A Universal Nonlinear Control Law for the Synchronization of Arbitrary 4-D Continuous-Time Quadratic Systems

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Abstract: In this letter we show the existence of a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 4-D continuous-time quadratic systems.

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1. Introduction

There are several methods of chaos synchronization. For example in [9], a method is introduced to synchronize two identical chaotic systems with different initial conditions. An adaptive control approach is presented in [10], a backstepping design is given in [13], an active control method is presented in [5-11,12], and a nonlinear control scheme is given in [4,6,8]. In fact, there are many applications of chaos synchronization in physical, chemical, and ecological systems, and in secure communications [1,2,3,7,9,10]. For 4-D continuous-time quadratic systems, some of these methods were applied to Lorenz-Stenflo systems, Qi systems, and other hyperchaotic quadratic systems as shown in [14,15,16,17,18,19,20]. In this letter, we apply nonlinear control theory to synchronize

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two arbitrary, 4-D, continuous-time, quadratic systems. This control law is a universal synchronization approach since it does not need any conditions on the considered systems.

2. Synchronization Using A Universal Nonlinear Control Law

In this section, we consider two arbitrary, 4-D, continuous-time, quadratic systems. The one with variables x_1, y_1, z_1, and u_1 will be controlled to be the new system given by

$$\begin{cases} x_1' = \mu_1 + \beta_{11}x_1 + \beta_{12}y_1 + \beta_{13}z_1 + \beta_{14}u_1 + f_1(x_1, y_1, z_1, u_1) \\ y_1' = \mu_2 + \beta_{21}x_1 + \beta_{22}y_1 + \beta_{23}z_1 + \beta_{24}u_1 + f_2(x_1, y_1, z_1, u_1) \\ z_1' = \mu_3 + \beta_{31}x_1 + \beta_{32}y_1 + \beta_{33}z_1 + \beta_{34}u_1 + f_3(x_1, y_1, z_1, u_1) \\ u_1' = \mu_4 + \beta_{41}x_1 + \beta_{42}y_1 + \beta_{43}z_1 + \beta_{44}u_1 + f_4(x_1, y_1, z_1, u_1) \end{cases} \tag{1}$$

where

$$\begin{cases} f_1 = a_4x_1^2 + a_5y_1^2 + a_6z_1^2 + a_7u_1^2 + a_8x_1y_1 + a_9x_1z_1 + p_1 \\ f_2 = b_4x_1^2 + b_5y_1^2 + b_6z_1^2 + b_7u_1^2 + b_8x_1y_1 + b_9x_1z_1 + p_2 \\ f_3 = c_4x_1^2 + c_5y_1^2 + c_6z_1^2 + c_7u_1^2 + c_8x_1y_1 + c_9x_1z_1 + p_3 \\ f_4 = d_4x_1^2 + d_5y_1^2 + d_6z_1^2 + d_7u_1^2 + d_8x_1y_1 + d_9x_1z_1 + p_4 \\ p_1 = a_{10}y_1z_1 + a_{11}x_1u_1 + a_{12}z_1u_1 + a_{13}y_1u_1 \\ p_2 = b_{10}y_1z_1 + b_{11}x_1u_1 + b_{12}z_1u_1 + b_{13}y_1u_1 \\ p_3 = c_{10}y_1z_1 + c_{11}x_1u_1 + c_{12}z_1u_1 + c_{13}y_1u_1 \\ p_4 = d_{10}y_1z_1 + d_{11}x_1u_1 + d_{12}z_1u_1 + d_{13}y_1u_1 \end{cases} \tag{2}$$

and the one with variables x_2, y_2, z_2, and u_2 as the response system

$$\begin{cases} x_2' = \delta_1 + \rho_{11}x_2 + \rho_{12}y_2 + \rho_{13}z_2 + \rho_{14}u_2 + g_1(x_2, y_2, z_2, u_2) + v_1(t) \\ y_1' = \delta_2 + \rho_{21}x_2 + \rho_{22}y_2 + \rho_{23}z_2 + \rho_{24}u_2 + g_2(x_2, y_2, z_2, u_2) + v_2(t) \\ z_1' = \delta_3 + \rho_{31}x_2 + \rho_{32}y_2 + \rho_{33}z_2 + \rho_{34}u_2 + g_3(x_2, y_2, z_2, u_2) + v_3(t) \\ u_1' = \delta_4 + \rho_{41}x_2 + \rho_{42}y_2 + \rho_{43}z_2 + \rho_{44}u_2 + g_4(x_2, y_2, z_2, u_2) + v_4(t) \end{cases} \tag{3}$$

where

$$\begin{cases} g_1 = h_4x_2^2 + h_5y_2^2 + h_6z_2^2 + h_7u_2^2 + h_8x_2y_2 + h_9x_2z_2 + p_5 \\ g_2 = m_4x_2^2 + m_5y_2^2 + m_6z_2^2 + m_7u_2^2 + m_8x_2y_2 + m_9x_2z_2 + p_6 \\ g_3 = r_4x_2^2 + r_5y_2^2 + r_6z_2^2 + r_7u_2^2 + r_8x_2y_2 + r_9x_2z_2 + p_7 \\ g_4 = s_4x_2^2 + s_5y_2^2 + s_6z_2^2 + s_7u_2^2 + s_8x_2y_2 + s_9x_2z_2 + p_8 \\ p_5 = h_{10}y_2z_2 + h_{11}x_2u_2 + h_{12}z_2u_2 + h_{13}y_2u_2 \\ p_6 = m_{10}y_2z_2 + m_{11}x_2u_2 + m_{12}z_2u_2 + m_{13}y_2u_2 \\ p_7 = r_{10}y_2z_2 + r_{11}x_2u_2 + r_{12}z_2u_2 + r_{13}y_2u_2 \\ p_8 = s_{10}y_2z_2 + s_{11}x_2u_2 + s_{12}z_2u_2 + s_{13}y_2u_2 \end{cases} \tag{4}$$

Here $(\mu_i, \delta_i)_{1\le i\le 4} \in \mathbb{R}^8$ and $(\beta_{ij}, \rho_{ij})_{1\le i,j\le 4} \in \mathbb{R}^{16}$ and $(a_i, b_i, c_i, d_i, h_i, m_i, r_i, s_i)_{4\le i\le 13} \in \mathbb{R}^{80}$ are bifurcation parameters, and $v_1(t), v_2(t), v_3(t)$, and $v_4(t)$ are the unknown nonlinear controller such that two systems (1)-(2) and (3)-(4) can be synchronized.

First, let us define the following quantities depending on the above two systems:

$$\begin{cases} \eta_1 = h_7u_1^2 - a_9u_1x_2 + h_{13}u_1y_2 - a_{12}u_1z_2 + \rho_{14}u_1 - a_7u_2^2 + h_9u_2x_1 \\ \eta_2 = -a_{13}u_2y_1 + h_{12}u_2z_1 - \beta_{14}u_2 + h_4x_1^2 + h_8x_1y_2 + h_9x_1z_2 + \rho_{11}x_1 \\ \eta_3 = -a_4x_2^2 - a_8x_2y_1 - a_9x_2z_1 - \beta_{11}x_2 + h_5y_1^2 + h_{10}y_1z_2 + \rho_{12}y_1 - a_5y_2^2 \\ \eta_4 = -a_{10}y_2z_1 - \beta_{12}y_2 + h_6z_1^2 + \rho_{13}z_1 - a_6z_2^2 - \beta_{13}z_2 - \mu_1 + \delta_1 \\ \eta_5 = m_7u_1^2 - b_9u_1x_2 + m_{13}u_1y_2 - b_{12}u_1z_2 + \rho_{24}u_1 - b_7u_2^2 + m_9u_2x_1 \\ \eta_6 = -b_{13}u_2y_1 + m_{12}u_2z_1 - \beta_{14}u_2 + m_4x_1^2 + m_8x_1y_2 + m_9x_1z_2 + \rho_{21}x_1 \\ \eta_7 = -b_4x_2^2 - b_8x_2y_1 - b_9x_2z_1 - \beta_{21}x_2 + m_5y_1^2 + m_{10}y_1z_2 + \rho_{22}y_1 - b_5y_2^2 \\ \eta_8 = -b_{10}y_2z_1 - \beta_{22}y_2 + m_6z_1^2 + \rho_{23}z_1 - b_6z_2^2 - \beta_{23}z_2 - \mu_2 + \delta_2 \end{cases} \tag{5}$$

$$\begin{cases} \eta_9 = r_7u_1^2 - c_9u_1x_2 + r_{13}u_1y_2 - c_{12}u_1z_2 + \rho_{34}u_1 - c_7u_2^2 + r_9u_2x_1 \\ \eta_{10} = -c_{13}u_2y_1 + r_{12}u_2z_1 - \beta_{34}u_2 + r_4x_1^2 + r_8x_1y_2 + r_9x_1z_2 + \rho_{11}x_1 \\ \eta_{11} = -c_4x_2^2 - c_8x_2y_1 - c_9x_2z_1 - \beta_{31}x_2 + r_5y_1^2 + r_{10}y_1z_2 + \rho_{32}y_1 - c_5y_2^2 \\ \eta_{12} = -c_{10}y_2z_1 - \beta_{32}y_2 + r_6z_1^2 + \rho_{33}z_1 - c_6z_2^2 - \beta_{33}z_2 - \mu_3 + \delta_3 \\ \eta_{13} = s_7u_1^2 - d_9u_1x_2 + s_{13}u_1y_2 - d_{12}u_1z_2 + \rho_{44}u_1 - d_7u_2^2 + s_9u_2x_1 \\ \eta_{14} = -d_{13}u_2y_1 + s_{12}u_2z_1 - \beta_{44}u_2 + s_4x_1^2 + s_8x_1y_2 + s_9x_1z_2 + \rho_{41}x_1 \\ \eta_{15} = -d_4x_2^2 - d_8x_2y_1 - d_9x_2z_1 - \beta_{41}x_2 + s_5y_1^2 + s_{10}y_1z_2 + \rho_{42}y_1 - d_5y_2^2 \\ \eta_{16} = -d_{10}y_2z_1 - \beta_{42}y_2 + s_6z_1^2 + \rho_{43}z_1 - d_6z_2^2 - \beta_{43}z_2 - \mu_4 + \delta_4 \end{cases} \tag{6}$$

$$\begin{cases} \xi_1 = \beta_{11} + \rho_{11} + (a_4 + h_4)(x_1 + x_2) + a_9 u_1 + a_8 y_1 + a_9 z_1 + h_9 u_2 + h_8 y_2 + h_9 z_2 \\ \xi_2 = \beta_{12} + \rho_{12} + (a_5 + h_5)(y_1 + y_2) + a_{10} z_1 + z_2 h_{10} \\ \xi_3 = \beta_{13} + \rho_{13} + (a_6 + h_6)(z_1 + z_2) + u_1 a_{12} + u_2 h_{12} \\ \xi_4 = \beta_{14} + \rho_{14} + (a_7 + h_7)(u_1 + u_2) + y_1 a_{13} + y_2 h_{13} \\ \xi_5 = \eta_1 + \eta_2 + \eta_3 + \eta_4 \end{cases} \tag{7}$$

$$\begin{cases} \xi_6 = \beta_{21} + \rho_{21} + (b_4 + m_4)(x_1 + x_2) + b_9 u_1 + b_8 y_1 + b_9 z_1 + m_9 u_2 + m_8 y_2 + m_9 z_2 \\ \xi_7 = \beta_{22} + \rho_{22} + (b_5 + m_5)(y_1 + y_2) + b_{10} z_1 + z_2 m_{10} \\ \xi_8 = \beta_{23} + \rho_{23} + (b_6 + m_6)(z_1 + z_2) + u_1 b_{12} + u_2 m_{12} \\ \xi_9 = \beta_{24} + \rho_{24} + (b_7 + m_7)(u_1 + u_2) + y_1 b_{13} + y_2 m_{13} \\ \xi_{10} = \eta_5 + \eta_6 + \eta_7 + \eta_8 \end{cases} \tag{8}$$

$$\begin{cases} \xi_{11} = \beta_{31} + \rho_{31} + (c_4 + r_4)(x_1 + x_2) + c_9 u_1 + c_8 y_1 + c_9 z_1 + r_9 u_2 + r_8 y_2 + r_9 z_2 \\ \xi_{12} = \beta_{32} + \rho_{32} + (c_5 + r_5)(y_1 + y_2) + c_{10} z_1 + z_2 r_{10} \\ \xi_{13} = \beta_{33} + \rho_{33} + (c_6 + r_6)(z_1 + z_2) + u_1 c_{12} + u_2 r_{12} \\ \xi_{14} = \beta_{34} + \rho_{34} + (c_7 + r_7)(u_1 + u_2) + y_1 c_{13} + y_2 r_{13} \\ \xi_{15} = \eta_9 + \eta_{10} + \eta_{11} + \eta_{12} \end{cases} \tag{9}$$

$$\begin{cases} \xi_{16} = \beta_{41} + \rho_{41} + (d_4 + s_4)(x_1 + x_2) + d_9 u_1 + d_8 y_1 + d_9 z_1 + s_9 u_2 + s_8 y_2 + s_9 z_2 \\ \xi_{17} = \beta_{42} + \rho_{42} + (d_5 + s_5)(y_1 + y_2) + d_{10} z_1 + z_2 s_{10} \\ \xi_{18} = \beta_{43} + \rho_{43} + (d_6 + s_6)(z_1 + z_2) + u_1 d_{12} + u_2 s_{12} \\ \xi_{19} = \beta_{44} + \rho_{44} + (d_7 + s_7)(u_1 + u_2) + y_1 d_{13} + y_2 s_{13} \\ \xi_{20} = \eta_{13} + \eta_{14} + \eta_{15} + \eta_{16} \end{cases} \tag{10}$$

Now let the error states be $e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1$, and $e_4 = u_2 - u_1$. Then the error system is given by

$$\begin{cases} e_1' = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 + \xi_5 + v_1(t) \\ e_2' = \xi_6 e_1 + \xi_7 e_2 + \xi_8 e_3 + \xi_9 e_4 + \xi_{10} + v_2(t) \\ e_3' = \xi_{11} e_1 + \xi_{12} e_2 + \xi_{13} e_3 + \xi_{14} e_4 + \xi_{15} + v_3(t) \\ e_4' = \xi_{16} e_1 + \xi_{17} e_2 + \xi_{18} e_3 + \xi_{19} e_4 + \xi_{20} + v_4(t) \end{cases} \tag{11}$$

In this letter, we propose the following universal control law for the system (3)-(4):

$$\begin{cases} v_1(t) = -(1+\xi_1)e_1 - \xi_5 \\ v_2(t) = -(\xi_2+\xi_6)e_1 - (1+\xi_7)e_2 - \xi_{10} \\ v_3(t) = -(\xi_3+\xi_{11})e_1 - (\xi_8+\xi_{12})e_2 - (1+\xi_{13})e_3 - \xi_{15} \\ v_4(t) = -(\xi_4+\xi_{16})e_1 - (\xi_9+\xi_{17})e_2 - (\xi_{14}+\xi_{18})e_3 - (1+\xi_{19})e_4 - \xi_{20} \end{cases} \quad (12)$$

Then the two 4-D, continuous-time, quadratic systems (1)-(2) and (3)-(4) approach synchronization for any initial condition, since the error system (11) becomes

$$\begin{cases} e_1' = -e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 \\ e_2' = -\xi_2 e_1 - e_2 + \xi_8 e_3 + \xi_9 e_4 \\ e_3' = -\xi_3 e_1 - \xi_8 e_2 - e_3 + \xi_{14} e_4 \\ e_4' = -\xi_4 e_1 - \xi_9 e_2 - \xi_{14} e_3 - e_4 \end{cases} \quad (13)$$

and if we consider the Lyapunov function $V = \frac{e_1^2+e_2^2+e_3^2+e_4^2}{2}$, then it is easy to verify the asymptotic stability of the error system (13) by Lyapunov stability theory since we have $\frac{dV}{dt} = -e_1^2 - e_2^2 - e_3^2 - e_4^2 < 0$ for all $(\mu_i, \delta_i)_{1\leq i\leq 4} \in \mathbb{R}^8$ and $(\beta_{ij}, \rho_{ij})_{1\leq i,j\leq 4} \in \mathbb{R}^{16}$ and $(a_i, b_i, c_i, d_i, h_i, m_i, r_i, s_i)_{4\leq i\leq 13} \in \mathbb{R}^{80}$ and for all initial conditions. If the two systems (1)-(2) and (3)-(4) are chaotic, then the control law (12) guaranties also their synchronization for any initial condition. An elementary example of this situation can be found in [14]. Also, we notice that any 4-D, continuous-time, quadratic, chaotic system can be stabilized (resp. controlled) to a stable 4-D, continuous-time, quadratic system that converges to an equilibrium point (resp. to a 4-D, continuous-time, quadratic system that converges to a periodic solution). Furthermore, any 4-D, continuous-time, quadratic system can be chaotified to a chaotic 4-D, continuous-time, quadratic system.

Conclusion

We have presented a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 4-D, continuous-time, quadratic systems. This control law (12) can be considered either as a stabilization, or as a control, or as a chaotification approach for a general 4-D, continuous-time, quadratic system.

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