Synchronization between integer-order chaotic systems and a class of fractional-order chaotic systems via sliding mode control

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In this paper, we focus on the synchronization between integer-order chaotic systems and a class of fractional-order chaotic system using the stability theory of fractional-order systems. A new sliding mode method is proposed to accomplish this end for different initial conditions and number of dimensions. More importantly, the vector controller is one-dimensional less than the system. Furthermore, three examples are presented to illustrate the effectiveness of the proposed scheme, which are the synchronization between a fractional-order Chen chaotic system and an integer-order T chaotic system, the synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order hyperchaotic system, and the synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order Lorenz chaotic system. Finally, numerical results are presented and are in agreement with theoretical analysis. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4721996]

Fractional-order chaotic systems are an extension of integer-order chaotic systems developed by mathematicians and are more universal. However, there have been relatively few applications of such systems. Examples of the synchronization of integer-order chaotic systems and the synchronization of fractional-order chaotic systems have been widely reported. There are few results on the synchronization between a fractional-order chaotic system and an integer-order chaotic system to our best knowledge. Therefore, this paper focuses on the synchronization between integer-order chaotic systems and a class of fractional-order chaotic system to expand the applicability of the theory. A new sliding mode control method with few control terms is proposed to illustrate the effectiveness of the scheme. We report results from numerical computations and theoretical analysis which are a perfect bridge between fractional-order chaotic systems and integer-order chaotic systems. As the synchronization of integer-order chaotic systems and fractional-order chaotic systems are employed extensively in research and engineering applications, we expect our theory to be potentially useful.

I. INTRODUCTION

Since Pecora and Carroll[1] proposed a synchronization method of chaotic systems in 1990, chaos synchronization has been extensively studied in a variety of contexts such as power systems,[2] engineering,[3] physics,[4] biology,[5] and chemistry.[6] Synchronization is based on the closeness of the frequencies of periodic oscillations in two systems, one of which is the drive and the other is the response. Many methods have been used to synchronize chaotic systems including sliding mode control,[7–10] linear feedback control,[11,12] adaptive control theory,[13] back-stepping control,[14,15] active control,[16–18] and fuzzy control.[19,20]

Fractional calculus is a much older classical mathematical notion with the same three-hundred year history as integer calculus. In recent years, it has found application in many areas of physics[21] and engineering.[22] At the same time, control and synchronization of fractional-order chaotic systems have made great contributions. Some papers discuss the synchronization of general fractional-order chaotic systems,[23–25] while others consider special classes of fractional-order chaotic systems.[26–28]

However, there are few previous papers considering synchronization between integer-order chaotic systems and a class of fractional-order chaotic system with different structure and dimensions. To the best of our knowledge, none of the previous studies employ vector controllers. Obviously, the synchronization between integer-order chaotic systems and fractional-order chaotic systems is more difficult than the synchronization between integer-order chaotic systems or fractional-order chaotic systems for different order of their error dynamical system.

Motivated by the above discussion, there are four advantages of our approach. First, based on sliding mode control (SMC) and the stability theorem, a new method for chaos synchronization between integer-order chaotic systems and a class of fractional-order chaotic system is presented. Second, it has only *n*-1 vector controllers, where *n* is the number of equation dimensions, but it produces a globally and exponentially asymptotic synchronization. Third, two chaotic systems are synchronized with different structure and dimension. Finally, it is easier to achieve synchronization with a saturation function.

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The rest of the paper is organized as follows: Section II introduces the integer-order chaotic systems and a class of fractional-order chaotic system. Section III proposes a compensation controller and vector controller based on sliding mode control theory. Furthermore, the controller design scheme and the stability analysis of the closed loop system are included in this section. Section IV provides results of numerical simulations and Section V gives brief comments and conclusions.

II. SYSTEM DESCRIPTION

Consider the n-dimensional, integer-order chaotic drive system

$$\frac{d^d x}{dt^d} = f(x), \tag{1}$$

where $x \in R^n, f : R^n \to R^n$ are differentiable functions.

Then consider the n-dimensional, fractional-order chaotic response system

$$\begin{cases} \dfrac{d^q y_i}{dt^q} = a(y_j - y_i) \\ \dfrac{d^q y}{dt^q} = g(y) \end{cases}, \tag{2}$$

where $y \in R^n$, $g : R^{n-1} \to R^{n-1}$ are differentiable functions. The dimensions $q = (q_1, q_2, ..., q_n)^T (0 < q_i < 1)$ may be equal or not, and the response system (2) is an integer-order system if $q_i = 1 (i \in [1, n])$. The constant a is positive.

III. PROBLEM FORMULATION AND CONTROL DESIGN

System (1) represents the drive system, and the controller $u(t) \in R^n$ is added into the response system (2) according to

$$\frac{d^q y}{dt^q} = g(y) + u(t). \tag{3}$$

We define the synchronization errors as $e = y - x$. The aim is to choose suitable control signals $u(t) \in R^n$ such that the states of the master and response systems are synchronized (i.e., $\lim_{t\to\infty} \|e\| = 0$, where $\|\cdot\|$ is the Euclidean norm).

Now let the controller $u(t)$ be

$$u(t) = u_1(t) + u_2(t), \tag{4}$$

where $u_2(t) \in R^{n-1}$ is a vector control function that will be designed later. The $u_1(t) \in R^n$ is a compensation controller, and $u_1 = \frac{d^q x}{dt^q} - g(x)$. Using Eq. (4), the response system (3) can be rewritten as

$$\frac{d^q e}{dt^q} = g(y) + \frac{d^q x}{dt^q} - g(x) + u_2 - \frac{d^q x}{dt^q} = g(y) - g(x) + u_2. \tag{5}$$

To control the chaotic systems easily, the modified compensation controller u_1 can be represented as

$$u_1 = \frac{d^q x}{dt^q} - g(y - e) \tag{6}$$

and the modified error dynamics (5) can be represented as

$$\frac{d^q e}{dt^q} = h(e, y) + u_2, \tag{7}$$

where $h(e, y) = g(y) - g(y - e)$.

In order to make the controller u_2 effective, we assume

$$h(e, y) = \begin{pmatrix} A_1 e_1 + f_1(e_1, e_2, y) \\ A_2 e_2 + f_{21}(e_1, e_2, y) + f_{22}(e_1, e_2, y) \end{pmatrix}, \tag{8}$$

where $f_1(e_1, e_2, y) \in R^m$ $(1 \le m < n)$, $f_{21}(e_1, e_2, y) \in R^{n-m}$, and $f_{22}(e_1, e_2, y) \in R^{n-m}$; $A_1 \in R^{m\times m}$ and $A_2 \in R^{(n-m)\times(n-m)}$ are constant real matrixes; $e_1 = (e_{t_1}, ..., e_{t_m})^T; e_2 = (e_{t_{m+1}}, ..., e_{t_{mn}})^T$

$$\text{Which } \lim_{e_1 \to 0} f_{21}(e_1, e_2, y) = 0. \tag{9}$$

Two steps are required to design a sliding mode controller. First, we construct a sliding surface that represents a desired system dynamic. Then, we develop a switching control law such that a sliding mode exists on every point of the sliding surface, and any states outside the surface are driven to reach the surface in a finite time.[29] As a choice for the sliding surface, we take

$$\begin{cases} s_j = e_j + \dfrac{d^{-q}}{dt^{-q}}(k_1 e_j + a e_i) \\ s_r = e_r + \dfrac{d^{-q}}{dt^{-q}} k_p e_r \\ \ldots\ldots \end{cases}, \tag{10}$$

where $r \in [1, n], r \notin (i, j)$, and k_1, k_p $(p \in [2, n-1])$ is a positive constant vector and $k_1 \neq a$. For the sliding mode method, the sliding surface and its derivative must satisfy

$$s(t) = 0, \quad \dot{s}(t) = 0. \tag{11}$$

Consider

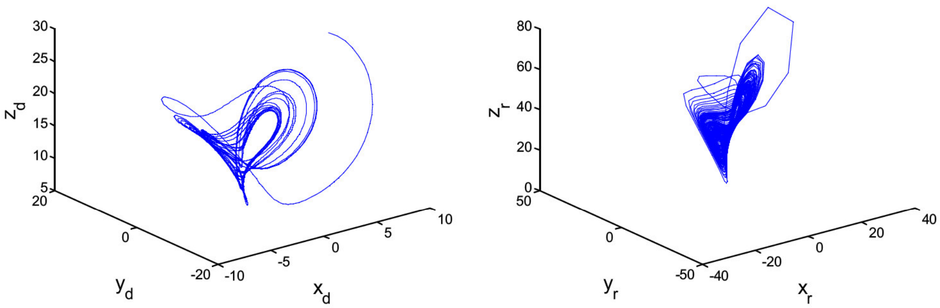
$$\dot{s}(t) = D^{1-q}(D^q s(t)) = 0 \Rightarrow D^q s(t) = 0 \tag{12}$$

from which it follows that

$$\begin{cases} \dfrac{d^q}{dt^q} s_j = \dfrac{d^q}{dt^q} e_j + (k_1 e_j + a e_i) = 0 \\ \dfrac{d^q}{dt^q} s_r = \dfrac{d^q}{dt^q} e_r + k_p e_r = 0 \\ \ldots\ldots \end{cases} \tag{13}$$

and

$$\begin{cases} \dfrac{d^q}{dt^q} e_i = a(e_j - e_i) \\ \dfrac{d^q}{dt^q} e_j = -(k_1 e_j + a e_i) \\ \dfrac{d^q}{dt^q} e_r = -k_p e_r \\ \ldots\ldots. \end{cases}. \tag{14}$$



a. Integer-order T chaotic system. b. Fractional-order Chen's chaotic system

FIG. 1. Chaotic attractors for the integer-order T chaotic system and the fractional-order Chen's chaotic system.

In accordance with active control design procedure, the nonlinear part of the error dynamics is eliminated by the following choice of the input vector:

$$\begin{cases} \dfrac{d^q y_i}{dt^q} = a(y_j - y_i) + u_{1i} \\ \dfrac{d^q y_j}{dt^q} = g(y) + u_{1j} + u_{2j} \\ \dfrac{d^q y}{dt^q} = g(y) + u_{1r} + u_{2r} \\ \ldots\ldots \end{cases} . \tag{15}$$

Now, according to Eq. (14), we can define $e_1 = \begin{pmatrix} e_i \\ e_j \end{pmatrix}$ and $e_2 = \begin{pmatrix} e_r \\ \ldots \end{pmatrix}$. So the system (14) can be rewritten as follows:

$$\begin{cases} \dfrac{d^q}{dt^q} e_1 = A_1 e_1 + f_1(e_1, e_2, y) \\ \dfrac{d^q}{dt^q} e_2 = A_2 e_2 + f_{21}(e_1, e_2, y) + f_{22}(e_1, e_2, y) \end{cases}, \tag{16}$$

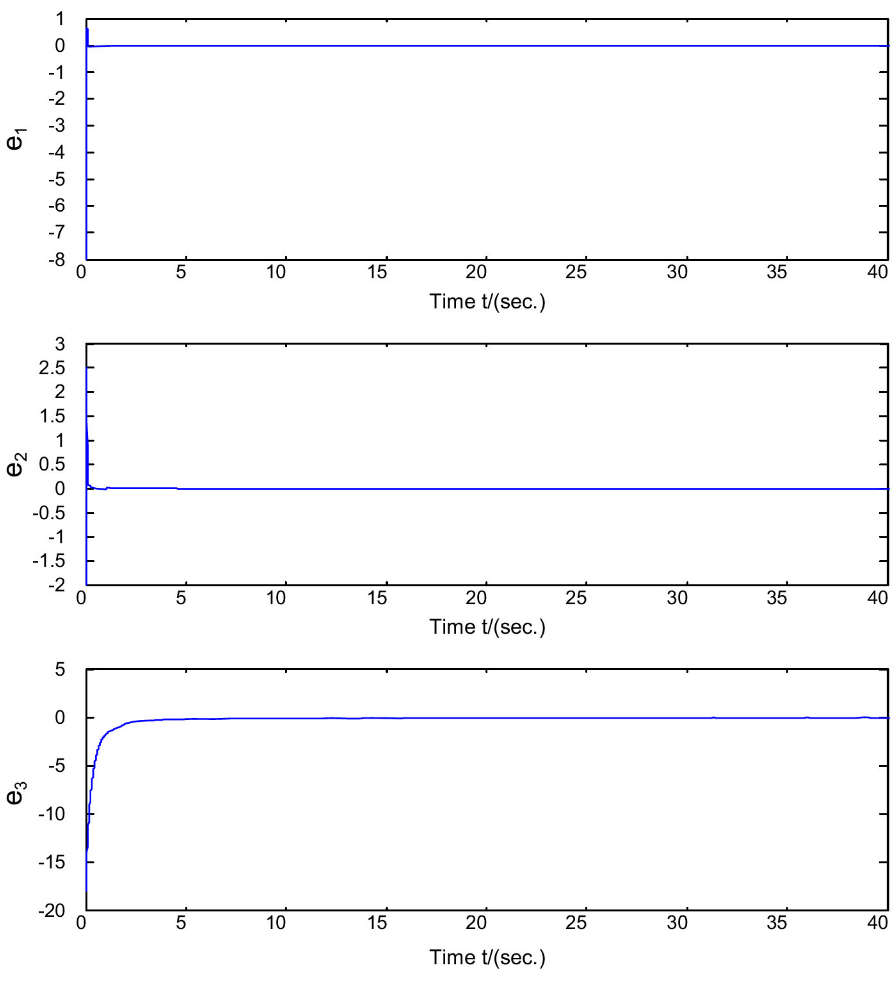


FIG. 2. Synchronization errors between the integer-order T chaotic system and the fractional-order Chen's chaotic system ($e_1 = x_r - x_d$, $e_2 = y_r - y_d, e_3 = z_r - z_d$).

where $f_1(e_1, e_2, y) = \begin{pmatrix} 0 \\ h_j(e, y) \end{pmatrix}$ and $A_1 = \begin{bmatrix} -a & a \\ -a & -k_1 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} -k_r & 0 & \dots & 0 \\ 0 & -k_{r+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -k_{n-1} \end{bmatrix}.$$

Now, system (16) can be described as follows:

$$\begin{cases} \dfrac{d^q}{dt^q} e_i = a(e_j - e_i) \\ \dfrac{d^q}{dt^q} e_j = -(k_1 e_j + a e_i) \\ \dfrac{d^q}{dt^q} e_2 = -A_2 e_2 + f_{21}(e_1, e_2, y) \end{cases}, \tag{17}$$

So, according to Eq. (17), the controller is designed as follows:

$$\begin{cases} u_{2j} = -h_j(e, y) - (k_1 e_j + a e_i) - \eta\,\mathrm{sat}(s_j/h) \\ u_{2r} = -f_{22}(e, y) - k_p e_r - \eta\,\mathrm{sat}(s_r/h) \\ \dots\dots \end{cases}, \tag{18}$$

where η is the gain of the controller and

$$\mathrm{sat}(S/h) = \begin{cases} \mathrm{sign}(S), & |S| > h \\ S/h, & |S| \le h \end{cases}, \tag{19}$$

with h a positive constant that must be small for a good approximation.

According to function (2) and controls (6) and (18), the error is given by

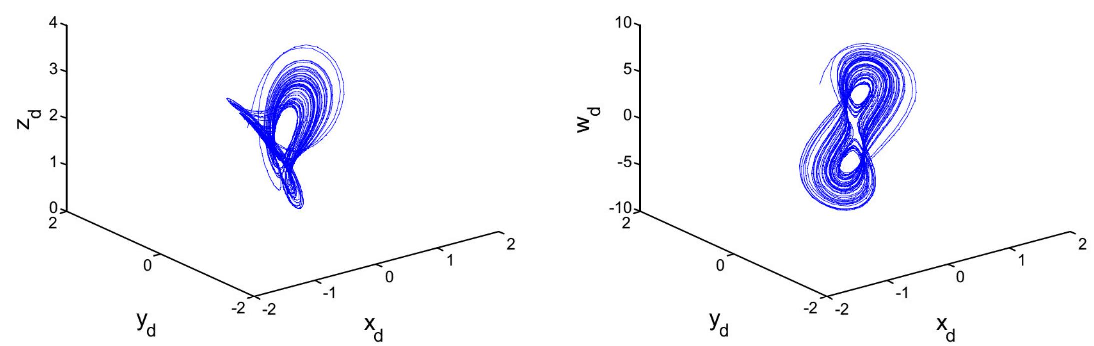
$$\begin{cases} \dfrac{d^q}{dt^q} e_i = a(e_j - e_i) \\ \dfrac{d^q}{dt^q} e_j = -(k_1 e_j + a e_i) - \eta\,\mathrm{sat}(s_j/h) \\ \dfrac{d^q}{dt^q} e_2 = -A_2 e_2 + f_{21}(e_1, e_2, y) - \eta\, SAT \end{cases}, \tag{20}$$

where $SAT = (sat(s_r/h), \dots, sat(s_{n-1}/h))^T$.

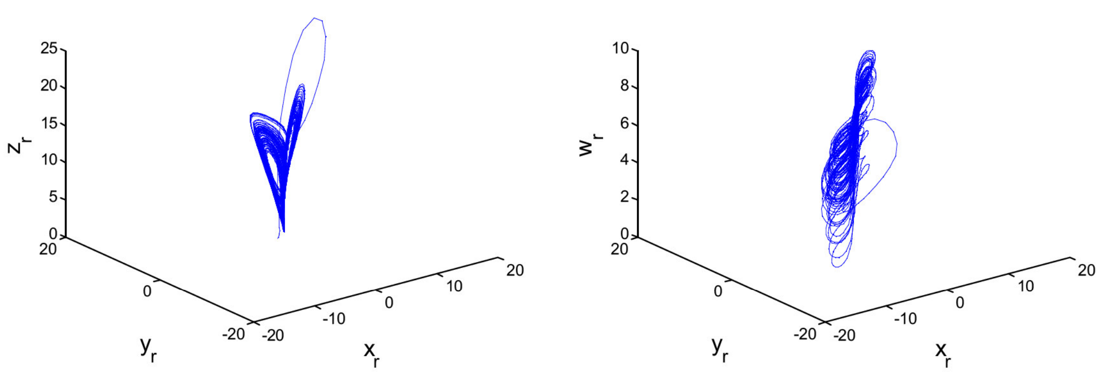
Theorem. Consider the error function (17). If A_1 is a stable matrix, the error between response system (3) and drive system (1) can be determined.

Proof. According to functions (2), (6), (18), and (20), A_1 is given by the matrix

$$A_1 = \begin{bmatrix} -a & a \\ -a & -k_1 \end{bmatrix}. \tag{21}$$



a. Integer-order hyperchaotic system.



b. Fractional-order hyperchaotic system based on Chen's system.

FIG. 3. Chaotic attractors of an integer-order hyperchaotic system and a fractional-order hyperchaotic system based on Chen's system.

Thus, the eigenvalues are easily got as

$$\begin{cases} \lambda_i = -\frac{1}{2}a + \frac{1}{2}k_1 - \sqrt{-3a^2 + 2ak_1 + k_1^2}) \\ \lambda_j = -\frac{1}{2}a + \frac{1}{2}k_1 + \sqrt{-3a^2 + 2ak_1 + k_1^2}) \end{cases}. \tag{22}$$

When $k_1 \neq a$, all eigenvalues of matrix A_1 satisfy $\mathrm{Re}(\lambda) < 0$, which implies $|\arg\lambda| > \frac{\pi}{2} > \frac{q\pi}{2}$. According to the stability theory of fractional-order systems,[30–32] the equilibrium point $e = 0$ in function (20) is asymptotically stable

$$\lim_{t\to+\infty} e_1 = 0. \tag{23}$$

According to Eqs. (9) and (23), Eq. (20) can be rewritten as follows:

$$\frac{d^q e}{dt^q} = Ae - \eta\,\mathrm{sat}(s/h). \tag{24}$$

According to Eqs. (18), (20), (17), and (24), A is given by

$$A = \begin{bmatrix} -a & a & 0 & \dots & 0 & \dots & 0 \\ -a & -k_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & -k_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & -k_r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -k_{n-1} \end{bmatrix}. \tag{25}$$

And the eigenvalues are

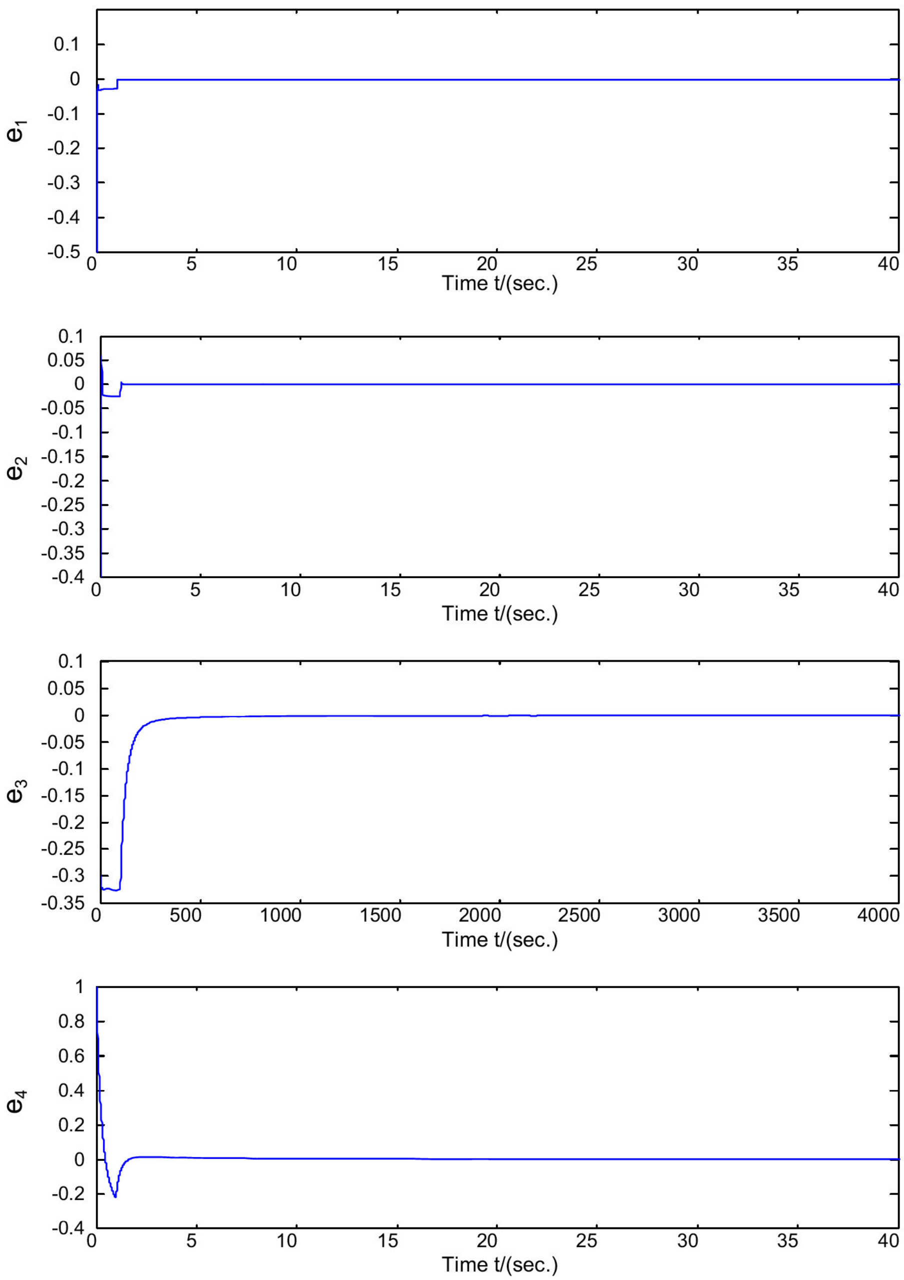


FIG. 4. Synchronization errors between an integer-order hyperchaotic system based on the Lorenz system and a fractional-order hyperchaotic system based on Chen's system ($e_1 = x_r - x_d, e_2 = y_r - y_d, e_3 = z_r - z_d, e_4 = w_r - w_d$).

$$\begin{cases} \lambda_i = -\dfrac{1}{2}(a + k_1 + \sqrt{-3a^2 - 2ak_1 - k_1^2}) \\ \lambda_j = -\dfrac{1}{2}(a + k_1 - \sqrt{-3a^2 - 2ak_1 - k_1^2}) \\ \lambda_r = -k_r \\ \ldots\ldots \end{cases}. \tag{26}$$

Thus, all eigenvalues of matrix A satisfy $\mathrm{Re}(\lambda) < 0$, which implies $|\arg\lambda| > \frac{\pi}{2} > \frac{q\pi}{2}$. According to the stability theory of fractional-order systems,[30–32] the equilibrium point $e = 0$ in function (24) is asymptotically stable

$$\lim_{t\to+\infty} e = \lim_{t\to+\infty} (y - x) = 0. \tag{27}$$

IV. NUMERICAL SIMULATION

This section of the paper presents three illustrative examples to verify and demonstrate the effectiveness of the proposed control scheme. In case I, a three-dimensional integer-order system is synchronized with a fractional-order system having a different structure. In case II, a four-dimensional integer-order system is synchronized with a fractional-order system. In case III, a three-dimensional integer-order system is synchronized with a four-dimensional fractional-order system. The numerical simulation results were carried out in MATLAB using the Caputo version and a predictor-corrector algorithm for fractional-order differential equations, which is the generalization of Adams-Bashforth-Moulton one.[33,34]

Case I. Synchronization between a fractional-order Chen chaotic system and an integer-order T chaotic system

The integer-order T chaotic system[35] is described by

$$\begin{cases} \dfrac{dx_d}{dt} = a(y_d - x_d) \\ \dfrac{dy_d}{dt} = (c - a)x_d - ax_d z_d \\ \dfrac{dz_d}{dt} = -bz_d + x_d y_d \end{cases}. \tag{28}$$

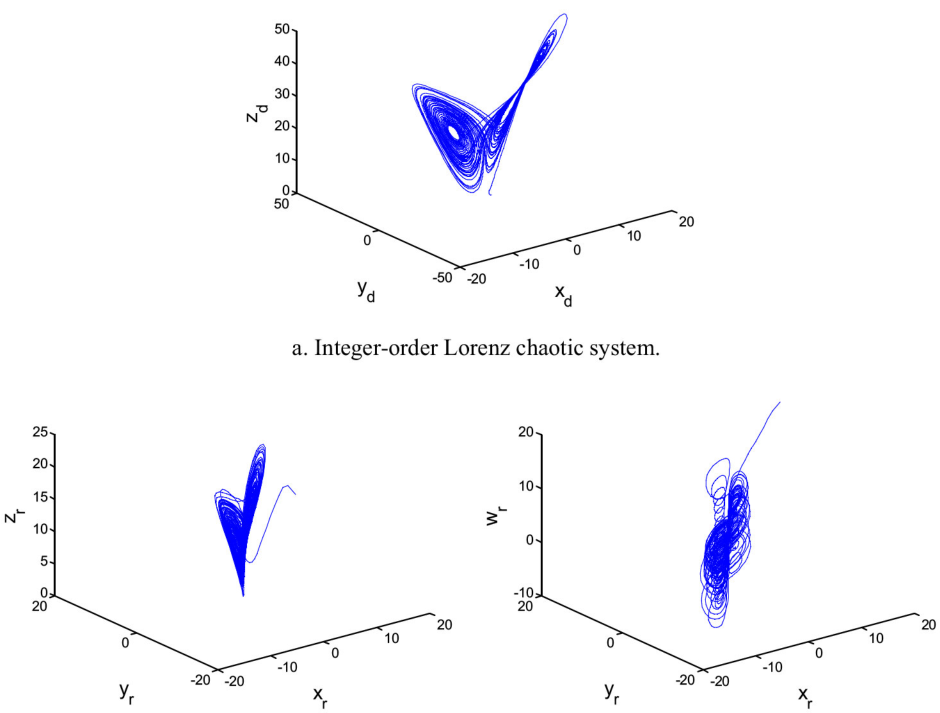
The system exhibits chaotic behavior for the parameters $(a, b, c) = (2.1, 0.6, 30)$ with initial conditions $[x_d, y_d, z_d]^T = [10, 5, 25]^T$ and a chaotic attractor as shown in Fig. 1(a).

The fractional-order Chen chaotic system[36] is

$$\begin{cases} \dfrac{d^{q_1}x_r}{dt} = a(y_r - x_r) \\ \dfrac{d^{q_2}y_r}{dt} = (c - a)x_r - x_r z_r + cy_r \\ \dfrac{d^{q_3}z_r}{dt} = -bz_r + x_r y_r \end{cases} \tag{29}$$

and exhibits chaotic behavior for $q_{r1} = q_{r2} = q_{r3} = 0.90$ and $(a, b, c) = (35, 3, 28)$ with initial conditions $[x_r, y_r, z_r]^T = [2, 3, 7]^T$ and a chaotic attractor as shown in Fig. 1(b).

Here, the controller parameters $K_1 = K_2 = 3$ and $\eta = 1$ are chosen, and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (-22.5 + 32.6917i, -22.5 - 32.6917i, -10)$ are located in the stable region. As described above, we can obtain the controller $u(t)$ for the response systems (6) and (18) as follows:



a. Integer-order Lorenz chaotic system.

b. Fractional-order hyperchaotic system based on Chen's system

FIG. 5. Chaotic attractors of the integer-order Lorenz chaotic system and a fractional-order hyperchaotic system based on Chen's system.

(i) Compensation controller

$$\begin{cases} u_{11} = 2.1(y_d - x_d) - 35[(y_r - e_2) - (x_r - e_1)] \\ u_{12} = 27.9x_d - 2.1x_d z_d + 7(x_r - e_1) + (x_r - e_1)(z_r - e_3) \\ -28(y_r - e_2) \\ u_{13} = -0.6z_d + x_d y_d - (x_r - e_1)(y_r - e_2) + 3(z_r - e_3) \end{cases} \quad (30)$$

(ii) Vector controller

$$\begin{cases} u_{22} = 7e_1 - e_1 e_3 + x_r e_3 + z_r e_1 - 28e_2 - (k_1 e_2 + a e_1) \\ -\eta\,\mathrm{sat}(s_1/h) \\ u_{23} = 3e_3 - k_2 e_3 - \eta\,\mathrm{sat}(s_2/h) \end{cases} \quad (31)$$

The synchronization errors are shown in Fig. 2, which demonstrates that the proposed method is successful in synchronizing the two systems.

Case II: Synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order four-dimensional chaotic system.

The integer-order four-dimensional chaotic system[37] is given by

$$\begin{cases} \dfrac{dx_d}{dt} = a(y_d - x_d) \\ \dfrac{dy_d}{dt} = bx_d - 20x_d z_d + w_d \\ \dfrac{dz_d}{dt} = 20x_d y_d - cz_d \\ \dfrac{dw}{dt} = 20y_d z_d - dw_d \end{cases} \quad (32)$$

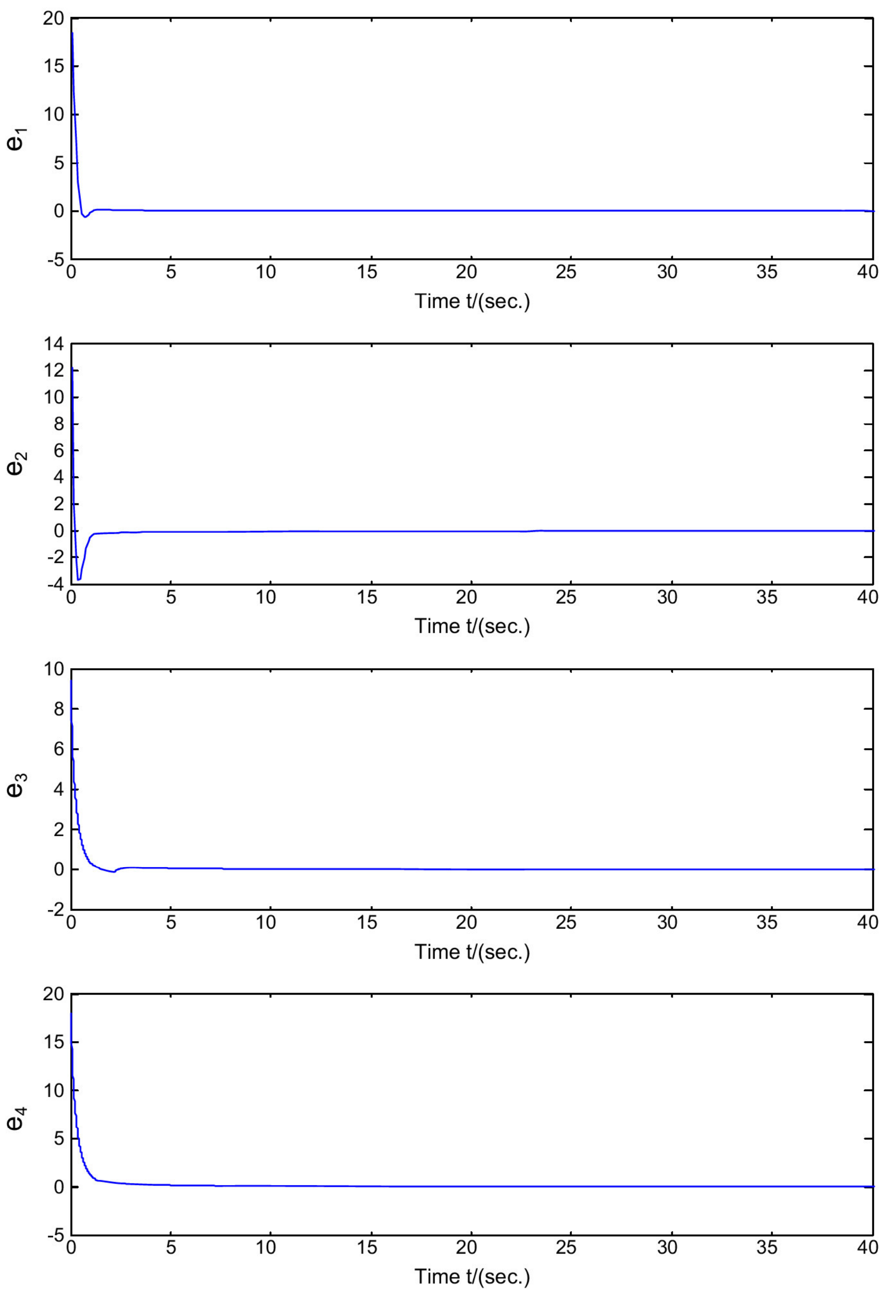


FIG. 6. Synchronization errors between the integer-order Lorenz chaotic system and a fractional-order hyperchaotic system based on Chen's system ($e_1 = x_r - x_d, e_2 = y_r - y_d$, $e_3 = z_r - z_d, e_4 = w_r - w_d$).

This system exhibits chaotic behavior for the parameters $(a, b, c, d) = (10, 40, 2.5, 5)$ with initial conditions $[x_d, y_d, z_d, w_d]^T = [0.5, 1.4, 1.2, 0.7]^T$ and a chaotic attractor as shown in Fig. 3(a).

The fractional-order hyperchaotic system based on Chen's system[38] is

$$\begin{cases} \dfrac{d^{q_{r1}}x_d}{dt^{q_{r1}}} = a(y_r - x_r) \\ \dfrac{dy_d^{q_{r2}}}{dt^{q_{r2}}} = dx_r - x_r z_r + cy_r - w_r \\ \dfrac{d^{q_{r3}}z_d}{dt^{q_{r3}}} = x_r y_r - bz_r \\ \dfrac{d^{q_{r4}}w}{dt^{q_{r4}}} = x_r + k \end{cases}. \quad (33)$$

This system exhibits chaos for $q_{r1} = q_{r2} = q_{r3} = 0.90$ and $(a, b, c, d, k) = (36, 3, 28, -16, 0.5)$ with initial conditions $[x_r, y_r, z_r, w_r]^T = [0, 1.0, 0.9, 1.7]^T$ and a chaotic attractor as shown in Fig. 3(b).

The controller parameters $K_1 = K_2 = K_3 = 3$ and $\eta = 1$ are chosen, and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-23 + 33.5708i, -23 - 33.5708i, -10, -10)$ are located in the stable region. As before, we can obtain the controller $u(t)$ for the response systems (6) and (18) as follows:

(i) Compensation controller

$$\begin{cases} u_{11} = 10(y_d - x_d) - 36[(y_r - e_2) - (x_r - e_1)] \\ u_{12} = 40x_d - 20x_d z_d + w_d + 16(x_r - e_1) \\ \qquad + (x_r - e_1)(z_r - e_3) - 28(y_r - e_2) + (w_r - e_4) \\ u_{13} = 20x_d y_d - 2.5z_d - (x_r - e_1)(y_r - e_2) + 3(z_r - e_3) \\ u_{14} = 20y_d z_d - 5w_d - (x_r - e_1) - 0.5 \end{cases}. \quad (34)$$

(ii) Vector controller

$$\begin{cases} u_{22} = e_1 + x_r e_3 + z_r e_1 - 28e_2 + e_4 - (k_1 e_2 + ae_1) \\ -\eta\,\mathrm{sat}(s_1/h) \\ u_{23} = 3e_3 - k_2 e_3 - \eta\,\mathrm{sat}(s_2/h) \\ u_{24} = -k_3 e_4 - \eta\,\mathrm{sat}(s_3/h) \end{cases}. \quad (35)$$

The synchronization errors are shown in Fig. 4, which demonstrates that the proposed method is successful in synchronizing the two systems.

Case III. Synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order Lorenz chaotic system.

The Lorenz chaotic system[39] is described by

$$\begin{cases} \dfrac{dx_d}{dt} = a(y_d - x_d) \\ \dfrac{dy_d}{dt} = rx_d - y_d - x_d z_d \\ \dfrac{dz_d}{dt} = -bz_d + x_d y_d \end{cases} \quad (36)$$

and exhibits chaotic behavior for the parameters $(a, b, r) = (10, 8/3, 28)$ with initial conditions $[x_d, y_d, z_d]^T = [1.5, 0.8, 1.3]^T$ and a chaotic attractor as shown in Fig. 5(a).

The fractional-order hyperchaotic system based on Chen's system[36] is

$$\begin{cases} \dfrac{d^{q_{r1}}x_d}{dt^{q_{r1}}} = a(y_r - x_r) \\ \dfrac{dy_d^{q_{r2}}}{dt^{q_{r2}}} = dx_r - x_r z_r + cy_r - w_r \\ \dfrac{d^{q_{r3}}z_d}{dt^{q_{r3}}} = x_r y_r - bz_r \\ \dfrac{d^{q_{r4}}w}{dt^{q_{r4}}} = x_r + k \end{cases} \quad (37)$$

and exhibits chaotic behavior for $q_{r1} = q_{r2} = q_{r3} = 0.90$ and $(a, b, c, d, k) = (36, 3, 28, -16, 0.5)$ with initial conditions $[x_r, y_r, z_r, w_r]^T = [20, 13, 9, 18]^T$ and a chaotic attractor as shown in Fig. 5(b).

The controller parameters $K_1 = K_2 = K_3 = 7$ and $\eta = 1$ are chosen, and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-21.5 + 32.9507i, -21.5 - 32.9507i, -7, -7)$ are located in the stable region. As mentioned above, we can obtain the controller $u(t)$ for the response systems (6) and (18) as follows:

(i) Compensation controller

$$\begin{cases} u_{11} = 10(y_d - x_d) - 36[(y_r - e_2) - (x_r - e_1)] \\ u_{12} = 28x_d - y_d - x_d z_d + 16(x_r - e_1) \\ \qquad + (x_r - e_1)(z_r - e_3) - 28(y_r - e_2) \\ u_{13} = -\dfrac{8}{3}z_d + x_d z_d - (x_r - e_1)(y_r - e_2) + 3(z_r - e_3) \\ u_{14} = -(x_r - e_1) - 0.5 \end{cases}. \quad (38)$$

(ii) Vector controller

$$\begin{cases} u_{22} = e_1 + x_r e_3 + z_r e_1 - 28e_2 + e_4 - (k_1 e_2 + ae_1) \\ -\eta\,\mathrm{sat}(s_1/h) \\ u_{23} = 3e_3 - k_2 e_3 - \eta\,\mathrm{sat}(s_2/h) \\ u_{24} = -k_3 e_4 - \eta\,\mathrm{sat}(s_3/h) \end{cases}. \quad (39)$$

The synchronization errors are shown in Fig. 6, which demonstrates that the proposed method is successful in synchronizing the two systems.

V. CONCLUSION AND DISCUSSION

In this paper, the synchronization between fractional-order chaotic systems and integer-order chaotic systems was achieved based on sliding mode control. The proposed synchronization approach is theoretically rigorous and pervasive. Furthermore, three typical examples were shown: the synchronization (1) between different three-dimensional chaotic systems, (2) between different four-dimensional chaotic systems, and (3) between a three-dimensional chaotic system and a four-dimensional chaotic system. Numerical results using the Caputo version and a predictor-corrector algorithm for fractional-order differential equations illustrated the effectiveness of the proposed scheme. These theoretical and numerical results provide a bridge between integer-order chaotic system and fractional-order chaotic systems and lend theoretical support for fractional-order chaotic systems.

More and better methods for the synchronization between integer-order chaotic systems and fractional-order chaotic systems should be studied. Moreover, this knowledge should be applied in engineering to fields such as communications and that will be a subject of our future work.

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