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# Chaos in the Planar Two-Body Coulomb Problem with a Uniform Magnetic Field

Vladimir Zhdankin
Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA
e-mail: zhdankin@wisc.edu

J. C. Sprott
Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, USA
email: sprott@physics.wisc.edu

**Abstract**

The dynamics of the classical two-body Coulomb problem in a uniform magnetic field are explored numerically in order to determine when chaos can occur. The analysis is restricted to the configuration of planar particles with an orthogonal magnetic field, for which there is a four-dimensional phase space. Parameters of mass and charge are chosen to represent physically motivated systems. To check for chaos, the largest Lyapunov exponent and Poincaré section are determined for each case. We find chaotic solutions when particles have equal signs of charge. We find cases with opposite signs of charge to be numerically unstable, but a Poincaré section shows that chaos occurs in at least one case.

**Keywords:** low-dimensional chaos, non-linear dynamics, Hamiltonian systems.

## 1 Introduction

Chaotic systems are bounded dynamical systems that exhibit a sensitive dependence to initial conditions. Generally, they are aperiodic and have governing equations that are nonlinear [1]. There is an interest in studying chaos in systems with a small number of degrees of freedom, such as the three-body gravitational problem [2]. The presence of chaos in such simple systems suggests that it is a fundamental feature of nature. By studying the simplest chaotic systems, we can better understand how chaos arises.

The two-body Coulomb problem in a uniform magnetic field is one of the simplest classical systems that can exhibit chaos. This is particularly true for the two-dimensional case, where the charged particles undergo planar motion and the magnetic field is directed orthogonal to the plane. If the system is simplified any further, for example by removing the magnetic field or one particle, then the equations become integrable. Thus the problem is interesting for its simplicity.

The two-body Coulomb problem in a uniform magnetic field is also applicable to a number of physical situations. It is commonly studied in the context of the classical hydrogen atom in a magnetic field [3]. Although quantum mechanics is essential for making accurate predictions for systems on the atomic scale, the classical model can approximate high energy states and be useful in trying to understand quantum chaos [4]. The problem may also be relevant for describing the interactions of ions in a strong magnetic field. This is appropriate, for example, in the magnetic fields found near white dwarfs and neutron stars, where the properties of matter are drastically modified [5, 6, 7]. In this case, the magnetic field is approximately uniform at the level of particle interactions. Information about the microscopic interactions in such systems could lead to observable global consequences, similar to what has been found in large populations of coupled oscillators [8].

The problem has been studied to various extents by both analytical and numerical means. On the analytical side, Curilef and Claro obtained solutions for the two-dimensional problem in the special case where particles have equal mass and equal magnitude of charge [9]. In addition, Pinheiro and MacKay have performed a mathematically rigorous analysis of the general problem in a recent series of two papers [10, 11]. For the two-dimensional problem, they found that all solutions are bounded in space and that the special case of equal gyrofrequencies ($q_1/m_1 = q_2/m_2$) is integrable. Furthermore, they inferred that chaos exists in cases with opposite signs of charge unless gyrofrequencies sum to zero. However, they did not establish what happens when the gyrofrequencies sum to zero and whether there is chaos for cases with equal signs of charge. Due to the complicated nature of the general solutions, the problem is well suited for numerical analysis. Previous numerical studies have been performed primarily for the special case of the classical hydrogen atom in three-dimensional space, which was found to be chaotic by Schmelcher and Cederbaum [12], as well as Friedrich and Wintgen [3]. However, numerical studies of the problem for other sets of particles are scant or nonexistent.

The goal of this paper is to investigate the question of what dynamics are possible in the general solution of the two-dimensional problem. Since the special case in which charges sum to zero (which includes proton-electron, positron-electron) has been studied by others [9, 10, 12, 3], we consider cases in which the two charges do not sum to zero. The charges and masses are chosen to represent cases of physical significance, which reduces the size of the parameter space to be explored. The equations are then solved numerically and two methods are used to test for chaos. The first is computation of the largest Lyapunov exponent, which measures the rate at which nearby trajectories diverge and is positive ($\lambda > 0$) for chaotic solutions. The second is construction of the Poincaré section, which exhibits a chaotic sea for chaotic solutions. As a result of our numerical analysis, we discover many chaotic solutions when charges have equal signs. On the other hand, we find that cases with opposite signs of charge are numerically unstable, which prevents the Lyapunov exponent from converging accurately. However, a Poincaré section suggests that chaos does occur in at least one of these cases. We also investigate the special case in which gyrofrequencies sum to zero, but are unable to find any chaotic solutions.

## 2 Equations of motion

In this section, the equations of motion for the problem are presented. First, consider the problem in Cartesian coordinates, where two charged particles confined to the $x$-$y$ plane interact via the Coulomb force in the presence of a uniform magnetic field oriented in the $z$ direction. Then there is an eight-dimensional phase space, which can be written in terms of the positions and kinetic momenta of the particles ($x_1$, $y_1$, $p_{x1}$, $p_{y1}$, $x_2$, $y_2$, $p_{x2}$, $p_{y2}$). The equations of motion read

$$\begin{aligned}
\dot{p}_{x1} &= \frac{k_e q_1 q_2 (x_1 - x_2)}{[(x_1-x_2)^2 + (y_1-y_2)^2]^{3/2}} + \frac{q_1 B}{m_1} p_{y1} \\
\dot{p}_{y1} &= \frac{k_e q_1 q_2 (y_1 - y_2)}{[(x_1-x_2)^2 + (y_1-y_2)^2]^{3/2}} - \frac{q_1 B}{m_1} p_{x1} \\
\dot{x}_1 &= \frac{1}{m_1} p_{x1} \\
\dot{y}_1 &= \frac{1}{m_1} p_{y1} \\
\dot{p}_{x2} &= \frac{k_e q_1 q_2 (x_2 - x_1)}{[(x_1-x_2)^2 + (y_1-y_2)^2]^{3/2}} + \frac{q_2 B}{m_2} p_{y2} \\
\dot{p}_{y2} &= \frac{k_e q_1 q_2 (y_2 - y_1)}{[(x_1-x_2)^2 + (y_1-y_2)^2]^{3/2}} - \frac{q_2 B}{m_2} p_{x2} \\
\dot{x}_2 &= \frac{1}{m_2} p_{x2} \\
\dot{y}_2 &= \frac{1}{m_2} p_{y2}
\end{aligned} \tag{1}$$

where $B$ is the magnetic field strength, $m_1$ and $m_2$ are the masses of the particles, $q_1$ and $q_2$ are the electrical charges of the particles, and $k_e$ is Coulomb's constant. There are four conserved quantities: the energy $E$, the $x$-component of linear momentum $P_x$, the $y$-component of linear momentum $P_y$, and angular momentum $L$. These quantities are given by

$$\begin{aligned}
E &= \frac{p_{1x}^2 + p_{1y}^2}{2m_1} + \frac{p_{2x}^2 + p_{2y}^2}{2m_2} - \frac{k_e q_1 q_2}{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}} \\
P_x &= p_{1x} + p_{2x} - q_1 B y_1 - q_2 B y_2 \\
P_y &= p_{1y} + p_{2y} + q_1 B x_1 + q_2 B x_2 \\
L &= (x_1 p_{1y} - y_1 p_{1x}) + (x_2 p_{2y} - y_2 p_{2x}) \\
&\quad + \frac{1}{2} B q_1 (x_1^2 + y_1^2) + \frac{1}{2} B q_2 (x_2^2 + y_2^2)
\end{aligned} \tag{2}$$

Although the full equations of motion (Eq. 1) can be solved numerically, it is preferable to use equations in a reduced phase space. Coordinate transformations derived by Pinheiro and MacKay [10] use the conservation laws to reduce the number of phase space dimensions from eight to four. The following transformations will require that charges have a nonzero sum, $q_1 + q_2 \neq 0$. The case for $q_1 + q_2 = 0$ must be treated separately because the center of mass undergoes a drift, and that transformation will not be considered here. The reduced phase space consists of the variables $r$, $p_r$, $\phi$, and $p_\phi$, defined by

$$\begin{aligned} r &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ p_r &= \frac{[p_{1x} - (q-1)p_{2x}](x_1 - x_2) + [p_{1y} - (q-1)p_{2y}](y_1 - y_2)}{qr} \\ \phi &= \frac{1}{2\mu} \arctan\left(\frac{(p_{1x} + p_{2x})(x_1 - x_2) + (p_{1y} + p_{2y})(y_1 - y_2)}{(p_{1y} + p_{2y})(x_1 - x_2) - (p_{1x} + p_{2x})(y_1 - y_2)}\right) \\ p_\phi &= (p_{1x} + p_{2x})^2 + (p_{1y} + p_{2y})^2 \end{aligned} \tag{3}$$

and the new parameters are defined as

$$\begin{aligned} m &= \frac{1 + m_2}{m_2} \\ q &= \frac{1 + q_2}{q_2} \\ \mu &= B(1 + q_2) \\ p_\theta &= -2B(1 + q_2)L + P_x^2 + P_y^2 \\ \epsilon &= \frac{q_2 B}{1 + q_2} = \frac{q-1}{q^2}\mu \end{aligned} \tag{4}$$

where units have been chosen such that $m_1 = 1$ and $q_1 = 1$. It is clear that $r > 0$ and $p_\phi > 0$, with singularities located at $r = 0$ and at $p_\phi = 0$. Also note that $\phi$ is periodic on the interval $(0, \frac{\pi}{\mu})$.

The equations of motion can obtained by differentiating Eq. 3 with respect to time and substituting the Cartesian equations of motion (Eq. 1), or alternatively they can be derived from the Hamiltonian (representing conserved energy),

$$\begin{aligned} H = {} & \frac{m}{2}p_r^2 + \frac{m}{8}\frac{(p_\theta - p_\phi)^2}{\mu^2 r^2} + \frac{m\epsilon^2}{8}r^2 + \frac{m\epsilon}{4\mu}(p_\theta + p_\phi) + \left(1 - \frac{m}{q}\right)\left(\frac{q-2}{2q}\right)p_\phi \\ & + \frac{k_e}{(q-1)r} + \left(1 - \frac{m}{q}\right)\left[p_r \sin(2\mu\phi) - \frac{1}{2}(\epsilon r + \frac{p_\theta - p_\phi}{\mu r})\cos(2\mu\phi)\right]p_\phi^{1/2} \end{aligned} \tag{5}$$

by using Hamilton's equations for each component of momentum and position,

$$\dot{x}_i = \frac{\partial H}{\partial p_i}\ , \ \dot{p}_i = -\frac{\partial H}{\partial x_i} \tag{6}$$

The equations of motion in the reduced coordinates are then

$$\begin{aligned}
\dot{r} &= mp_r + (1 - \frac{m}{q})\sin(2\mu\phi)p_\phi^{1/2} \\
\dot{p}_r &= \frac{m}{4}\frac{(p_\theta - p_\phi)^2}{\mu^2 r^3} - \frac{m\epsilon^2}{4}r + \frac{k_e}{(q-1)r^2} + \frac{1}{2}(1 - \frac{m}{q})(\epsilon - \frac{p_\theta - p_\phi}{\mu r^2})\cos(2\mu\phi)p_\phi^{1/2} \\
\dot{\phi} &= (1 - \frac{m}{q})(\frac{q-2}{2q}) + \frac{m}{4}(\frac{\epsilon}{\mu} - \frac{p_\theta - p_\phi}{\mu^2 r^2}) \\
&\quad + \frac{1}{2}(1 - \frac{m}{q})\left[p_r \sin(2\mu\phi) - \frac{1}{2}(\epsilon r + \frac{p_\theta - 3p_\phi}{\mu r})\cos(2\mu\phi)\right] p_\phi^{-1/2} \\
\dot{p}_\phi &= -2\mu(1 - \frac{m}{q})\left[p_r \cos(2\mu\phi) + \frac{1}{2}(\epsilon r + \frac{p_\theta - p_\phi}{\mu r})\sin(2\mu\phi)\right] p_\phi^{1/2}
\end{aligned} \tag{7}$$

The five parameters are the relative mass $m$ ($\geq 1$), the relative charge $q$, the strength of the magnetic field $\mu$, the strength of the Coulomb force $k_e$ ($\geq 0$), and the initial total momentum $p_\theta$. If the first particle is taken to be the less massive of the pair, then $1 \leq m \leq 2$.

To obtain numerical solutions, a fourth-order Runge-Kutta algorithm with an adaptive step size is used. We prefer this method over a symplectic integrator because it is much simpler to implement for the given equations of motion. The solutions are independently confirmed using MATLAB and PowerBASIC. The accuracy in each case is primarily checked by monitoring the energy. As numerical error accumulates, the energy given by Eq. 5 drifts away from the initial energy. Since energy is conserved in the actual solution, the energy drift is a signature of numerical error. Typically, we demand that the energy stays constant to six significant digits throughout the simulation.

# 3 Results

The parameters of $m$ and $q$ are chosen from the physically motivated cases in Table 1. The table also shows whether any chaotic solutions were found. These cases cover a range of parameter values, although there are other combinations of particles that may be interesting but were not considered here.

The particles shown in Table 1 are the proton $p$, electron $e$, deuteron $d$, triton $t$ (tritium nucleus), helion $h$ (helium-3 nucleus), and alpha particle $\alpha$ (helium-4 nucleus). Additionally, the antiparticles $\bar{p}$ and $\bar{d}$ are included in a case. The case of $\bar{d}$-$\alpha$ is very unlikely to occur in nature, but is interesting because of the fact that the gyrofrequencies sum to zero, for which Pinheiro and MacKay were unable to establish what happens [10].

Out of the cases listed in Table 1, four were found to exhibit chaos. Parameters and initial conditions that give a chaotic solution for these cases are listed in Table 2. An estimation of the largest Lyapunov exponent is also given. An example of possible trajectories in Cartesian coordinates is shown in Fig. 1.

The largest Lyapunov exponent could not be computed accurately for $\bar{p}$-$\alpha$. This is due to numerical error (as deduced from energy drift) that accumulates due to close approaches of the particles. Since the Coulomb force in this case is attractive, the particles can come arbitrarily close to the singularity at $r = 0$, and this causes problems over long

Table 1: Some possible two-body problems with $q_1 + q_2 \neq 0$

| 1 | 2 | $m_2/m_1$ | $q_2/q_1$ | $m$ | $q$ | Chaotic? |
|---|---|---|---|---|---|---|
| $p$ | $p$ | 1 | 1 | 2 | 2 | No, integrable |
| $p$ | $d$ | 2 | 1 | 1.5 | 2 | No |
| $p$ | $h$ | 3 | 2 | 4/3 | 1.5 | Yes |
| $p$ | $t$ | 3 | 1 | 4/3 | 2 | Yes |
| $p$ | $\alpha$ | 4 | 2 | 1.25 | 1.5 | No |
| $d$ | $t$ | 1.5 | 1 | 5/3 | 2 | Yes |
| $d$ | $\alpha$ | 2 | 2 | 1.5 | 1.5 | No, integrable |
| $e$ | $\alpha$ | 8000 | -2 | 1.000125 | 0.5 | No |
| $\bar{p}$ | $\alpha$ | 4 | -2 | 1.25 | 0.5 | Yes |
| $\bar{d}$ | $\alpha$ | 2 | -2 | 1.5 | 0.5 | No |

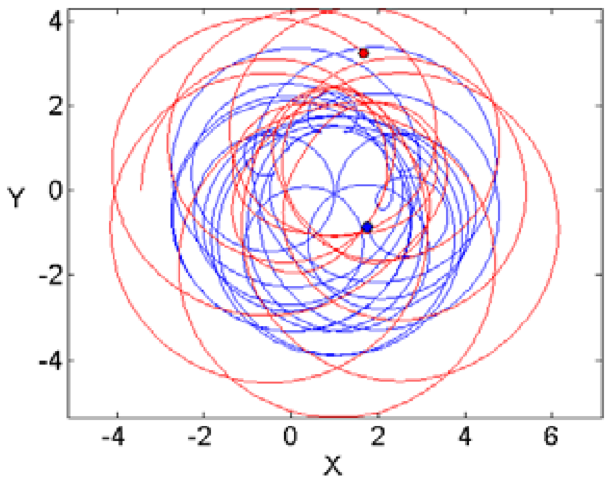


Figure 1: An example of possible trajectories for the two-body Coulomb problem in a uniform magnetic field. The parameters are taken from the chaotic case of a deuteron (blue) and triton (red).

simulations. This prevents computation of the Lyapunov exponent since averaging over a long simulation time is required, but it is still possible to construct a Poincaré section by collecting points until the error becomes too large.

Poincaré sections are constructed by plotting ($r$,$p_r$,$p_\phi$) when $\phi$ crosses a chosen value. This is performed for a set of representative initial conditions. To further reduce the dimensionality of the Poincaré section, the initial conditions are chosen to all have equal energies. This restricts the Poincaré section to a two-dimensional surface of constant energy. Chaotic solutions fill a region of the Poincaré section known as the chaotic sea, while quasiperiodic solutions show up as closed curves.

The surface of constant energy for $p$-$h$ is ellipsoidal, so it is possible to show the Poincaré section as two projections on the $r$-$p_r$ plane. These projections are shown in Fig. 2. The Poincaré section predominantly consists of a chaotic sea, with some islands of quasiperiodicity.

The surface of constant energy has a more complicated topology for the cases of $p$-$t$

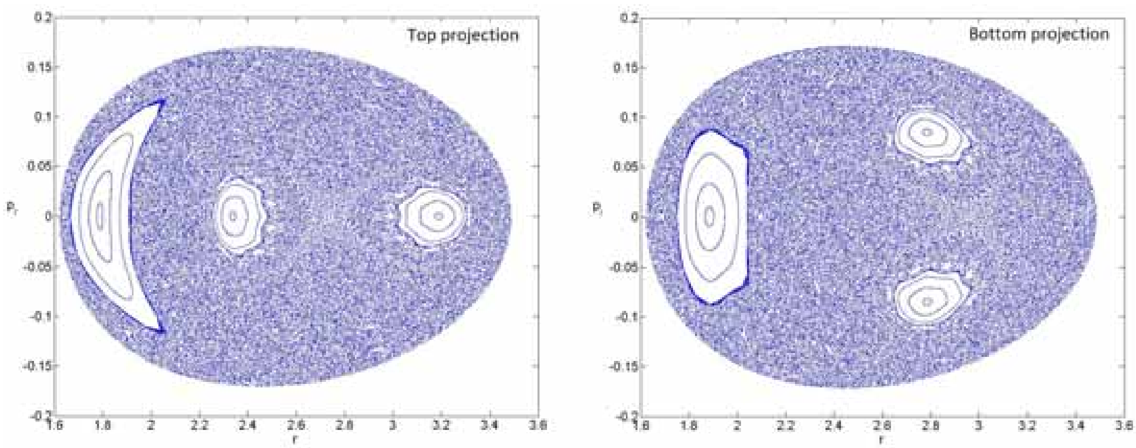


Figure 2: Poincaré section for the two-body system of a helion and proton taken at $\phi = \pi/2$. The surface of constant energy can be conveniently separated into two projections onto the $r$-$p_r$ plane.

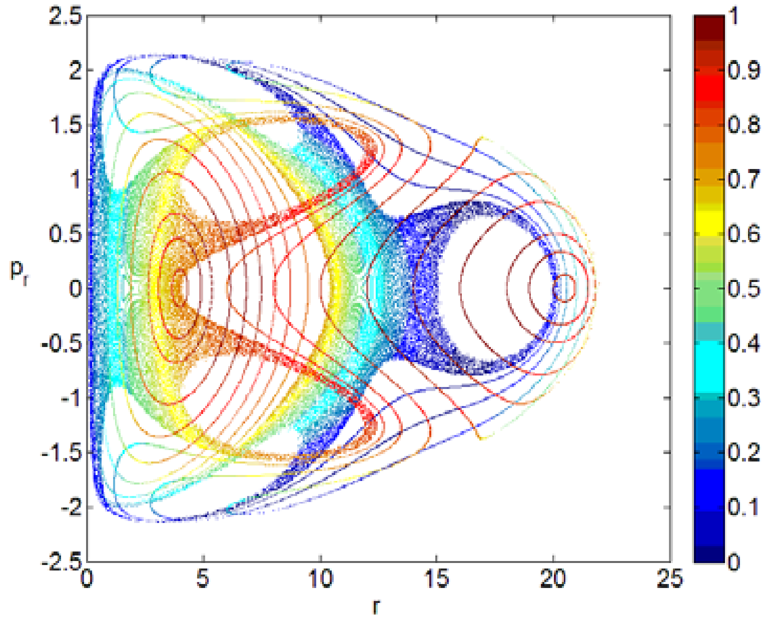


Figure 3: The projected Poincaré section for the two-body system of a proton and triton taken at $\phi = 0$. The color of each point indicates the $p_\phi$ position.

Table 2: Parameters and initial conditions for chaotic cases

| 1 | 2 | $m$ | $q$ | $\mu$ | $p_\theta$ | $k_e$ | $[r, p_r, \phi, p_\phi]_{t=0}$ | Remarks |
|---|---|---|---|---|---|---|---|---|
| $h$ | $p$ | 4 | 3 | -1 | 1 | 1 | $[2.8, 0, 0, 0.9370]$ | $\lambda \approx 0.0155$ |
| $t$ | $p$ | 4 | 2 | -1 | 2 | 1 | $[0.4, 0, 0, 3.1109]$ | $\lambda \sim 0.02$ |
| $d$ | $t$ | 5/3 | 2 | 1 | 3 | 1 | $[4.5, 0, 0, 4.8038]$ | $\lambda \approx 0.0175$ |
| $\bar{p}$ | $\alpha$ | 1.25 | 0.5 | -0.5 | 1 | 3 | $[3.0, 0, 0, 1.4102]$ | Numerically unstable |

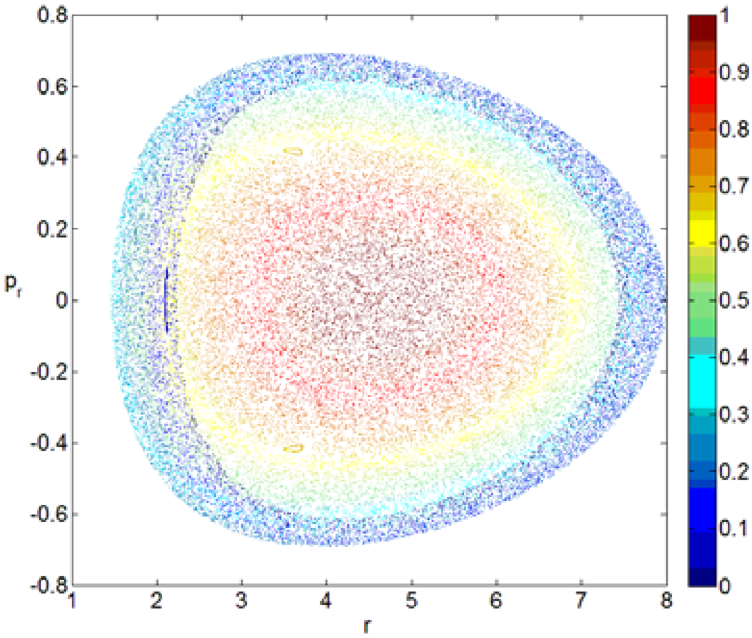


Figure 4: The projected Poincaré section for the two-body system of a deuteron and triton taken at $\phi = \frac{\pi}{2|\mu|}$. The color of each point indicates the $p_\phi$ position.

and $d$-$t$. In these cases, the projection of the Poincaré section onto the $r$-$p_r$ plane is taken, and then a coloring scheme is used to represent the $p_\phi$ value at each point. The resulting Poincaré sections are shown in Fig. 3 and Fig. 4. The case of $p$-$t$ has a phase space with a roughly equal amount of chaotic and quasiperiodic regions, while the case of $d$-$t$ almost exclusively consists of a chaotic sea.

The Poincaré section for $\bar{p}$-$\alpha$ is shown in Fig. 5. One notable difference from the other cases is that the Poincaré section now extends to infinity in $p_r$ as $r$ approaches zero. This is due to the attractive Coulomb force, which allows the particles to come arbitrarily close to each other.

No chaos was found for case of $\bar{d}$-$\alpha$. This is an interesting case because the gyrofrequencies sum to zero, so $q_1/m_1 + q_2/m_2 = 0$ or equivalently $q = 2 - m$. To answer the question posed by Pinheiro and MacKay about what happens in this case, a search for chaos in general cases with $q = 2 - m$ was made. No cases that exhibited chaos were found, which suggests that the solutions are all periodic or quasiperiodic.

An automated search algorithm was used to find chaotic solutions with $m$ and $q$ in the interval $[1, 2]$. In each case, initial conditions were taken randomly from a Gaussian

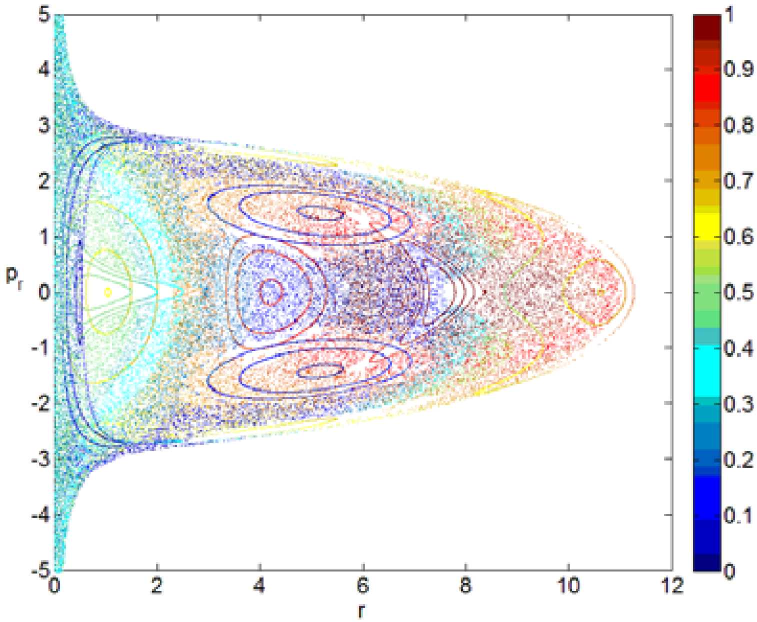


Figure 5: The projected Poincaré section for the two-body system of an antiproton and alpha particle taken at $\phi = 0$. The color of each point indicates the $p_\phi$ position. Note that as $r \to 0$, $p_r \to \infty$.

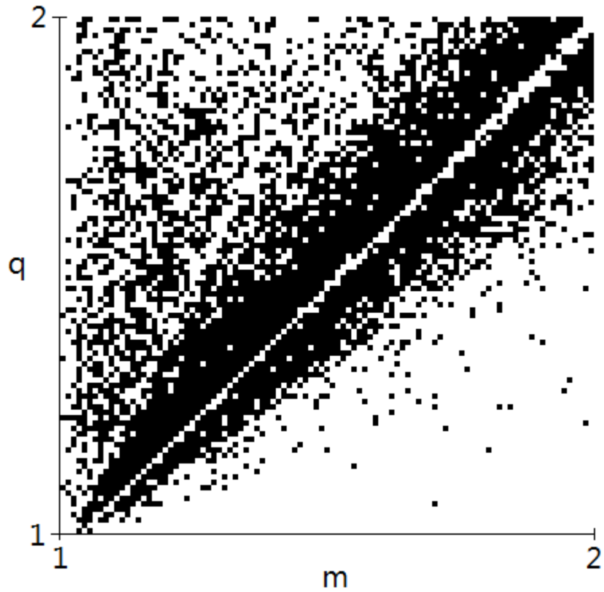


Figure 6: A plot showing the regions where positive Lyapunov exponents were measured by an automated search algorithm that varied $m$ and $q$ with random initial conditions. These regions are indicated by darkened pixels. There is an observed lack of chaos along the line $q = m$, which is an integrable case. Also, no chaos was found along the line $q = 2m - 1$.

distribution. A plot marking the locations where positive Lyapunov exponents were measured is shown in Fig. 6. There is a prominent lack of chaotic solutions on the line $q = m$, where solutions are integrable [10]. There is also a visible lack of chaotic solutions on the line $q = 2m - 1$, suggesting that those cases may also be integrable. However, points nearby both of these lines are often chaotic, suggesting that these integrable solutions are a set of measure zero.

# 4 Conclusion

The classical system of the two-body Coulomb problem in a uniform magnetic field was investigated numerically in the restricted case of planar particles with an orthogonal magnetic field. The goal was to determine under which conditions the system can exhibit chaos. The analysis was done in a four-dimensional phase space with values of mass and charge chosen to represent common physical particles.

Chaos was confirmed by computing a positive largest Lyapunov exponent and observing a chaotic sea in the Poincaré section. For the case of charges with equal signs, which has been largely unexplored in the past, we found several new chaotic solutions. However, these cases may be of limited physical interest since the charges would repel in the third dimension if perturbed out of the plane. For the case of charges with opposite signs, the largest Lyapunov exponent would not converge accurately due to numerical issues, but a Poincaré section showed that chaos occurs in the antideuteron-alpha particle case. The observation of chaos in this case is consistent with the analytical study of Pinheiro and MacKay [10], and can possibly be studied in greater detail if collisions are regularized. An automated search for chaotic solutions was performed for the problem where gyrofrequencies sum to zero, and no chaotic solutions were detected. Finally, the location of chaotic solutions in the region $1 < m < 2$ and $1 < q < 2$ were studied. There were two prominent regions with no chaotic solutions, along the lines $q = m$ and $q = 2m - 1$. The first of these is known to be integrable, but it is not clear why the second criterion would preclude chaos. Future analytical study of this case could be interesting, but is outside of the scope of the current paper.

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