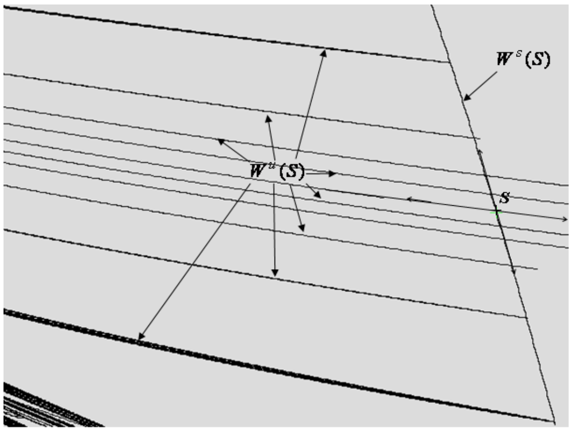

Figure 14: After the homoclinic bifurcation

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