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ABOUT UNIVERSAL BASINS OF ATTRACTION IN HIGH-DIMENSIONAL SYSTEMS

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In this letter, we will show the existence of invariant sets called *universal basins of attraction* for typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps. The method of analysis is based on the definition of an *equivalence class* between systems with the same number of neurons, the same number of time lags, and the same upper bound for one family of bifurcation parameters.

Keywords: High dimensions; single-layer recurrent neural networks; general dynamical systems; universal basins of attraction.

1. Introduction

The basin of attraction of an attractor of a dynamical system is the set of initial conditions leading to long-time behavior that approaches the attractor. Determining rigorously the basin of attraction (or a part of this basin) of a dynamical system is a very difficult problem. In this letter, we show the existence of basins of attraction for typical nonlinear high-dimensional dynamical systems based on their number of neurons, number of time lags, and the upper bound for one family of bifurcation parameters.

It was shown in [Albers & Sprott, 2006] that the use of approximation theorems of [Hornik *et al.*, 1990] and time-series embedding of [Sauer *et al.*, 1991] established an equivalence between single-layer recurrent neural networks of the form [Albers *et al.*, 2006]

$$F_{d,n,\beta} : x_t = \beta_0 + \sum_{i=1}^{n} \beta_i \tanh s \left(\omega_{i0} + \sum_{j=1}^{d} \omega_{ij} x_{t-j} \right) \tag{1}$$

and general dynamical systems. Indeed, the so-called *universal function approximators* define a class of dynamical systems. This class is universal in two equivalent senses: (a) In the limit that these systems have infinitely many parameters they are dense in C^r on compacta as shown in [Hornik *et al.*, 1990]. (b) These types of systems can approximate arbitrarily closely any C^r mapping and its derivatives on compacta. These systems are single-layer recurrent neural networks of the form (1). More detailed information about the procedure of approximation can be found in [Hornik *et al.*, 1990]. In this work, we give some conditions confirming that multilayer feedforward networks with a single hidden layer and an appropriately smooth hidden layer activation function are capable of providing a good and accurate approximation to an arbitrary function (including certain piecewise differentiable functions) and its derivatives. This procedure requires simultaneous approximation of a function and its derivatives.

In fact, models of the form (1) are maps from $\mathbb{R}^d$ to $\mathbb{R}$. Here n is the number of neurons, d is the number of time lags, which determines the system's input embedding dimension, and s is a scaling factor for the connection weights ω_{ij}. The initial condition is $(x_1, x_2, \ldots, x_d)$, and the state at time t is $(x_t, x_{t+1}, \ldots, x_{t+d-1})$. In [Albers *et al.*, 2006] the $k = n(d+2)+1$-dimensional parameter space was taken as follows: (i) $\beta_i \in [0,1]$ is uniformly distributed and rescaled to satisfy $\sum_{i=1}^{n} \beta_i^2 = n$, (ii) ω_{ij} is normally distributed with zero mean and unit variance, and (iii) the initial condition $x_j \in [-1,1]$ is uniform.

2. Universal Basins of Attraction

In this section, we will use the term *general* to indicate a discrete dynamical system of the form $x_{t+1} = T(x_t)$ and the word *universal* to describe a type of basin of attraction of these systems.

Let $T : U \subset \mathbb{R}^d \to \mathbb{R}$ be an arbitrary mapping defining a discrete dynamical system of the form $x_{t+1} = T(x_t)$. Here T can be a discontinuous map and can be a C^r diffeomorphism, etc. It is well known that the set of points in the space of system variables such that initial conditions chosen in this set dynamically evolve to a particular attractor is called the *basin of attraction*. More generally, a subset S of the domain U of a mapping T is an invariant set under the mapping when $x \in S$ implies that $T(x) \in S$. The basic topological structure of a basin of attraction or an invariant set can vary greatly from system to system. Generally, the shape or the structure of a basin of attraction is not well known for any system. For example, a basin of attraction can be an interval, a ball, the union of intervals, or the union of balls, and the basin boundary can be a smooth curve. For some systems, the shape of the basin of attraction is very complicated, and the basin boundary can be a fractal set. There are numerous examples of this situation in the current literature. See [McDonald *et al.*, 1985] for some details. The methods used for finding these basins are essentially based on numerical calculations. Finding a *preassigned* structure of a basin of attraction or a part of it for a system is very hard even for the simplest systems, and certainly becomes more difficult for high-dimensional systems.

The main issue addressed in this letter is the existence of some simple structure for these basins of attractions or parts of them for typical high-dimensional systems. This question is very complicated, and its solution is not immediately evident.

In this letter, we will show the existence of a *universal basin of attraction* $\Gamma_{d,n,\beta}$ for typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps.

For this purpose, consider the set $\Omega_{n,d,\alpha}$ as the space of single-layer recurrent neural networks of the form (1) with n neurons, d time lags, and $\alpha = (\beta_i)_i \in [0,1]$ as bifurcation parameters. We define an *equivalence relation* in order to make a partition of the set $\Omega_{n,d,\beta}$ in the form of several equivalence classes. Two elements of $\Omega_{n,d,\alpha}$ are considered equivalent if and only if they are elements of the same class. In this way, every element $F_{d,n,\alpha} \in \Omega_{n,d,\alpha}$ is a member of one and only one class. The intersection of any two different classes is empty, and the union of all the classes equals the original set $\Omega_{n,d,\alpha}$. Here all operations can be defined on the equivalence classes using representatives from each equivalence class. Furthermore, the expected results of these operations are independent of the selected class representatives. Hence the word *universal* used here implies the existence of a common set as the basin of attraction of all the elements of one equivalence class. As we show below, this basin of attraction is an open ball, depending on properties of the considered recurrent neural networks such as the number of neurons, time lags, and bifurcation parameters.

By using this idea, we will show that the basin of attraction of an element $F_{d,n,\alpha} \in \Omega_{n,d,\alpha}$ is an *equivalence class* between systems with the same number of neurons n, the same number of time lags d, and the same upper bound for the family of bifurcation parameters $\beta_i \in [0,1]$. Indeed, let $F_{d,n,\alpha} \in \Omega_{n,d,\alpha}$ (with its basin of attraction $\Gamma_{d,n,\alpha}$) and $G_{d',n',\alpha'}$ be two single-layer recurrent neural networks of the form (1). Define the *equivalence relation* $\Re$ as follows:

$$F_{d,n,\alpha} \Re G_{d',n',\alpha'} \Leftrightarrow d = d', n = n', \alpha = \alpha'.$$

This relation is reflexive, symmetric, and transitive. Thus an equivalence class of the relation $\Re$ is given by

$$\begin{aligned}\overline{F_{d,n,\alpha}} &= \{G_{d',n',\alpha'} : \mathbb{R}^d \to \mathbb{R} : F_{d,n,\alpha} \Re G_{d',n',\alpha'}\} \\ &= \{G_{d,n,\alpha} : \mathbb{R}^d \to \mathbb{R}\}. \end{aligned} \tag{2}$$

Thus the set $\Gamma_{d,n,\alpha}$ is the basin of attraction for any map $G_{d,n,\alpha} : \mathbb{R}^d \to \mathbb{R}$ equivalent to $F_{d,n,\alpha}$ with respect to $\Re$. We remark that the equivalence class $\overline{F_{d,n,\alpha}}$ does not depend on the values of the scaling factor s and the connection weights ω_{ij}. Hence these properties further justify the name *universal* used here for the required basins of attraction.

Now consider the possible shapes of these invariant sets. Indeed, since $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, then $|\tanh x| = \left|\frac{e^{2x}-1}{e^{2x}+1}\right| \leq 1$. Hence $|x_t| \leq \sum_{i=0}^{n} |\beta_i| \leq (n+1)\beta$, where $\beta = \max\{\beta_i, i = 0, \ldots, n\}$. If we set $X_t = (x_t, x_{t+1}, \ldots, x_{t+d-1})$, then $\|X_t\|_1 \leq d(n+1)\beta$ (here $\|X_t\|_1 = \sum_{i=0}^{d-1} |x_{t+i}|$). It is remarkable that the region $\Gamma_{d,n,\beta}$ defined by

$$\Gamma_{d,n,\beta} = \{x_0 \in \mathbb{R}^d : \|x_0\|_1 < d(n+1)\beta\} \tag{3}$$

is an invariant set for system (1) since by induction, if we assume that $\|X_t\|_1 \leq d(n+1)\beta$, then $\|X_{t+1}\|_1 \leq d(n+1)\beta$. Indeed, we have $|x_{t+1}| \leq \sum_{i=0}^{n} |\beta_i| \leq (n+1)\beta$, then $|x_{t+1}| + |x_{t+2}| + \cdots + |x_{t+d}| \leq d(n+1)\beta$, that is $\|X_{t+1}\|_1 \leq d(n+1)\beta$. Thus the set $\Gamma_{d,n,\beta}$ is the basin of attraction of system (1).

As noted in the introduction, since networks of the form (1) are universal function approximators, then the set $\Gamma_{d,n,\beta}$ is still the basin of attraction of typical nonlinear high-dimensional dynamical systems such as randomly sampled high-dimensional vector fields (ODEs) or maps.

Generally, it was supposed that the size of the basin of attraction is some kind of measure of how likely an attractor is to appear in the system that it is modeling. In real-world systems, the parameters are not usually constant, and so the existence of a nearby basin of attraction could signal the likelihood of the dynamics flipping to another attractor. In other words, the so-called *basin boundaries* arise in dissipative dynamical systems when two or more attractors are present. In this case, each attractor has its own basin of initial conditions. The sets that separate different basins are called the basin boundaries. In some cases the basin boundary is smooth, and in other cases the basin boundaries can have very complicated fractal structure and create an additional impediment to predicting long-term behavior. See [Zeraoulia & Sprott, 2008a] for more details and examples.

In the case where the parameters are constant, noise or other extrinsic factors could bump the dynamics into a different basin. Basically, the size of the basin of attraction is an indication of how *robust* the attractor is. Here the robustness means the degree of resistance of that attractor to small changes in initial conditions and other quantities related to that system. Generally, an attractor lies within its basin of attraction. In particular, chaotic attractors tend to lie close to their basin boundary somewhere in the state space, i.e. chaos tends to occur just before attractors collide with their basin boundary. For explanation, we begin by saying that chaotic dynamical systems display two kinds of chaotic attractors: (a) fragile chaos, in which case the attractors disappear with perturbations of a parameter or coexist with other attractors, and (b) robust chaos, which is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space. The existence of these windows means that small changes of the parameters would destroy the chaos, implying the fragility of this type of chaos. See [Zeraoulia & Sprott, 2008b] for more details and mathematical examples.

In this letter, we have proved that systems of the form (1) have basins of attraction described by Eq. (3). These basins have a simple geometry. The case where the basin geometry is more complex in geometry is very interesting, in view of theoretical studies and even real world applications. However, there are only a few examples of this situation obtained by numerical simulation. See [McDonald *et al.*, 1985; Zeraoulia & Sprott, 2008a, and references therein]. A mathematical formulation of these fractal basins is a very prominent open problem in the study of dynamical systems. Some examples are illustrated in Figs. 1–4. These figures show a plot of a 2-D cross-section of the d-dimensional basin of attraction ($d = 2, 4, 8$) with the parameters arbitrarily chosen to give a chaotic solution. The light blue is the basin of the strange attractor, and the red is the basin of some other attractor, most likely a stable equilibrium. The strange attractor is shown in black projected onto the plane, which explains why it appears to cross the basin boundary. For the case of 4-D neural net systems with two neurons,

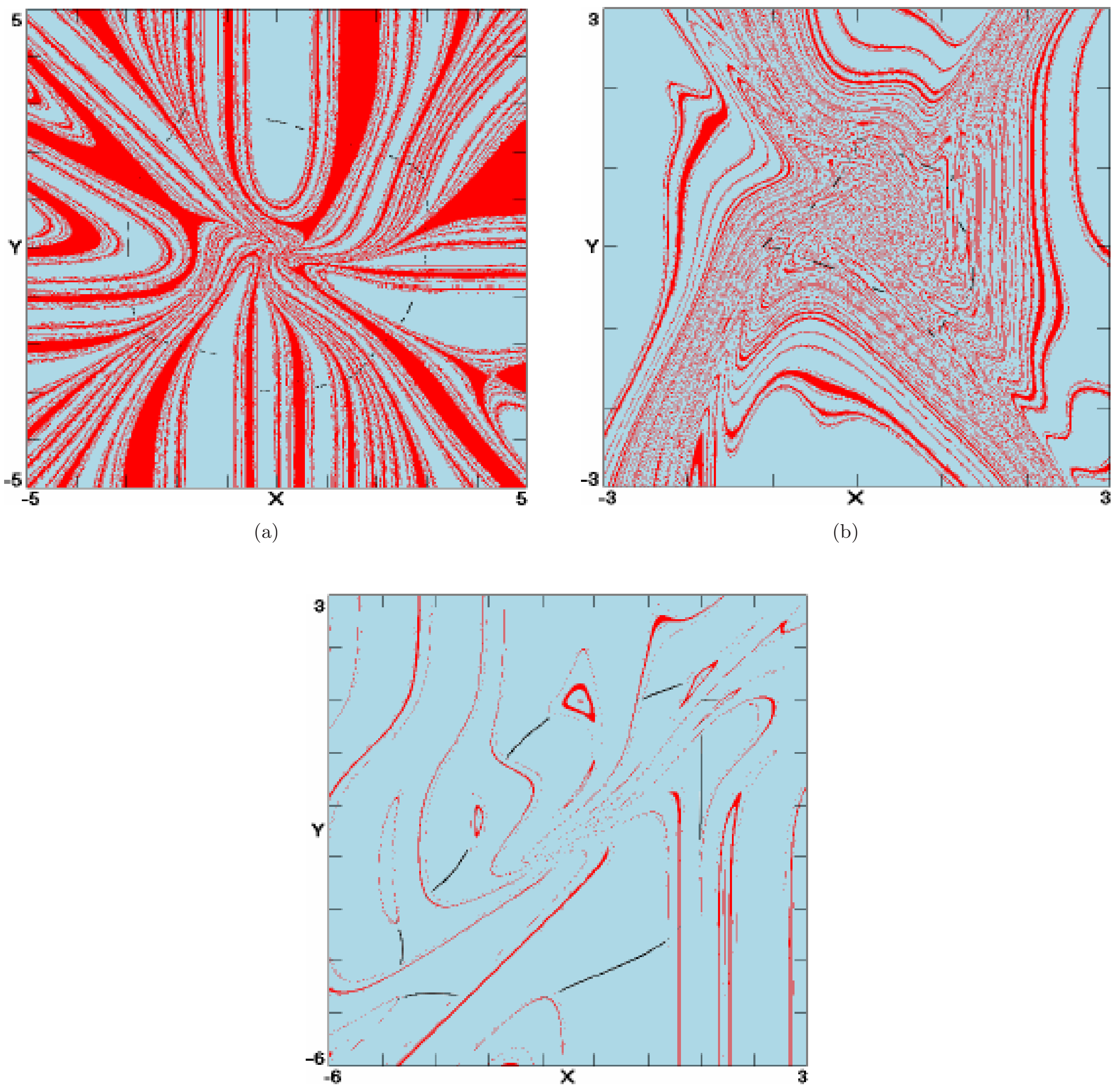
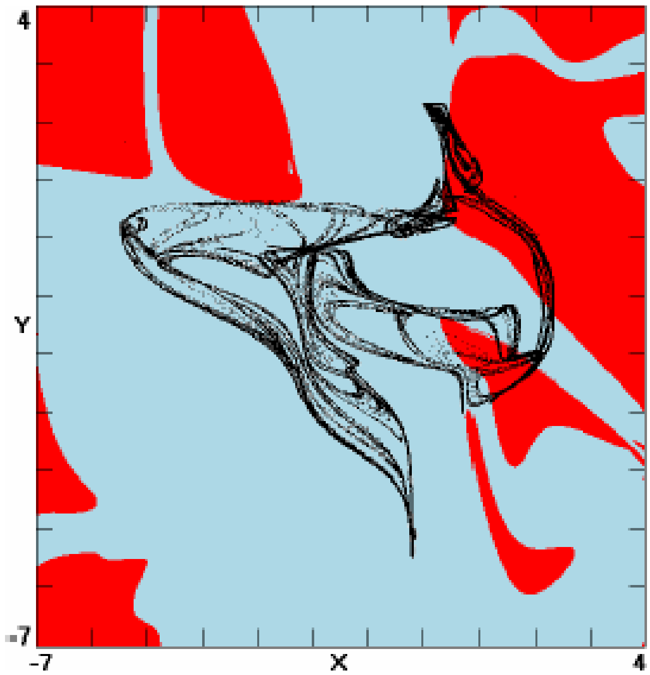


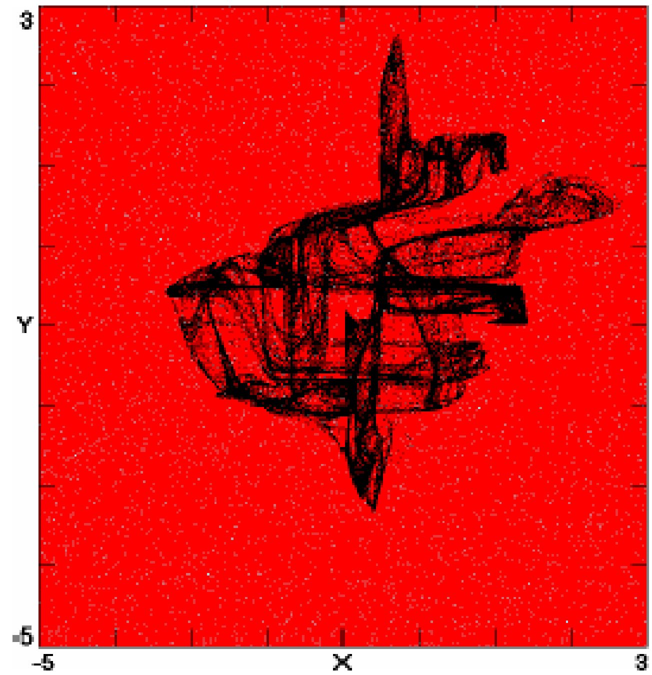
Fig. 1. 2-D cross-sections of a 2-D basin of attraction of system (1) with four neurons.

we found 76 cases. In almost all cases, the strange attractor is globally attracting with no other attractors. There are eight exceptions, some of which are shown in Fig. 2. We remark that some of the basins are apparently *riddled* (every point in the basin is arbitrarily close to another basin).

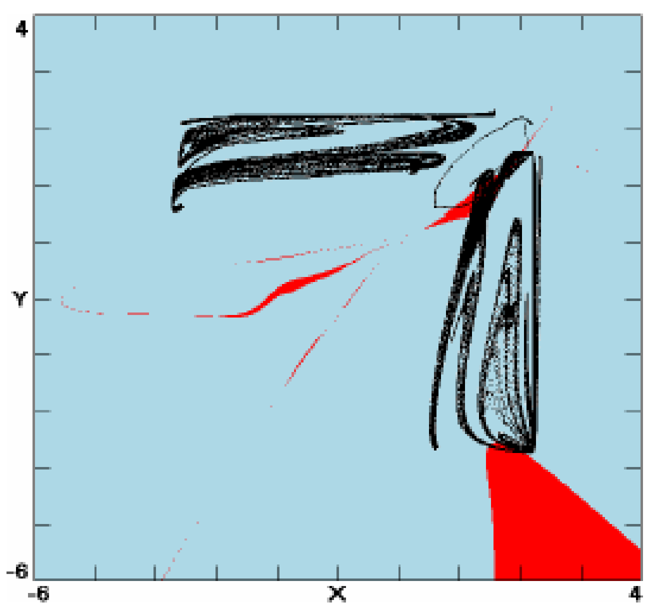
As a real world application, we notice that the notion of basin of attraction is widely used as a measure of the disturbance rejection for biped robots. This basin is a total set of state variables from which the walker can walk successfully as shown in [Ning *et al.*, 2007]. Now, if the underactuated biped robot system is approximated with a network of the form (1), then we can see that it has a universal basin of attraction in the form of an open ball. It is well known that if the size of the basin of attraction is large, then the stability of a biped robot system is strong. There are several methods used to



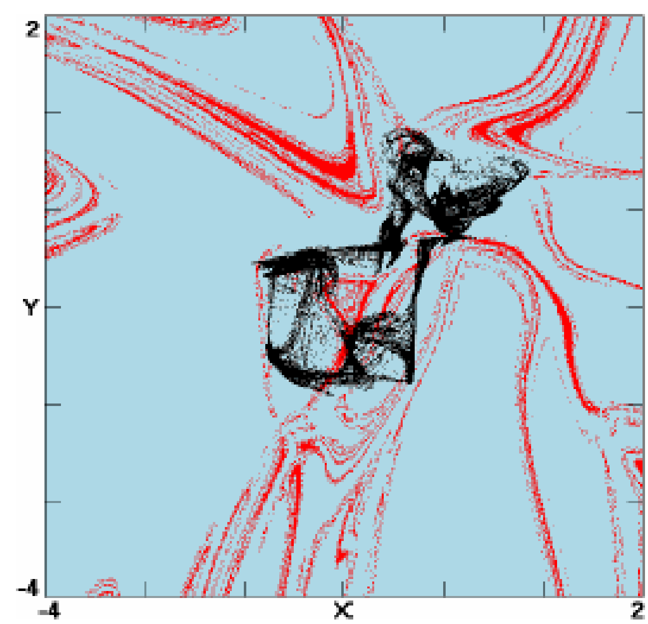
(a)



(b)



(c)



(d)

Fig. 2. 2-D cross-sections of a 4-D basin of attraction of system (1) with two neurons.

estimate the basin of attraction for the biped robot. For example, the *cell mapping method* was proposed in [Schwab & Wisse, 2001] to compute the basin of attraction for the simplest walking model with point feet and the planar model with round feet. Although this method is effective, it is time-consuming for multidimensional state space as shown in [Zhang *et al.*, 2009]. Since networks are universal function approximators, the approach presented in this letter can be considered as an alternate method for finding basins of attraction. All we need is to make a good approximation using networks of the form (1).

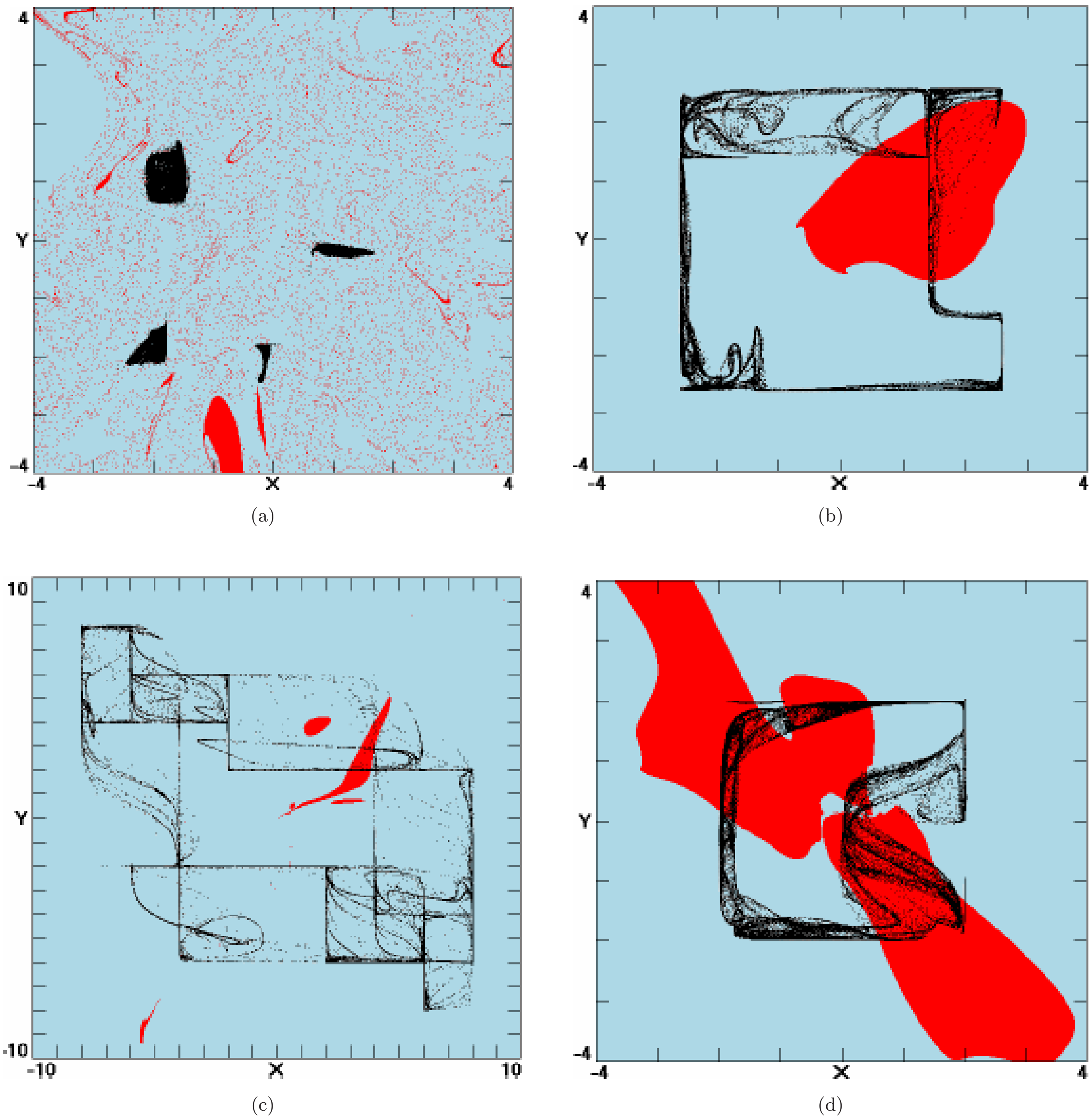
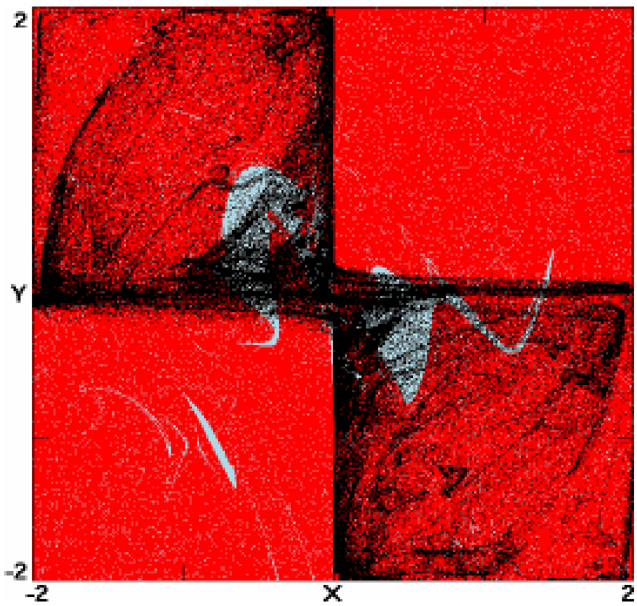
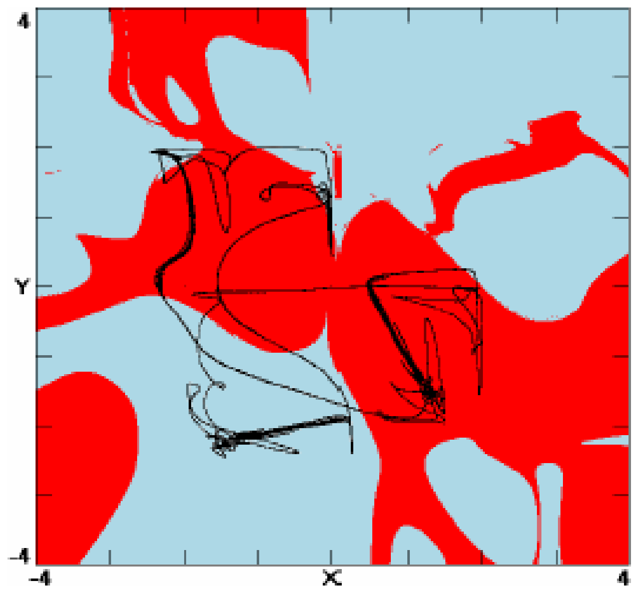


Fig. 3. 2-D cross-sections of an 8-D basin of attraction of system (1) with four neurons.



(a)



(b)

Fig. 4. 2-D cross-sections of the 8-D basin of attraction of system (1) with four neurons.

3. Conclusion

In this letter, we show the existence of invariant sets called universal basins of attraction for high-dimensional dynamics. The method of analysis is based on the definition of an equivalence class between systems with the same characteristics. The relevance of this result is that the types of invariant sets are unchanged between such systems. Hence a fundamental question is whether it is possible to classify high-dimensional systems (or a part of them) into equivalence classes. This question leaves open a nontrivial problem whose solution is not immediately evident.

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