A New Chaotic Jerk Circuit

J. C. Sprott

Abstract—**Much recent interest has been given to simple chaotic oscillators based on jerk equations that involve a third-time derivative of a single scalar variable. The simplest such equation has yet to be electronically implemented. This paper describes a particularly elegant circuit whose operation is accurately described by a simple variant of that equation in which the requisite nonlinearity is provided by a single diode and for which the analysis is particularly straightforward.**

Index Terms—**Analog computers, chaos, differential equations, feedback circuits, jerk functions, nonlinear systems, oscillators.**

I. Introduction

The development of Chua's circuit in 1983 [1] and its many variants [2], [3] launched a quest for other circuits that chaotically oscillate. Some of the most elegant examples of such circuits [4]–[14] were motivated by the discovery of simple third-order ordinary differential equations of the form $\dddot{x} = J(\ddot{x}, \dot{x}, x)$ whose solutions are chaotic [15], [16]. The nonlinear function J is called a "jerk," because it describes the third-time derivative of x, which would correspond to the first-time derivative of acceleration in a mechanical system [17]. Previous jerk circuits involved a nonlinearity only in the x term of the jerk function.

The simplest autonomous dissipative ordinary differential equation with a quadratic nonlinearity whose solutions are chaotic is given by [18]

$$\dddot{x} + A\ddot{x} + x \pm \dot{x}^2 = 0 \qquad (1)$$

which is a jerk equation with $J = -A\ddot{x} - x \mp \dot{x}^2$, where A is a bifurcation parameter leading to chaos over most of the narrow range $2.0168\ldots < A < 2.0577\ldots$. Its nonlinearity is in the $\dot{x}$ term. It has been rigorously shown [19] that no simpler case can exist. Unfortunately, the quadratic nonlinearity is not easily implemented electronically with standard components. In addition, such a circuit would be rather delicate since the chaos exists over a very narrow range of component values.

Recently, Munmuangsaen *et al.* [20] have shown that (1) can be generalized to

$$\dddot{x} + \ddot{x} + x + f(\dot{x}) = 0 \qquad (2)$$

in which chaos occurs for a wide variety of nonlinear functions, including $f(\dot{x}) = \alpha^2 \exp(\dot{x}/\alpha)$ with $\alpha < 0.27$, except for many mostly small periodic windows. Of particular interest is that the chaos persists in the limit of $\alpha \to 0$, where $f(\dot{x})$ has the characteristics of an ideal diode. In fact, the behavior can be reproduced using a two-segment piecewise linear approximation in which $f(\dot{x}) = 0$ for $\dot{x} < 1$ and infinite otherwise, in which case (2) can be solved exactly in the region $\dot{x} < 1$, where the equation is linear, subject to a boundary condition in which $\ddot{x}$ reverses sign whenever $\dot{x} = 1$. The dynamics in the $x - \dot{x}$ plane thus resembles a ball chaotically bouncing on a floor. A number of other piecewise linear circuits have been studied [21]–[25] with similar dynamics, although their representation in terms of jerk functions tend to be rather complicated.

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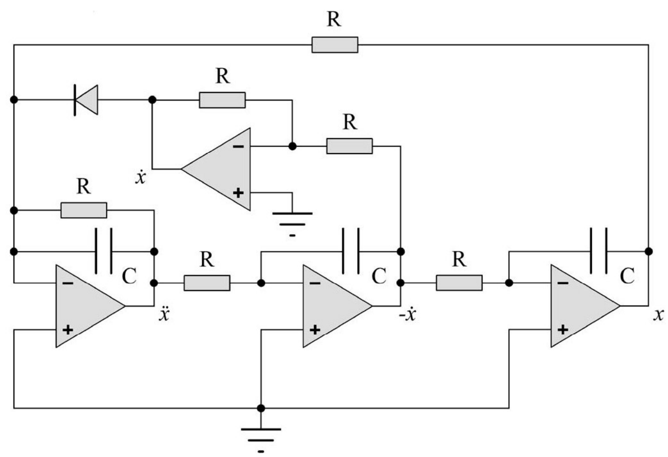

Fig. 1. Chaotic circuit schematic.

II. Circuit Realization

Equation (2) can be electronically implemented in a circuit, as shown in Fig. 1. The circuit consists of three successive active integrators in a feedback loop plus a second nonlinear feedback loop involving only two of the integrators and an inverter with a diode. It can be viewed as a chaotic phase-shift oscillator with gain control or as an analog computer solving (2).

Unlike many other chaotic circuits, the component values are not critical, but all the resistors have the same arbitrary values, which are here taken as $R = 1$ kΩ, as do all the capacitors, which are here taken as $C = 1$ μF, with all components having 10% tolerance. The nonlinear feedback is provided by a 1N4001 silicon p-n-junction diode, although the circuit works equally well with germanium and other diodes.

The circuit was constructed in breadboard style using Tektronix AM501 operational amplifier modules, as shown in Fig. 2. The chaos is readily apparent in the oscilloscope trace, which shows x on the horizontal axis and $\dot{x}$ on the vertical axis,

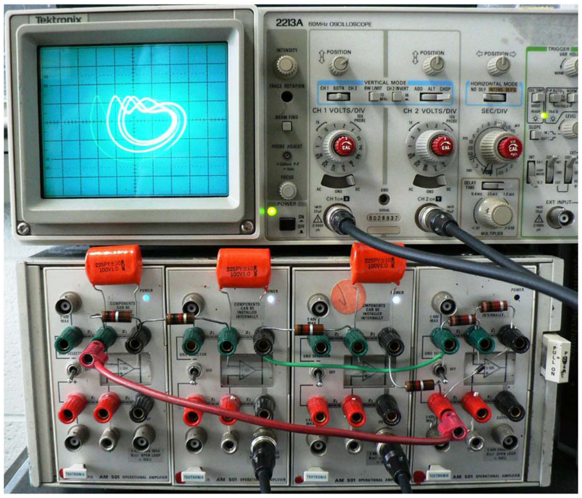

Fig. 2. Actual circuit in operation.

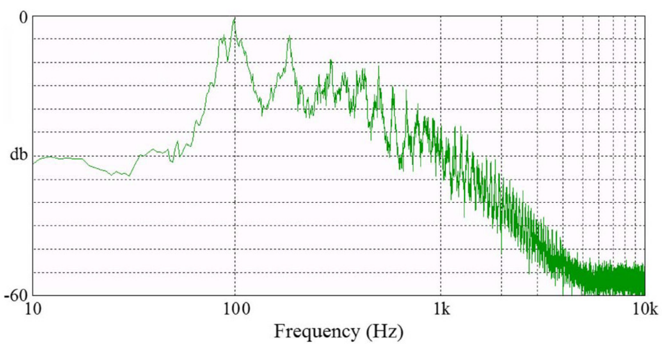

Fig. 3. Frequency spectrum of the x output of the chaotic circuit.

both at 0.5 V/division. The x voltage can be connected to an audio amplifier and speaker to hear the chaos. The diode can be replaced with a light-emitting diode to display a chaotic flicker resembling a candle flame.

As further evidence of chaos, the x voltage was digitized using the sound card of a personal computer and stored as a .WAV file, which is available at http://sprott.physics.wisc.edu/chaos/newckt.wav. From the file, a frequency spectrum was produced, as shown in Fig. 3, using the freeware Visual Analyzer program available at http://www.sillanumsoft.com. The broadband nature of the spectrum is clearly evident with a dominant peak near 100 Hz and its harmonics but with significant signal over the entire audio range.

The component values were chosen to make the circuit oscillate in the audio range, so that the chaos can be easily heard and displayed on an oscilloscope, although the frequency can be scaled up or down as desired over several decades. No attempt was made to find the upper frequency limit of operation since that would largely depend on the choice of components, but one should note that fast switching of the diode is not essential for operation of the circuit but only for ease of analysis. Furthermore, the circuit should more favorably scale with frequency than circuits that use op-amp comparators where slew rate is a serious limitation.

The circuit is robust to parameter variations and requires no careful tuning of component values, although there are small

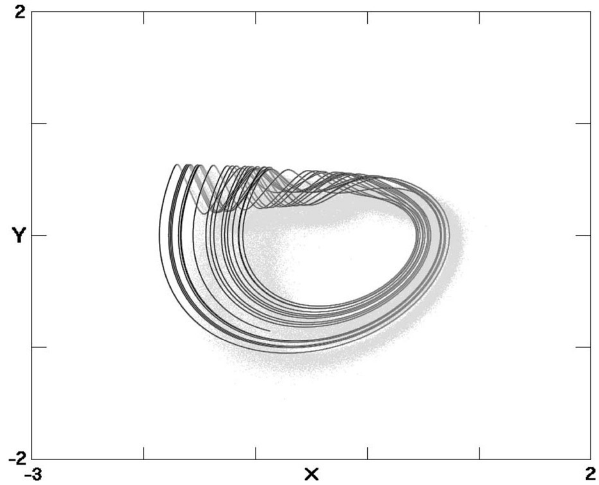

Fig. 4. Numerically calculated phase space plot on the same scale as Fig. 2.

periodic windows that one must avoid. Almost any of the resistors could be made variable to exhibit and study bifurcations and routes to chaos. For example, the chaos persists when the leftmost resistor in Fig. 1 is varied over the range of about 780–2080 Ω, with period doubling of a limit cycle that is clearly visible and audible at both ends of the wide range.

III. Analysis

Except for the brief instants in which the diode conducts, the dynamics are governed by the linear equation

$$\dddot{x} + \ddot{x} + x = 0 \tag{3}$$

whose eigenvalues satisfy the cubic characteristic equation

$$\lambda^3 + \lambda^2 + 1 = 0 \tag{4}$$

and are given by $\lambda = -1.465571,\ 0.232786 \pm 0.792552i$. Thus, the origin is a saddle point with a 1-D stable manifold and a 2-D unstable manifold. The dominant frequency of oscillation is expected to be $f = 0.792552/2\pi RC = 126$ Hz, which is slightly higher than the 100 Hz experimentally observed, with the difference resulting from the nonlinearity and perhaps from the 10% components.

The nonlinear circuit was numerically analyzed by writing (2) as three first-order differential equations

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= -z - x - 10^{-9}\left[\exp(y/0.026) - 1\right]\end{aligned} \tag{5}$$

using a fourth-order Runge–Kutta integrator with adaptive step size [26]. The adaptive step size is important because of the rapid variation in $z(t)$ when the diode briefly conducts. The diode model is not critical but is here taken as $f(y) = I_0 R[\exp(y/\alpha) - 1]$ with $I_0 = 10^{-12}$ A and $\alpha = 0.026$ V, giving a forward voltage drop of ~0.6 V (at room temperature), as is typical for a silicon diode. The factor of R comes from the fact that time is measured in units of RC, and it makes $f(y)$ have units of volts, as required. Initial conditions are not

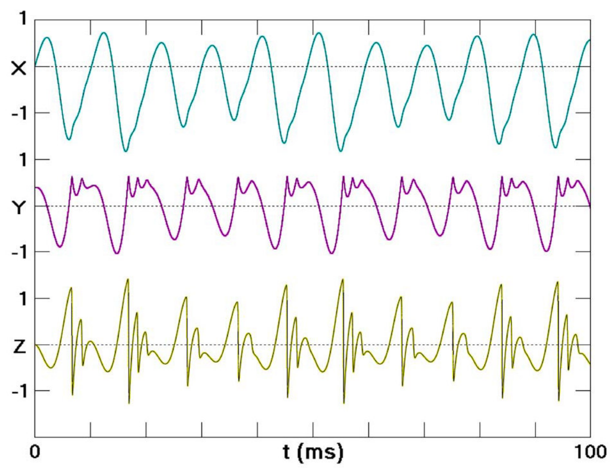

Fig. 5. Numerically calculated waveforms of x and its successive derivatives.

critical, but values of $(x_0, y_0, z_0) = (0, 0.4, 0)$ lie close to the attractor. The resulting phase space plot (y versus x) is shown in Fig. 4 on the same scale as the oscilloscope trace in Fig. 2 and shows good agreement.

Fig. 5 shows the time variation of the voltage x and its successive derivatives y and z. Note that, for $RC = 10^{-3}$ s, time is in units of milliseconds, and the dominant period is approximately 10 ms, corresponding to a frequency of 100 Hz, as expected.

The Lyapunov exponents are calculated [27] to be (0.0735, 0, -1.0735), giving a Kaplan–Yorke dimension [28] of $D_{KY} = 2.0685$ for the resulting strange attractor. In real units, this means that information about the initial state of the system, or equivalently, the accuracy of the prediction of its future state, exponentially decays, on average, with a time constant of $RC/0.0735 \simeq 13.6$ ms.

Another way to display the dynamics is by means of a Poincaré section, one example of which in Fig. 6 shows the value of (x, y) every time y reaches a local maximum or, equivalently, whenever z crosses zero in the downward direction. This choice corresponds to the point of maximum conduction of the diode. The resulting plot has a particularly simple form, with three branches corresponding to the three bounces that typically occur during each cycle. The underlying fractal structure is evident by zooming into one of the lines by a factor of 2×10^4, revealing that it consists of a pair of closely spaced lines, each of which presumably consists of a pair of lines, and so forth, to arbitrarily small scale.

IV. Bifurcations

To examine bifurcations and routes to chaos, a variable parameter A can be added to the equations, e.g., in the damping term to give

$$\dddot{x} + A\ddot{x} + x + f(\dot{x}) = 0 \tag{6}$$

which corresponds to varying the leftmost resistor in Fig. 1 such that $R(\mathrm{k}\Omega) = 1/A$. A numerical solution of (6) using the diode model previously described for $f(\dot{x})$ leads to the bifurcation diagram in Fig. 7. The period doubling at both large and small A

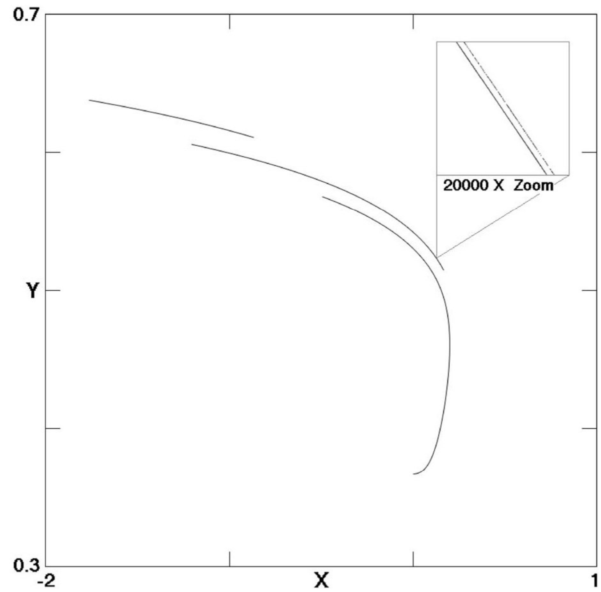

Fig. 6. Poincaré section in the $x-y$ plane at the instant of maximum diode conduction.

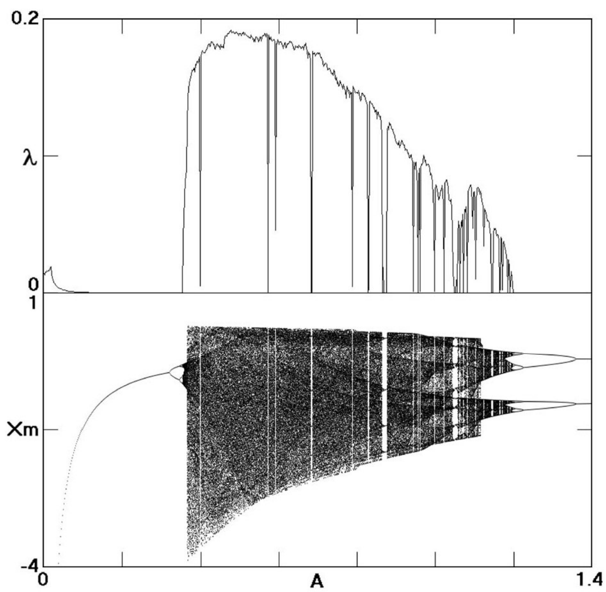

Fig. 7. Largest Lyapunov exponent and the value of x when $\dot{x}$ is a local maximum versus the bifurcation parameter A in (6).

is evident, as well as narrow windows of various periods where the largest Lyapunov exponent is zero. The maximum chaos occurs at about $A = 0.5$ where the largest Lyapunov exponent is about 0.19, the Kaplan-Yorke dimension is about 2.16, the amplitude is large, and periodic windows are less evident.

For most values of A, the equilibrium at the origin ($x = \dot{x} = \ddot{x} = 0$) lies off the attractor, but for five values given by $A = 0.7043, 0.5633, 0.4823, 0.4284,$ and 0.3890, the attractor intersects the origin, and a homoclinic connection exists, as shown in Fig. 8, for the two extreme cases, which correspond to one and five bounces, respectively. For $A = 0.7043$, the eigenvalues for the saddle point at the origin are given by $\lambda = -1.2979,\ 0.2968 \pm 0.8261i$. Since $|-1.2979| > |0.2968|$, the Shilnikov condition [29] is satisfied as it is also for the other four values, thereby constituting a proof of chaos in the system.

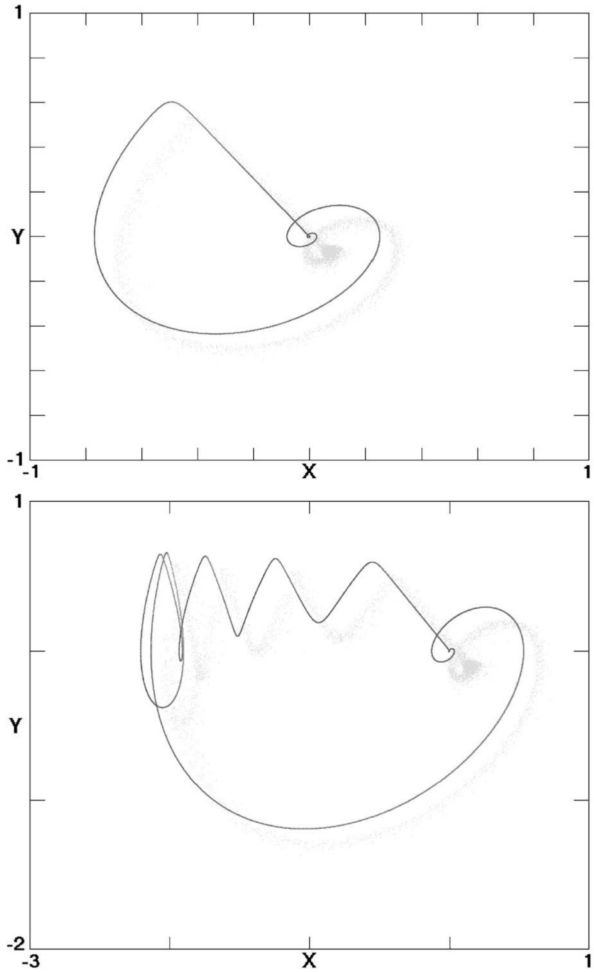

Fig. 8. Homoclinic orbits, with the upper for $A = 0.7043$ and the lower for $A = 0.3890$ in (6).

V. Conclusion

A new chaotic jerk circuit has been described, which was motivated and well predicted by a variant of the simplest differential equation whose solutions are chaotic. The circuit uses no special components, can be scaled over a wide range of frequencies, requires no careful tuning, and is strongly and robustly chaotic. The only reactive components are three identical capacitors, and the only nonlinear element is a diode whose characteristics are not critical. The circuit is easily amenable to numerical and theoretical analysis, particularly in the ideal diode limit where the dynamics are linear subject to a reflecting boundary condition when the diode briefly conducts. It is thus an attractive candidate for all applications of chaotic circuits. Further circuit simplifications might be possible.

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