International Journal of Bifurcation and Chaos, Vol. 21, No. 9 (2011) 2391–2394

DOI: 10.1142/S021812741103009X

A PROPOSED STANDARD FOR THE PUBLICATION OF NEW CHAOTIC SYSTEMS

J. C. SPROTT
Department of Physics, University of Wisconsin,
Madison, WI 53706, USA
sprott@physics.wisc.edu

Received April 15, 2011

With the abundance of chaotic systems that have now been identified and studied, it is prudent to establish a standard for the publication of new examples of such systems and to develop acceptable criteria for their characterization.

Keywords: Chaos; attractor; bifurcation; Lorenz system.

1. Background

By now, hundreds of examples of low-dimensional discrete-time and continuous-time mathematical models have been identified and studied in which the solutions are chaotic, as evidenced, for example, by a positive Lyapunov exponent [Sprott, 2010]. New examples continue to be discovered and submitted for publication in a plethora of nonlinear dynamics journals including this one. These submissions range in detail from simply showing what appears to be an aperiodic orbit in state space to a thorough characterization of the bifurcations and routes to chaos over the entire parameter space with calculation of the spectrum of Lyapunov exponents and other dynamical and topological quantities. Rarely are these papers manifestly wrong, but they often provide little more than just another example of behavior that is well known and thoroughly understood.

2. Proposed Standard

To be considered for publication in a reputable journal, such papers ought to satisfy at least one of the following criteria:

(1) The system should credibly model some important unsolved problem in nature and shed insight on that problem.
(2) The system should exhibit some behavior previously unobserved.
(3) The system should be simpler than all other known examples exhibiting the observed behavior.

These criteria represent a necessary but not sufficient condition for publication. The celebrated Lorenz [1963] system satisfied all three criteria when first published since it dealt with atmospheric turbulence, exhibited sensitive dependence on initial conditions, and was the simplest system in its time known to have these properties. Lorenz [1993] devoted considerable effort to simplifying the system from the original seven-dimensional system with 13 quadratic nonlinearities that was suggested by Saltzman [1962].

3. Simplification

Simplicity is not a well-defined mathematical concept, but without doubt, the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz\end{aligned} \tag{1}$$

with three variables, seven terms, and two quadratic nonlinearities is a simplification of the Saltzman system. It also has three parameters (r, σ, and

b) that Lorenz chose to correspond to physically meaningful quantities. The fact that there are exactly three parameters is no accident. By a linear rescaling of x, y, z, and t, four of the coefficients of the seven terms can be set to 1.0 without loss of generality. Three is thus the minimum number of parameters required to completely characterize the system and the maximum number that avoids redundancy. Absent a compelling reason, newly published systems should include sufficient but not extraneous parameters. When the parameter values are not of order unity (such as 10 or 28), that is a clue that rescaling is likely to produce a simplification.

The choice of where to put the parameters is largely a matter of taste, but some choices permit further simplification. For example, if Lorenz had chosen the coefficient of the y term in the $\dot{y}$ equation as one of the parameters, he might have noticed that chaos persists even when that parameter is set to zero as reported by Zhou *et al.* [2008] or that the rx term is unnecessary if the sign of the parameter is reversed as reported by Lü *et al.* [2002] and further discussed by Sun and Sprott [2009]. An optimal simplification of the coefficients, allowing either sign, leads to the reduced chaotic system

$$\begin{aligned} \dot{x} &= x - y \\ \dot{y} &= xz - 2y \\ \dot{z} &= xy - z, \end{aligned} \tag{2}$$

which is completely specified by only two parameters, one of which has a value of 2, and the other has a value of 1 and can be chosen in five different ways.

However, one can do even better, as shown by van der Schrier and Maas [2000] and further by Munmuangsaen and Srisuchinwong [2009]. By rescaling the variables $(x, y, z) \rightarrow (\sigma x, \sigma y, \sigma z + r)$ and $t \rightarrow t/\sigma$ and taking the limit $r, \sigma \rightarrow \infty$ but in such a way that $R = br/\sigma^2$ remains finite, Eq. (1) can be reduced to the single-parameter system

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -xz \\ \dot{z} &= xy - R, \end{aligned} \tag{3}$$

in which chaos occurs even for $R = 1$.

None of this is meant as a criticism of Lorenz, who would have embraced these simplifications, but it illustrates that advances in computers over the past 50 years now permit us to simplify our systems before rushing to publication.

4. Bifurcations

Given that every system has a unique number of parameters that completely characterize it without redundancy, a comprehensive study of a system requires that the entire parameter space be explored to identify all types of dynamic behaviors and the bifurcation boundaries that separate them. When there are more than two parameters, the current state of the art makes this impractical, and even if it could be done, there is a visualization problem when using the printed page. Perhaps the best one can do is to examine one or more one-dimensional trajectories through the parameter space or a two-dimensional slice through the space. With two parameters, one should attempt a complete characterization, even if only at low resolution in which some features are inevitably missed.

5. Initial Conditions

A complication of making bifurcation diagrams is that the initial conditions must be within the basin of attraction or chaotic sea and there may be multiple attractors. Thus the bifurcation diagrams may look very different depending on how the initial conditions are chosen, and this information is often missing in publications, making it difficult to interpret and reproduce the results. A simple way to remain within the basin of attraction as a parameter is varied is to use the values of the variables at the previous value of the parameter as the initial condition for each new and slightly altered parameter value. In such a case, multiple attractors can often be detected by varying the parameter in the opposite direction, with the coexisting attractors showing up through hysteresis in the vicinity of a bifurcation boundary. Such a method also helps to identify errors due to intermittency and long transients. Better yet is to use many different initial conditions for each parameter value and calculate the spectrum of some quantity that is likely to be different for different coexisting attractors such as their size or Lyapunov exponent.

Papers describing new chaotic systems should include evidence for or against coexisting attractors. Numerous papers in the literature have falsely claimed the absence of chaos in a system based on the existence of a stable equilibrium. As Strogatz [1994] points out, even the Lorenz system in Eq. (1) has a strange attractor coexisting with a stable equilibrium for $24.06 < r < 24.74$ with $\sigma = 10$ and $b = 8/3$.

6. Optimization

There was a time when it sufficed to select an arbitrary value of the parameters which gives rise to chaos, but modern computers now allow us to find those values that optimize the chaoticity. An obvious choice is to maximize the largest Lyapunov exponent, but this choice is poor for a continuous-time system since the Lyapunov exponent can be made as large as desired through a linear rescaling of time that otherwise has no effect on the dynamics. Such an optimization requires a constraint and depends on the parametrization and thus fails to give a unique result. Better yet is a dimensionless parameter such as the Kaplan–Yorke dimension [Kaplan & Yorke, 1979] whose value bears an obvious and important relation to the dimension of the state space. For example, the Lorenz system in Eq. (1) with $(\sigma, r, b) = (10, 28, 8/3)$ has a Kaplan–Yorke dimension of 2.062, whereas the optimized form in Eq. (3) has a dimension of 2.235 at $R = 3.4693$ [Sprott, 2007]. If the goal of a paper is to show that a system exhibits chaos, why not take the additional step of finding out how chaotic it can be?

7. Uncertainty

Most papers reporting new chaotic systems are based on numerical experiments. It has long been standard practice in science to include an estimate of the uncertainty in experimental results, but we have fallen out of the habit of doing this. The chaos literature is filled with estimates of Lyapunov exponents and other related quantities with no indication of the uncertainty and often with many more digits than are credible.

Uncertainties arise from systematic and statistical errors. Systematic errors have many sources, especially for continuous-time systems where discretization and round-off errors occur in the numerical procedures used to calculate the orbit and the Lyapunov exponent. These errors are best estimated by benchmarking the results against well-established values for similar model systems.

Statistical errors usually arise from using a finite-time approximation of a quantity that is properly described by an infinite-time integral. The proper way to estimate such errors is to perform many realizations of the calculation for different orbits on the attractor or in the chaotic sea and to calculate the mean and standard deviation of these values. Fortunately, for a chaotic system, only a tiny change in initial condition is required to produce a highly uncorrelated orbit because of the sensitive dependence on initial conditions. Even doing the calculation twice will usually suffice to determine the approximate number of significant digits in a result. If putting error bars on the results is too onerous, let us at least quote only those digits that are significant and have a firm basis for deciding.

8. Figures

Even inexpensive computers are now capable of producing very accurate results and making high-quality graphics. Yet many submissions include Poincaré sections with dots that are too large, bifurcation diagrams in which the bifurcation parameter is adjusted in overly coarse steps, initial transients are not allowed to decay, and axes that are inadequately labeled or labeled with lettering that is too small to read. Authors should develop or acquire tools that produce good quality figures with appropriately sized dots, high resolution, legible lettering, and anti-aliasing so that diagonal lines do not appear jagged. Publication quality figures may require many hours or days of computation, but that is a small fraction of the total writing and publication time. Since most articles are now downloaded as pdf files, there is no reason to avoid using color, although one must consider how the figure will look in the printed journal. As a final check, be sure the paper includes all the information required for the reader to replicate the calculations that produced the figures such as parameter values, initial conditions, scale factors, step size, and the numerical method.

9. Conclusion

If nonlinear science is to enjoy a good reputation, it is important that our publications meet a minimum standard and that we strive for completeness and accuracy. The suggestions here cover many of the common faults with submitted manuscripts and the reasons such papers often receive unfavorable reviews. Let us work to improve the quality of our research and publications for our common good.

References

Kaplan, J. & Yorke, J. [1979] "Chaotic behavior of multidimensional difference equations," in *Functional Differential Equations and Approximation of Fixed Points*, eds. Peitgen, H.-O. & Walther, H.-O., Lecture Notes in Mathematics, Vol. 730, pp. 477–482.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Lorenz, E. N. [1993] *The Essence of Chaos* (University of Washington Press, Seattle).

Lü, J., Chen, G., Cheng, D. & Celikovsky, S. [2002] "Bridge the gap between the Lorenz system and the Chen system," *Int. J. Bifurcation and Chaos* **12**, 2917–2926.

Munmuangsaen, B. & Srisuchinwong, B. [2009] "A new five-term simple chaotic attractor," *Phys. Lett. A* **373**, 4038–4043.

Saltzman, B. [1962] "Finite amplitude free connection as an initial value problem — I," *J. Atmos. Sci.* **19**, 329–341.

Sprott, J. C. [2007] "Maximally complex simple attractors," *Chaos* **17**, 033124-1–6.

Sprott, J. C. [2010] *Elegant Chaos: Algebraically Simple Chaotic Flows* (World Scientific, Singapore).

Strogatz, S. H. [1994] *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering* (Addison-Wesley Publishing Company, Reading, MA), p. 330.

Sun, K. & Sprott, J. C. [2009] "Dynamics of a simplified Lorenz system," *Int. J. Bifurcation and Chaos* **19**, 1357–1366.

van der Schrier, G. & Maas, L. R. M. [2000] "The diffusionless Lorenz equations; Shil'nikov bifurcations and reduction to an explicit map," *Phys. Nonlin. Phenom.* **141**, 19–36.

Zhou, W., Xu, Y., Lu, H. & Pan, L. [2008] "On dynamics analysis of a new chaotic attractor," *Phys. Lett. A* **372**, 5773–5777.