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BIFURCATIONS AND CHAOS IN FRACTIONAL-ORDER SIMPLIFIED LORENZ SYSTEM

KEHUI SUN* and XIA WANG†
School of Physics Science and Technology, Central South University, Changsha 410083, P. R. China
**kehui@csu.edu.cn*
†wangxiacsu@163.com

J. C. SPROTT
Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA
sprott@physics.wisc.edu

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The dynamics of fractional-order systems have attracted increasing attention in recent years. In this paper, we numerically study the bifurcations and chaotic behaviors in the fractional-order simplified Lorenz system using the time-domain scheme. Chaos does exist in this system for a wide range of fractional orders, both less than and greater than three. Complex dynamics with interesting characteristics are presented by means of phase portraits, bifurcation diagrams and the largest Lyapunov exponent. Both the system parameter and the fractional order can be taken as bifurcation parameters, and the range of existing chaos is different for different parameters. The lowest order we found for this system to yield chaos is 2.62.

Keywords: Chaos; fractional-order calculus; simplified Lorenz system; time-domain methods; bifurcations.

1. Introduction

In recent years, many scientists have become aware of the potential use of chaotic dynamics in engineering applications, such as electrical engineering, information processing, secure communications, etc. With the development of the fractional-order calculus, fractional-order systems have received much attention. Typically, chaotic systems remain chaotic when their equations become fractional [Hartley *et al.*, 1995; Li & Chen, 2004a, 2004b; Ahmad & Sprott, 2003; Zhang & Yang, 2009; Chen *et al.*, 2008; Li & Peng, 2004; Wu *et al.*, 2008; Sheu *et al.*, 2008; Yu *et al.*, 2009; Tam & Tou, 2008; Mohammad & Mohammad, 2008].

Two methods have been mainly used in the previous literatures to solve fractional-order differential equations: frequency-domain methods [Sun *et al.*, 1984] and time-domain methods [Diethelm *et al.*, 2002; Deng, 2007a, 2007b]. The frequency-domain methods have been most frequently used to investigate chaos in fractional-order systems (used in [Hartley *et al.*, 1995; Li & Chen, 2004a, 2004b; Ahmad & Sprott, 2003; Zhang & Yang, 2009; Chen *et al.*, 2008]). Unfortunately, it has been shown that this approach is not always reliable for detecting chaos in such systems [Tavazoei & Haeri, 2008, 2007]. In light of the limitations of the frequency-domain approach, the numerical simulations of this paper are done using the time-domain approach. Our choice within this category is the improved version of the Adams–Bashforth–Moulton algorithm which is based on the predictor–corrector scheme. This method was introduced by Diethelm *et al.* [2002] and has been employed in [Li & Peng, 2004; Wu *et al.*, 2008; Sheu *et al.*, 2008; Yu *et al.*, 2009; Tam & Tou, 2008; Mohammad & Mohammad, 2008]. In addition, Hartley *et al.* [1995] studied the effects of fractional dynamics in Chua's systems by varying the total system order incrementally from 2.6 to 3.7, which demonstrated that systems of "order" more than three can exhibit chaos as well as other nonlinear behavior. However, considering the undesirable effects of using the frequency-domain approximation, the conclusions from that paper are doubtful. To the best of our knowledge, that was the first report concerning fractional-order systems with a total order more than three. In this paper, we report the first investigation of a fractional-order system with an order greater than three using the time-domain method. It will be shown from bifurcation diagrams of the system that chaos exists when the total order is 3.3.

Recently, a simplified Lorenz chaotic system was reported by Sun and Sprott [2009]. It has the same features as the Lü system, but a simpler algebraic form. In particular, the adjustable parameter occurs in only two of the terms rather than in four. However, the fractional-order variant of this system has not been studied, and it is an ideal candidate for examining bifurcations since it has a single adjustable parameter. In this paper, we focus on the dynamic behaviors of this fractional-order simplified Lorenz system. The paper is organized as follows. Fractional-order derivatives and the numerical algorithm for the solution of fractional-order dynamical systems are presented in Sec. 2. In Sec. 3, a time domain scheme is applied to the fractional-order simplified Lorenz system. Bifurcations and the largest Lyapunov exponent of the fractional-order system are presented. Finally, we summarize the results and indicate future directions.

2. Numerical Algorithm for Fractional-Order Dynamical Systems

According to the different definitions of fractional derivatives [Butzer & Westphal, 2000; Podlubny, 1999; Samko *et al.*, 1993], two approaches have been primarily used to solve the fractional-order equations: the frequency-domain method and the time-domain method. The Caputo derivative definition involves a time-domain computation in which nonhomogenous initial conditions are needed, and those values are readily determined [Samko *et al.*, 1993]. This popular definition is given by

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= J^{n-\alpha}\frac{d^n x(t)}{dt^n}, \quad \text{or} \\ D_*^\alpha x(t) &= J^{n-\alpha}x^{(n)}(t), \end{aligned} \tag{1}$$

where $n := \lceil \alpha \rceil$ is the first integer which is not less than α and $\alpha > 0$ but not necessarily $\alpha \in N, x^{(n)}(t)$ is the ordinary nth derivative of x(t), and J^θ is the θ-order Riemann–Liouville (R–L) integral operator given by

$$J^\theta \varphi(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-\tau)^{\theta-1}\varphi(\tau)d\tau, \tag{2}$$

where $\Gamma(\theta)$ is the Gamma function with $0 < \theta \le 1$. The operator D_*^α is commonly called the Caputo differential operator of order α. Compared with the R–L derivative, the Caputo derivative is much preferred since it is more popular in real applications. We can specify the initial values of $x(0), x'(0), \ldots, x^{(m-1)}(0)$, which typically have a well-understood physical meaning and can be measured [Diethelm *et al.*, 2002].

Here, we consider the fractional differential equation with initial conditions

$$\begin{cases} D_*^\alpha x(t) = f(t, x(t)), & 0 \le t < T \\ x^{(k)}(0) = x_0^{(k)}, & k = 0, 1, 2, \ldots, n-1 \end{cases}. \tag{3}$$

It is equivalent to the Volterra integral equation [Diethelm & Ford, 2002].

$$x(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0^{(k)}\frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(t))d\tau. \tag{4}$$

Set $h = T/N, t_j = jh, j = 0, 1, \ldots, N \in Z^+$. Then Eq. (4) can be discretized as follows

$$x_h(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0^{(k)}\frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j)) \tag{5}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases} \tag{6}$$

$$x_h^p(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{i,n+1} f(t_j, x_h(t_j)) \tag{7}$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n-j+1)^\alpha - (n-j)^\alpha), \quad 0 \le j \le n \tag{8}$$

The error estimate in this method is

$$e = \max_{j=0,1,\ldots,N} |x(t_j) - x_h(t_j)| = O(h^p) \tag{9}$$

in which $p = \min(2, 1+\alpha)$.

This is called the Adams–Bashforth–Moulton predictor–corrector scheme [Diethelm *et al.*, 2002; Diethelm & Ford, 2002; Diethelm, 1997], which is a time-domain approach and is more effective for investigating the dynamics of fractional-order systems.

3. Dynamical Behaviors of the Fractional-Order Simplified Lorenz System

In this section, the chaotic dynamics of the fractional-order simplified Lorenz system are analyzed. In particular, we identify a novel bifurcation parameter, that is, the fractional-order α, β, γ of the derivative. The dynamics of the system are also analyzed when c is considered as a bifurcation parameter.

3.1. *The fractional-order simplified Lorenz system*

The simplified Lorenz system with a single adjustable parameter c is described by Sun and Sprott [2009],

$$\begin{cases} \dot{x} = 10(y-x) \\ \dot{y} = -xz + (24-4c)x + cy. \\ \dot{z} = xy - \dfrac{8z}{3} \end{cases} \tag{10}$$

Now, consider the fractional-order simplified Lorenz system given by

$$\begin{cases} \dfrac{d^\alpha x}{dt^\alpha} = 10(y-x) \\ \dfrac{d^\beta y}{dt^\beta} = -xz + (24-4c)x + cy, \\ \dfrac{d^\gamma z}{dt^\gamma} = xy - \dfrac{8z}{3} \end{cases} \tag{11}$$

where α, β, γ determine the fractional order of the equation, and $\alpha, \beta, \gamma > 0$, but not necessarily $\alpha, \beta, \gamma \in N$. When $\alpha = \beta = \gamma = 1$, system (11) becomes the original integer order simplified Lorenz system (10). By exploiting the Adams–Bashforth–Moulton scheme, the fractional-order simplified Lorenz equations (11) can be written as

$$\begin{cases} x_{n+1} = x_0 + \dfrac{h^\alpha}{\Gamma(\alpha+2)} \left\{ 10[y_{n+1}^p - x_{n+1}^p] + \displaystyle\sum_{j=0}^{n} a_{1,j,n+1} 10(y_j - x_j) \right\} \\ y_{n+1} = y_0 + \dfrac{h^\beta}{\Gamma(\beta+2)} \left\{ [-x_{n+1}^p z_{n+1}^p + (24-4c)x_{n+1}^p + cy_{n+1}^p] \right. \\ \qquad \left. + \displaystyle\sum_{j=0}^{n} a_{2,j,n+1}(-x_j z_j + (24-4c)x_j + cy_j) \right\} \\ z_{n+1} = z_0 + \dfrac{h^\gamma}{\Gamma(\gamma+2)} \left\{ \left[x_{n+1}^p y_{n+1}^p - \dfrac{8z_{n+1}^p}{3} \right] + \displaystyle\sum_{j=0}^{n} a_{3,j,n+1} \left(x_j y_j - \dfrac{8z_j}{3} \right) \right\} \end{cases} \tag{12}$$

in which

$$\begin{cases} x^p_{n+1} = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{j=0}^{n} b_{1,j,n+1} 10(y_j - x_j) \\ y^p_{n+1} = y_0 + \dfrac{1}{\Gamma(\beta)} \displaystyle\sum_{j=0}^{n} b_{2,j,n+1} (-x_j z_j + (24-4c)x_j + c y_j) \\ z^p_{n+1} = z_0 + \dfrac{1}{\Gamma(\gamma)} \displaystyle\sum_{j=0}^{n} b_{3,j,n+1} \left(x_j y_j - \dfrac{8 z_j}{3} \right) \end{cases} \tag{13}$$

where

$$\begin{cases} b_{1,j,n+1} = \dfrac{h^\alpha}{\alpha}((n-j+1)^\alpha - (n-j)^\alpha), \quad 0 \le j \le n \\ b_{2,j,n+1} = \dfrac{h^\beta}{\beta}((n-j+1)^\beta - (n-j)^\beta), \quad 0 \le j \le n \\ b_{3,j,n+1} = \dfrac{h^\gamma}{\gamma}((n-j+1)^\gamma - (n-j)^\gamma), \quad 0 \le j \le n \end{cases} \tag{14}$$

$$\begin{cases} a_{1,j,n+1} = \begin{cases} n^\alpha - (n-\alpha)(n+1)^\alpha & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & 0 \le j \le n \end{cases} \\ a_{2,j,n+1} = \begin{cases} n^\beta - (n-\beta)(n+1)^\beta & j = 0 \\ (n-j+2)^{\beta+1} + (n-j)^{\beta+1} - 2(n-j+1)^{\beta+1} & 0 \le j \le n \end{cases} \\ a_{3,j,n+1} = \begin{cases} n^\gamma - (n-\gamma)(n+1)^\gamma & j = 0 \\ (n-j+2)^{\gamma+1} + (n-j)^{\gamma+1} - 2(n-j+1)^{\gamma+1} & 0 \le j \le n \end{cases} \end{cases} \tag{15}$$

By exploiting the capabilities offered by the predictor–corrector scheme mentioned above in our simulations, the bifurcation diagrams indicate chaos, which has been confirmed by calculating the largest Lyapunov exponent in some cases using the Wolf algorithm [Wolf *et al.*, 1985].

3.2. *Chaos and bifurcations with different system parameter c*

Let $\alpha = \beta = \gamma = 0.95$, and vary c from 2 to 8. The initial states of the fractional-order simplified Lorenz system are taken as $x(0) = -8.3458, y(0) = -10.6753$, and $z(0) = 12.3088$. The step size for c is 0.005, and the resulting bifurcation diagram is shown in Fig. 1(a). The chaotic motion identified is validated by the positive Lyapunov exponent (LE), which is calculated by Wolf algorithm and plotted in Fig. 1(b). Compared with that of the integer-order simplified Lorenz system as shown in Fig. 2 [Sun & Sprott, 2009], the fractional-order simplified Lorenz system has the same tendency as its integer-order counterpart, but the attractor is smaller. It is chaotic over most of the range $c \in [2.6, 7.4]$, where the largest Lyapunov exponents of the system are positive. The figure suggests that the transition to chaos is apparently different at the two extremes of c. As c increases from $-\infty$ to 2.5, the system abruptly becomes chaotic at about $c = 2.5$, whereas a decrease in c from $+\infty$ causes the fractional-order system to enter chaos by pitchfork and period-doubling bifurcations.

To observe the dynamic behavior, the phase-space trajectory is shown in Fig. 3. The phase-space diagrams have been plotted so as to visualize the pitchfork bifurcation. In this case, two sets of symmetrical initial conditions have been used to show two different attractors, which are almost completely superimposed in some cases. To distinguish one from another, we plot them in blue and red, respectively. From these figures, it is clear that as the value of parameter c decreases, a pitchfork bifurcation is observed. At $c = 8$, the origin loses stability by a supercritical pitchfork bifurcation, and a symmetric pair of attracting fixed points are born.

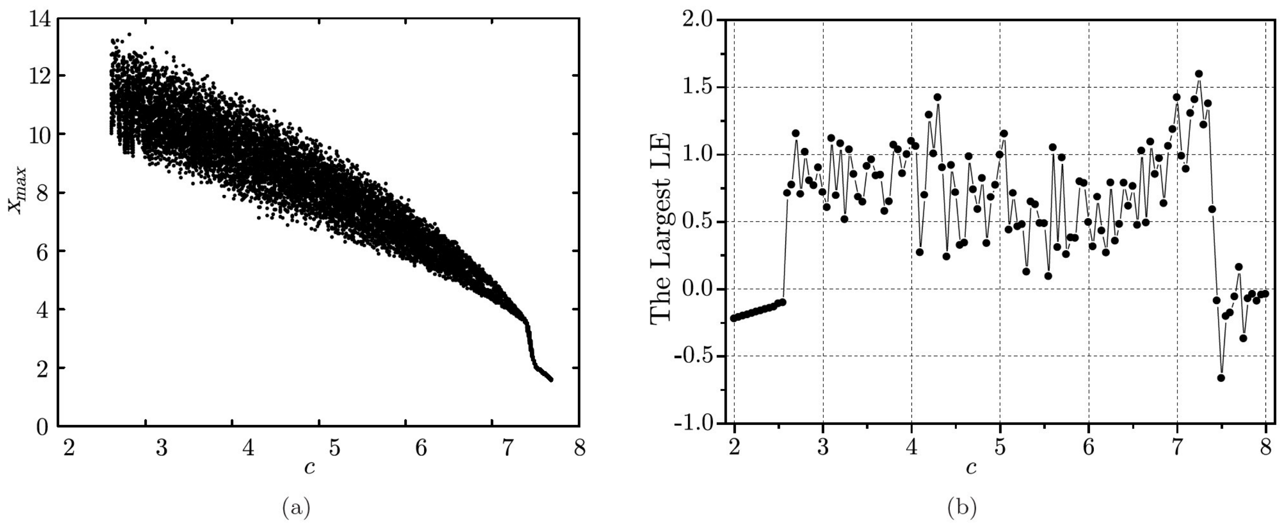


Fig. 1. Bifurcation diagram and the largest LE for the system (11) versus c for q= 0.95. (a) Bifurcation diagram, (b) largest LE.

At $c = 7.7$, a pair of coexisting stable limit cycles is born. As c decreases from 7.7, the two coexisting unstable limit cycles expand as shown in Figs. 3(a) and 3(b) and then merge at $c = 7.432$ as shown in Fig. 3(c). The merged limit cycle grows with decreasing c as shown in Fig. 3(d). When c decreases further, a pitchfork bifurcation occurs at $c = 7.25$, where the attractor splits into two as shown in Fig. 3(e). Then one circle of the attractor grows while the other shrinks with decreasing c as shown in Fig. 3(f). At $c = 7.1$, the limit cycles touch the saddle point and become homoclinic orbits, and the fractional-order system enters into chaos by a homoclinic bifurcation as shown in Fig. 3(g).

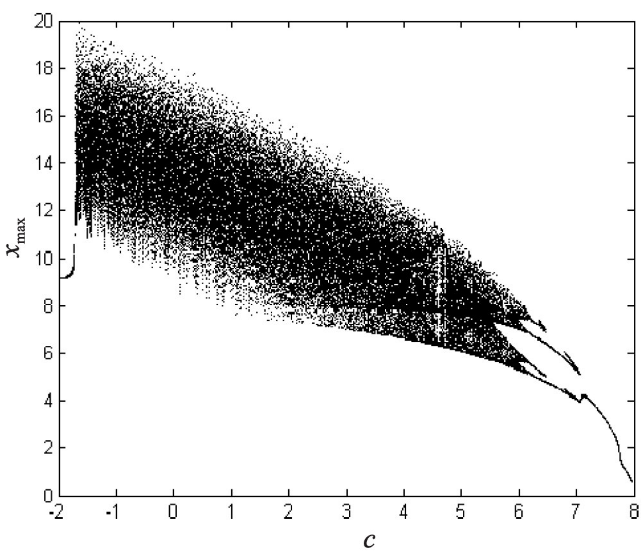


Fig. 2. The bifurcation diagram of the integer-order simplified Lorenz system versus c.

3.3. *Chaos and bifurcations with different fractional order q*

Now let $\alpha = \beta = \gamma = q$, and vary the fractional order q from 0.9 to 1.15, but with a fixed system parameter of $c = 5$. The initial states of the fractional-order simplified Lorenz system are the same as above. With an increment of q equal to 0.001, the bifurcation diagram is shown in Fig. 4(a). The corresponding largest Lyapunov exponent is shown in Fig. 4(b). The fractional-order simplified Lorenz system is chaotic over most of the range $q \in [0.93, 1.07]$, where the largest Lyapunov exponent is positive. At least, one periodic window is observed when $q \in [1.024, 1.033]$, where the largest Lyapunov exponent is zero.

To observe the dynamic behaviors, the periodic window is expanded in steps of 0.0001 as shown in Fig. 5(a). Similarly, two suitable sets of initial conditions have been selected, and the bifurcation diagrams have been plotted in blue and red, respectively, for visualizing the pitchfork bifurcation. There exist four kinds of bifurcation in the periodic window, including a tangent bifurcation, a flip bifurcation, an interior crisis and an attractor merging crisis. An interior crisis and an attractor merging crisis occur when $q = 1.024$. A flip bifurcation occurs when $q = 1.0315$ and $q = 1.0263$, and a tangent bifurcation occurs at $q = 1.033$. An attractor merging crisis occurs at $q = 1.051$. The interior crisis and attractor merging crisis are global bifurcations. A crisis occurs when a chaotic attractor collides with an unstable periodic orbit or its basin of attraction. Here, there exit two

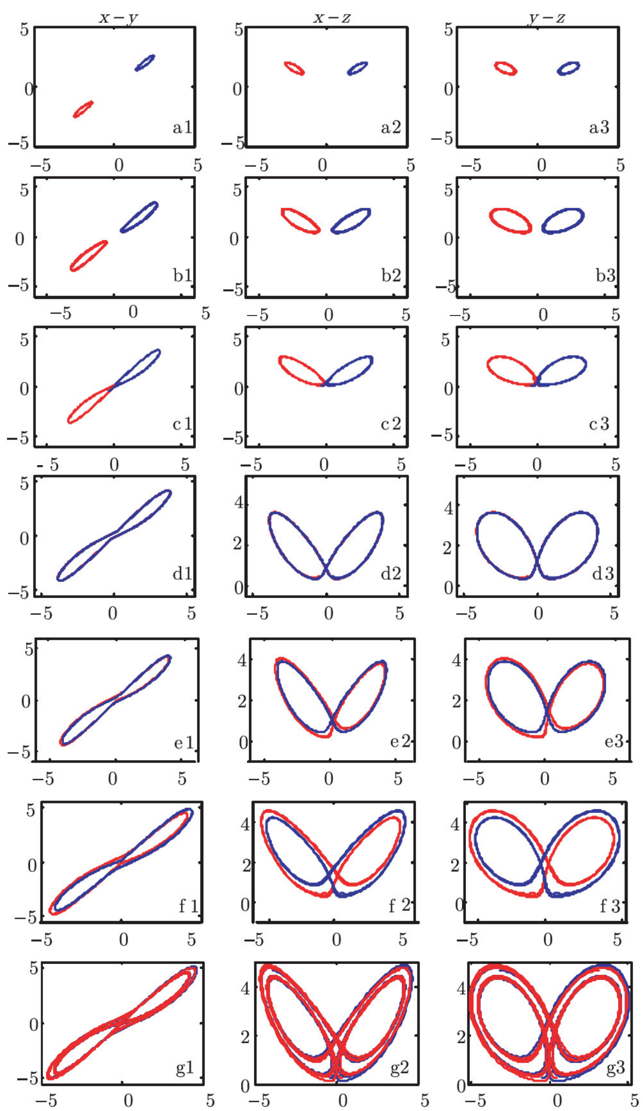
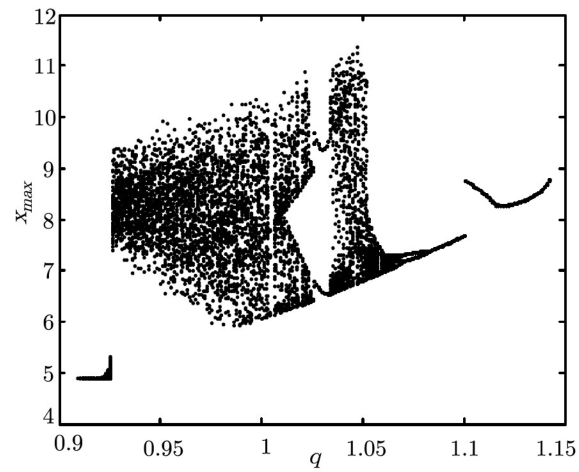
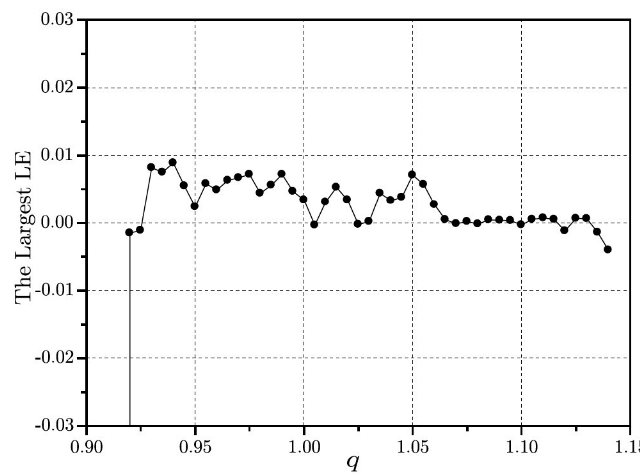


Fig. 3. State space plots for different values of c (blue and red attractors correspond to two symmetrical initial conditions) (a) $c = 7.5$, (b) $c = 7.44$, (c) $c = 7.432$, (d) $c = 7.3$, (e) $c = 7.25$, (f) $c = 7.15$, (g) $c = 7.1$.



(a)



(b)

Fig. 4. Bifurcation diagram and the largest LE of the system (11) versus q for $c = 5$. (a) Bifurcation diagram, (b) largest LE.

distinguishable crises: an interior crisis in which the attractor touches a periodic orbit within its basin, and an attractor merging crisis in which two or more attractors simultaneously touch a periodic orbit on the basin boundary that separates them [Sprott, 2003]. Consequently, the fractional-order parameter can be taken as a bifurcation parameter, just like the system parameter c.

As shown in Fig. 5(b) (with steps of 0.0005), when the bifurcation parameter q is decreased from 1.15, a pitchfork bifurcation occurs at $q = 1.12$. Two limit cycles, denoted in blue and red respectively, coexist until the period-doubling bifurcations occur at $q = 1.082$. Then the fractional-order system enters into chaos by a series of period-doubling bifurcations. When the fractional order is less than 0.93, the fractional-order system converges to a fixed point, so the lowest total order for the fractional-order system to yield chaos is 2.79 in this case.

3.4. *Chaos and bifurcations with different fractional orders* α, β, γ

To study the dynamics of system (11) with different fractional order, three cases are considered as follows.

(1) Fix $\beta = \gamma = 1, c = 5$, and let α vary. The system is calculated numerically for $\alpha \in [0.75, 1.15]$

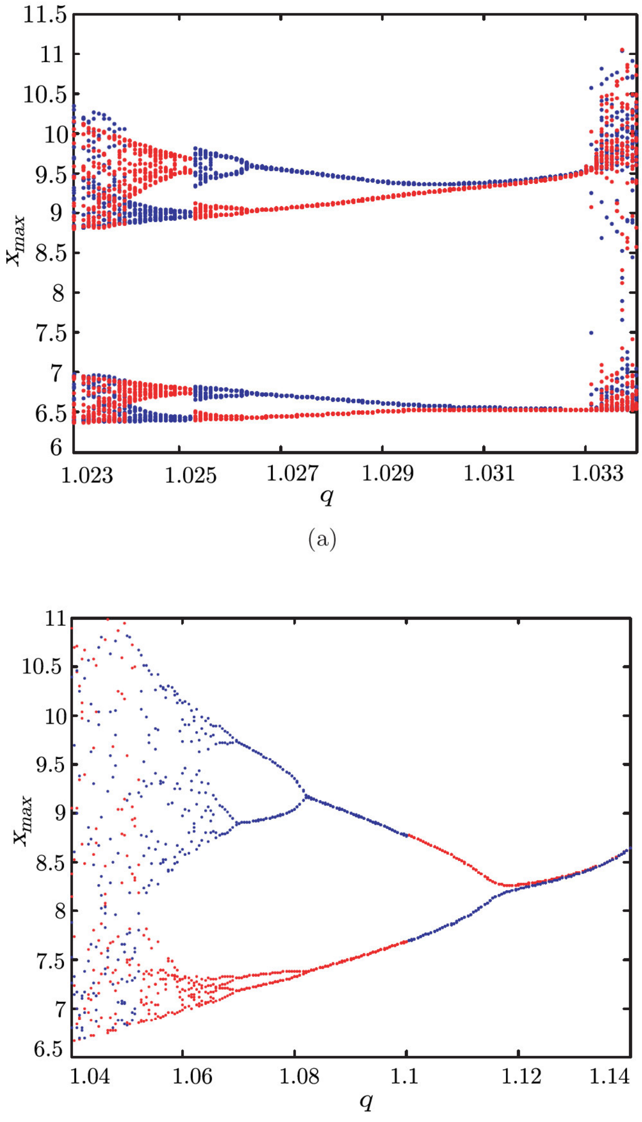


Fig. 5. Bifurcation diagrams of the system (11) versus q for $c = 5$ (blue and red attractors correspond to two symmetrical initial conditions). (a) $q \in [1.023, 1.034]$, (b) $q \in [1.04, 1.14]$.

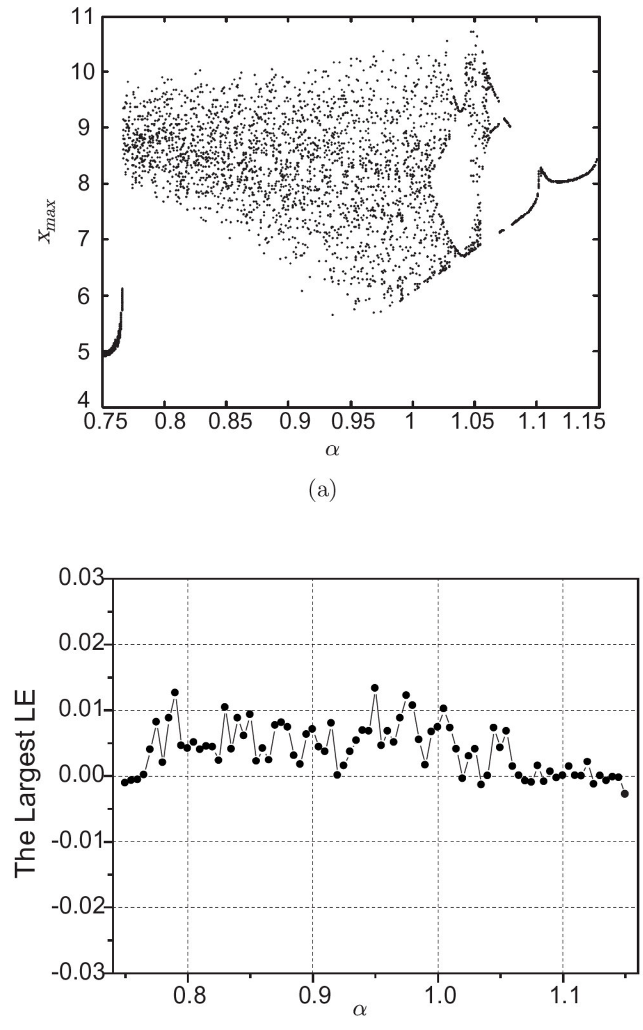
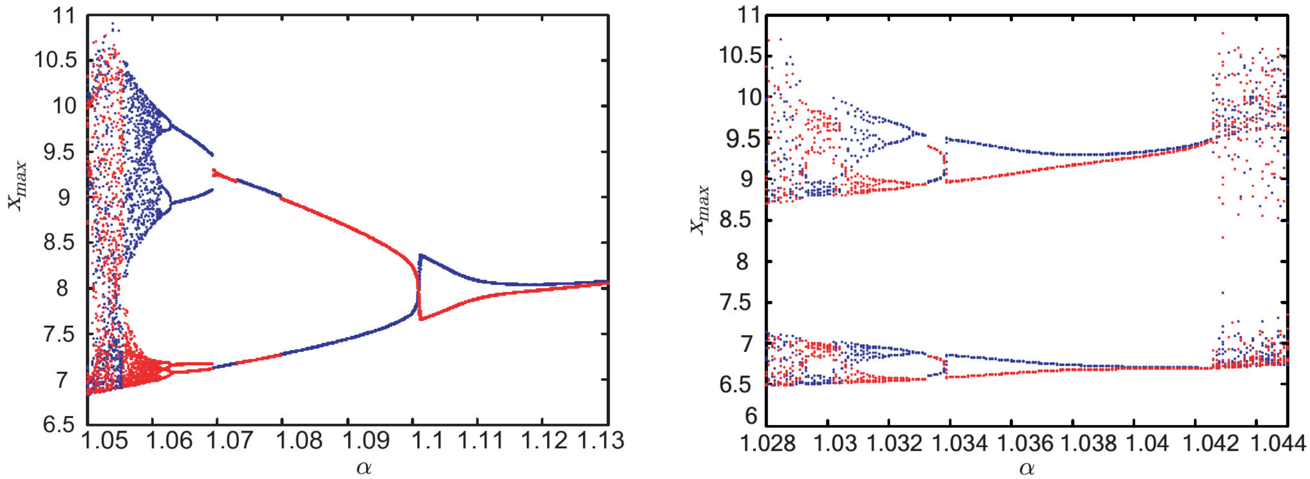


Fig. 6. Bifurcation diagram and the largest LE of the system (11) versus α for $\beta = \gamma = 1$. (a) Bifurcation diagram, (b) largest LE.

with an increment of α equal to 0.001. The initial states of the system are the same as above. The bifurcation diagram is shown in Fig. 6(a). The corresponding largest Lyapunov exponent versus α is plotted in Fig. 6(b). The fractional-order simplified Lorenz system is chaotic over most of the range $\alpha \in [0.77, 1.05]$, where the largest Lyapunov exponent is positive. One periodic window appears when the fractional order is $\alpha \in [1.03, 1.045]$, where the largest Lyapunov exponent is zero. Transient chaos is observed when α is less than 0.77.

Expanded bifurcation diagrams with two symmetrical initial conditions are shown in Fig. 7(a) with steps of 0.0001, denoted by blue and red, respectively. Considering the blue one first, when α decreases from 1.13 to 1.1, $x_{\max}$ gradually increases until it suddenly decreases at $\alpha = 1.1$. Three similar sudden changes occur when $\alpha \in [1.07, 1.08]$ with a period-doubling route to chaos. Similar phenomena take place for the red case. An attractor merging crisis occurs when $\alpha = 1.055$. An expanded periodic window with steps of 0.0001 is shown in Fig. 7(b). Here also exist four kinds of bifurcations, including a tangent bifurcation, a flip bifurcation, an interior crisis and an attractor merging crisis. Sudden changes are also observed in this figure. In this case, the lowest order for which chaos was found for the fractional-order system is 2.77.



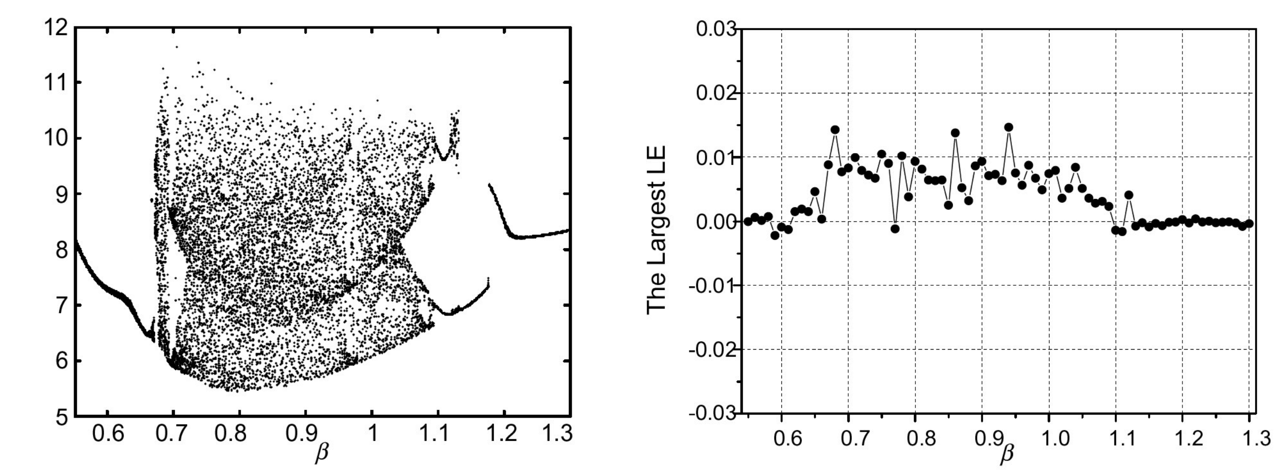
(a)

(b)

Fig. 7. Bifurcation diagram of the system (11) versus α for $\beta = \gamma = 1$ (blue and red attractors correspond to two symmetrical initial conditions). (a) $\alpha \in [1.05, 1.13]$, (b) $\alpha \in [1.028, 1.044]$.

(2) Fix $\alpha = \gamma = 1, c = 5$, and let β vary. The system is calculated numerically for $\beta \in [0.5, 1.3]$ with an increment of β equal to 0.001. The initial states of the system are the same as above. The bifurcation diagram is shown in Fig. 8(a). The largest Lyapunov exponent versus β is plotted in Fig. 8(b). As shown in Fig. 9(a), when β decreases from 1.3, a pitchfork bifurcation occurs at $\beta = 1.22$. Two limit cycles, denoted in blue and red, respectively, coexist until a period-doubling bifurcation occurs at $\beta = 1.138$. Then the fractional-order system enters into chaos by a series of period-doubling bifurcations. When β increases from 0.5, a similar route to chaos is shown in Fig. 9(c). When $\beta \in [0.62, 1.09]$, the fractional-order system is chaotic with three periodic windows at $\beta \in (0.692, 0.701), \beta \in (1.08, 1.085)$, and $\beta \in (1.09, 1.12)$, as shown in Figs. 9(b) and 9(d) with steps of 0.0001. As in case (1), a tangent bifurcation, flip bifurcation, interior crisis and attractor merging crisis exist in these periodic windows. In this case, the lowest order of the fractional-order system is 2.62, which is also the lowest order we found for this system to yield chaos.

(3) Fix $\alpha = \beta = 1, c = 5$, and let γ vary. The system is calculated numerically for $\gamma \in [0.7, 1.6]$ with an



(a)

(b)

Fig. 8. Bifurcation diagram and the largest LE of the system (11) with β for $\alpha = \gamma = 1$. (a) Bifurcation diagram, (b) largest LE.

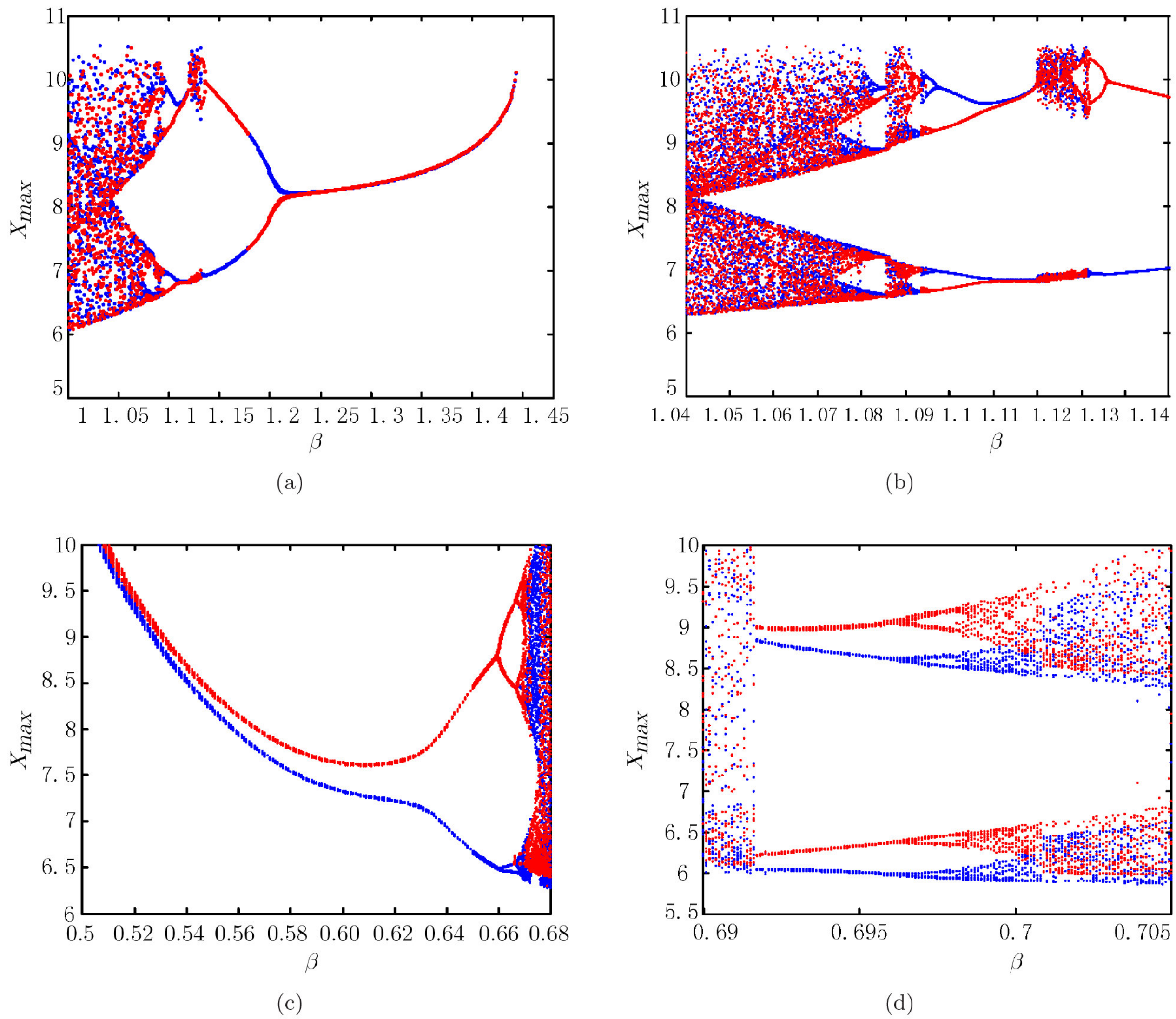


Fig. 9. Bifurcation diagrams of the system (11) with β for $\alpha = \gamma = 1$ (blue and red attractors correspond to two symmetrical initial conditions). (a) $\beta \in [1, 1.45]$, (b) $\beta \in [1.04, 1.14]$, (c) $\beta \in [0.5, 0.68]$, (d) $\beta \in [0.69, 0.705]$.

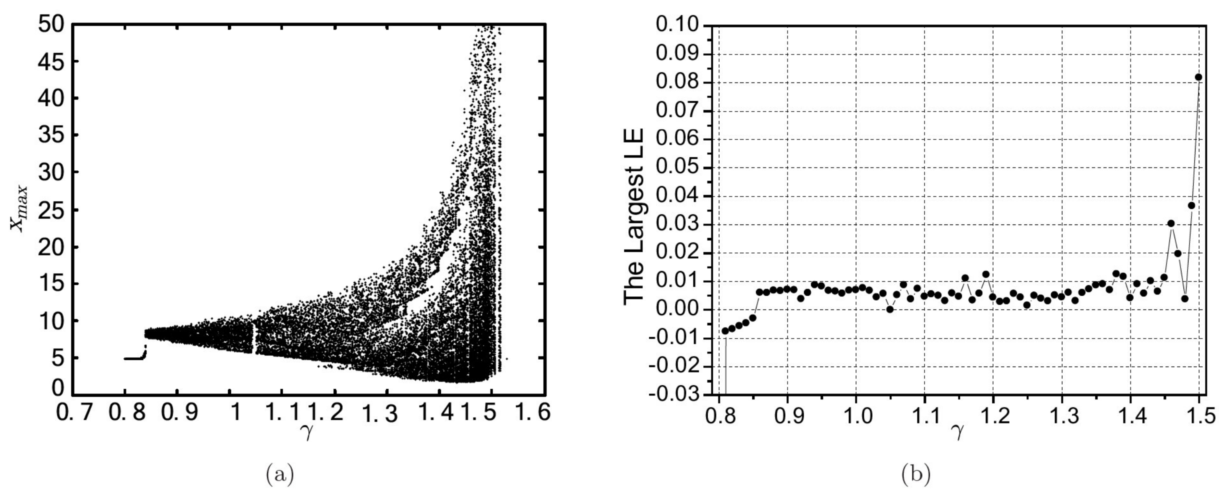


Fig. 10. Bifurcation diagram and the largest LE of the system (11) versus γ for $\alpha = \beta = 1$. (a) Bifurcation diagram, (b) largest LE.

increment of γ equal to 0.001. The initial states of the system are the same as above. The bifurcation diagram is shown in Fig. 10(a). The corresponding largest Lyapunov exponent is plotted versus γ in Fig. 10(b). Obviously, the dynamical behavior is simpler than that of the previous two cases. It shows that the fractional-order simplified Lorenz system is chaotic over most of the range $\gamma \in [0.86, 1.15]$ where the largest Lyapunov exponent is positive. There is a periodic window when the fractional order is $\gamma \in [1.05, 1.06]$ where the largest Lyapunov exponent is zero. The system converges to a fixed point at $\gamma = 0.86$, and the attractor expands in size as γ increases until $\gamma = 1.52$. The lowest order of system (11) for chaos is about 2.86 in this case.

4. Conclusions

In this paper, we studied the dynamics of the fractional-order simplified Lorenz system. The bifurcations and chaos in this system were numerically investigated by varying the system parameter c and the fractional-order q. Several typical bifurcations are observed such as flip bifurcations, tangent bifurcations, interior crisis bifurcations and attractor merging crises. Chaos does exist in the fractional-order simplified Lorenz system with a wide range of fractional orders, which are not only less than 3.0 but also greater than 3.0. The lowest order for this system to yield chaos is 2.62. Future work on the topic should include a theoretical analysis of the fractional-order system, as well as in-depth studies of synchronization control of this fractional-order chaotic system.

Acknowledgments

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