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Confirmation of Persistent Chaos in High Dimensions

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"*Order and chaos belong together in God's creation, but potential chaos of another kind was introduced when God created human beings endowed with freedom.*" –Bernhard W. Anderson.

In this letter, we prove rigorously the persistence property of chaos in high dimensions stated as a conjecture in [1]. The idea of the proof is based on a simple remark on the form of the variation of bifurcation parameters. The relevance of this result is that persistent chaos in high dimensions was observed and tested numerically, but without any rigorous proof. Also, this proof shows that persistent chaos still occurs in typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps.

1 Introduction

Dynamical persistence means that a behavior type, *i.e.*, equilibrium, oscillation, or chaos does not change with functional perturbation or parameter variation. Mathematically, *persistent chaos* (p-chaos) of degree p for a dynamical system can be defined as follows: Assume a map $f_\xi : X \to X \subset \mathbb{R}^d$ depends on a parameter $\xi \in \mathbb{R}^k$. The map f_ξ has chaos of degree-p on an open set $\mathcal{O} \subset X$ that is persistent for $\xi \in \mathcal{U} \subset \mathbb{R}^k$ if there is a neighborhood $\mathcal{N}$ of $\mathcal{U}$ such that $\forall$ $\xi \in \mathcal{N}$, the map f_ξ retains at least $p \geq 1$ positive Lyapunov characteristic exponents (LCEs) for Lebesgue almost every X in $\mathcal{O}$.

In this definition, the choice of p is arbitrary. For example, the condition where p equals the number of positive LCEs is a very strict constraint; specifying a minimum p or ratio of p to the maximum number of positive exponents are weaker. Flexibility allows one to analyze (say) systems with 10^6 unstable directions in which a change in 1% of the geometry is undetectable, but a 50% change is. Robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space. The existence of these windows in some chaotic regions means that small changes of the parameters would destroy the chaos, implying the fragility of this type of chaos [8]. Hence, the notion of persistent chaos differs from that of a robust chaotic attractor in several ways. In particular, the uniqueness of the attractor is not required on the set U since there is little physical evidence indicating that such strict forms of uniqueness are present in many complex physical systems. For a low-dimensional system, uniqueness is markedly more difficult to establish.

Some real-world systems show dynamical persistence, for example, the decline of biological species in natural habitats as shown in [5]. The main issues for decline of species are concerned with the effects of spatial synchrony and dynamical chaos. In this case, persistence can be viewed as a problem and eradication as an achievement. For this case, ecologists and epidemiologists use very similar mathematical structure of the population dynamics. In [6], it was shown that a model of a ninth-order truncated ordinary differential equation (ODE) model of 3-D incompressible convection displays *cycling chaos*, *i.e.*, the attractors consist of a heteroclinic cycle between chaotic sets. This behavior is robust to perturbations that preserve the symmetry of the system. In [7], it was observed that a model of discretionary consumption dynamics (*i.e.*, an endogenous transformation of a society inhabited by boundedly rational interactive consumers) shows the existence of a persistent chaotic regime of different social standards. Other examples can be found in [3-4].

Some results about persistence of chaos in high dimensions can be found in [1-2]. It was shown in [1] that as the dimension of a typical dissipative dynamical system is increased, the number of positive Lyapunov exponents increases monotonically, and the number of parameter windows with periodic behavior decreases. The method of analysis is an extensive statistical survey of *universal approximators* introduced in [9] given by single-layer recurrent neural networks of the form

$$x_t = f_{s,\beta,\omega} = \beta_0 + \sum_{i=1}^{n} \beta_i \tanh s \left(\omega_{i0} + \sum_{j=1}^{d} \omega_{ij} x_{t-j} \right) \tag{1.1}$$

which are maps from $\mathbb{R}^d$ to $\mathbb{R}$. Here n is the number of neurons, d is the number of time lags which determines the system's input embedding dimension, and s is a scaling factor for the connection weights ω_{ij}. The initial condition is $(x_1, x_2, ..., x_d)$, and the state at time t is $(x_t, x_{t+1}, ..., x_{t+d-1})$. In [1] the $k = n\,(d+2) + 1$-dimensional parameter space was taken as follows: (i) $\beta_i \in [0,1]$ is uniformly distributed and rescaled to satisfy $\sum\limits_{i=1}^{n} \beta_i^2 = n$, (ii) ω_{ij} is normally distributed with zero mean and unit variance, and (iii) the initial condition $x_j \in [-1,1]$ is uniform. The distributions of β_i and ω_{ij} are denoted by m_β and m_ω and form a product measure on the space of parameters and initial conditions. We note that the approximation theorems of [9] and time-series embedding of [10] establish an equivalence between these neural networks and general dynamical systems [2].

These results lead to the following conjecture for persistent chaos in high dimensions formulated in [1] as follows: *Given $f_{s,\beta,\omega}$, if k and d are large enough, the probability with respect to $m_\beta \times m_\omega$ of the set (β, ω), defined above (the* parameters β_i and ω_{ij}*) is large and approaches* 1 *as $k \to \infty$.*

This conjecture means that if $f_{s,\beta,\omega}$ is a network of the form (1.1) with a sufficiently large number d of dimensions and number of parameters $k = n(d+2)+1$, then there exists an open set of significant positive Lebesgue measure in parameter space $\mathbb{R}^k$ for which chaos will be degree-p persistent, with $p \to \infty$ as $d \to \infty$.

Indeed, if we consider two maps $f_{s,\beta,\omega}$ and $f_{r,\alpha,\varphi}$ of the form (1.1), then it is easy to remark that both maps display the same dynamical behavior (the same x_t) when

$$\begin{cases} \alpha_i = \beta_i, i = 0, n \\ \varphi_{ij} = \frac{\omega_{ij}}{r} s, i = 1, n, j = 0, d \end{cases} \tag{1.2}$$

That is, $x_t = f_{r,\beta,\frac{\omega}{r}s} = f_{s,\beta,\omega}$ for all r, α, φ satisfying (1.2). Thus the above conjecture is true not only for the announced values in [1], but for all $d, k \in \mathbb{N}$. We note that since networks are universal function approximators, then persistent chaos still occurs in typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps.

2 Conclusion

In this letter, a proof of the persistence property of chaos in high dimensions stated as a conjecture in [1] was given. The relevance of this result is that persistent chaos in high dimensions was previously observed and tested numerically, but without any rigorous proof. Also, this proof shows that persistent chaos still occurs in typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps.

References

[1] D. J. Albers, J. C. Sprott, J. P. Crutchfield, Persistent chaos in high dimensions, Phys. Rev. E. **74**, 057201 (2006).

[2] D. J. Albers, J. C. Sprott, Structural stability and hyperbolicity violation in high-dimensional dynamical systems, Nonlinearity. **19**, 1801–1847 (2006).

[3] F. R. Adler, Migration alone can produce persistence of host-parasitoid models, Am. Nat. **141**, 642-650 (1993).

[4] B. M. Bolker, B. T. Grenfell, Impact of vaccination on the spatial correlation and persistence of measles dynamics, Proc. Natn. Acad. Sci. USA. **93**, 12648-12653 (1996).

[5] D. J. D. Earn, P. Rohani, B. T. Grenfell, Population dynamic interference among childhood diseases, Proc. R. Soc. Lond. B. **265**, 7 2033-2041 (1998).

[6] P. Ashwin, A. M. Rucklidge, Cycling chaos: its creation, persistence and loss of stability in a model of nonlinear magnetoconvection, Physica D. **122**(1–4),134-154 (1998).

[7] F. F. Villafranca, D. Saura, F. J. Vazquez, Diversity, persistence and chaos in consumption patterns, J. Bioeconomics. **11**(1), 43-63 (2009).

[8] S. Banerjee, J. A. Yorke, C. Grebogi, Robust chaos, Phys. Rev. Lett. **80**, 3049–3052 (1998).

[9] K. Hornik, M. Stinchocombe, H. White, Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks, Neural Networks. **3**, 551-560 (1990).

[10] T. Sauer, J. Yorke, M. Casdagli, Embedology, J. Stat. Phys. **65**, 579-616 (1991).

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