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Is A Unifying Chaotic Dynamical System Possible?

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Abstract

Strange attractors are classified into three principal classes: hyperbolic, Lorenz-type, and quasi-attractors. This paper discusses the possibility of finding a rigorous mathematical model describing the three types of chaotic attractors and the transitions among them.

Keywords: *Strange attractors, unified model, system dimensions.*

1 Introduction

A global unification of processes in nature is a very important question. There are some known efforts toward unification of forces. The origin of this problem is the fact that interactions between objects are described with only four fundamental forces. Thus, the possibility of describing them in terms of one master force is under investigation since the pioneer imagination of Albert Einstein. In the domain of differential equations, we investigate the possibility of the same approach for unifiying different types of chaotic motions known in the current literature.

Generally, strange attractors can be classified into three principal classes: hyperbolic, Lorenz-type, and quasi-attractors, as described in [Anishchenko & Strelkova, 1998, Anishchenko, *et al.* 1998, Mira, 1997]. The definitions, detailed properties, and examples of these types of attractors are found in [Zeraoulia & Sprott, 2011]. Note that several examples of these of attractors display some properties of the other types. For example, it was shown in

[Bykov & Shilnikov, 1989] that the bifurcation diagram of the Lorenz system exhibits a region containing the Lorenz attractor, outside of which is a region containing a chaotic quasi-attractor, i.e., there is a bifurcational transition to a nonhyperbolic attractor as shown in [Anishchenko, *et al.* 1998]. Similar phenomena are observed for hyperbolic and quasi-attractors. See [Zeraoulia & Sprott, 2011] for more details.

This paper is concerned with the following problem: *Is a unifying chaotic dynamical system possible*? This question speaks to the possibility of finding a rigorous mathematical model describing the three types of chaotic attractors. For this approach, there are three main problems: continuous-time versus discrete-time, the dimensionality of systems, and the fact that many chaotic systems are not modeled or at least have no rigorous description in terms of mathematical equations. This last point is hard to analyze here, and it is the origin of the main question studied in this paper.

For systems with known mathematical equations, we propose in this paper an approach based on some homotopic functions and three known dynamical systems corresponding to the three types of chaotic attractors.

2 A unified model for systems with the same dimension

In this section, we propose a mathematical model as a unified model for maps (continuous-time systems) having the same dimension n. Indeed, consider f, g, and h as functions corresponding to three discrete mappings (continuous-time systems) displaying respectively hyperbolic, Lorenz-type, and quasi-attractors. The choice of these systems is arbitrary. The most important properties are that they have the same dimension and each of them displays the corresponding type of chaotic attractor, at least numerically.

The proposed systems for discrete maps and continuous-time systems are, respectively, $x_{k+1} = u(\alpha) f(x_k) + w(\alpha) g(x_k) + v(\alpha) h(x_k)$ and $x' = u(\alpha) f(x) + w(\alpha) g(x) + v(\alpha) h(x)$ where $\alpha \in [0,1]$ is the unified parameter and $u(\alpha) = 2(1-\alpha)\left(\frac{1}{2} - \alpha\right), w(\alpha) = 4\alpha(1-\alpha)$ and $v(\alpha) = 2\alpha\left(\alpha - \frac{1}{2}\right)$. Here we choose that for $\alpha = 0$, the unified map (resp, the unified continuous-time system) displays a hyperbolic attractor, for $\alpha = \frac{1}{2}$, the map (resp, the unified continuous-time system) displays a Lorenz-type attractor, and for $\alpha = 1$, the map (resp, the unified continuous-time system) displays a quasi-attractor, where $\alpha \in [0,1] - \left\{0, \frac{1}{2}, 1\right\}$. Finally, the unified chaotic map (resp, the unified continuous-time system) is chaotic with three different kinds of attractors. As an example of the above analysis in the plane, one can choose, respectively, any variant of the Anosov torus in $\mathbb{T}^2$, the Lozi mapping, and the Hénon mapping. Furthermore, it is possible to create any of these types of chaotic attractor at any point $\alpha \in [0,1]$ by the same approach and by a permutation of the functions u, v, and w. For example, to construct a Lorenz-type attractor at a point $\alpha_1 \in (0,1)$, one can choose the functions u, v, and w as follows: $u(\alpha) = \frac{(1-\alpha)(\alpha_1-\alpha)}{\alpha_1}$, $w(\alpha) = \frac{\alpha(1-\alpha)}{\alpha_1(1-\alpha_1)}$, and $v(\alpha) = \frac{\alpha(\alpha-\alpha_1)}{1-\alpha_1}$.

The above procedure can be generalized to any interval $[a,b]$ either by recopying the same steps or by using the fact that any interval $[a,b]$ is homeomorphic to the interval $[0,1]$.

3 A unified model for systems with different dimensions

In this section, we propose a mathematical model as a unified model for maps (continuous-time systems) with different dimensions. Indeed, consider f, g, and h as functions corresponding to three discrete mappings (continuous-time systems) displaying, respectively, hyperbolic, Lorenz-type, and quasi-attractors. The respective dimensions are n, m and q. Let $r(\alpha) = [u(\alpha)]\,n + [w(\alpha)]\,m + [v(\alpha)]\,q \in \mathbb{N}$ be a suitable dimension for the resulting unified system, where $[z]$ is the integer part of the real number z. We have $r(0) = n, r(\alpha_1) = m$ and $r(1) = q$. Consider three matrices $U(\alpha), W(\alpha)$, and $V(\alpha)$ with respective dimensions $r \times n, r \times m$, and $r \times q$. Hence the proposed systems are $x_{k+1} = U(\alpha) f(x_k) + W(\alpha) g(x_k) + V(\alpha) h(x_k)$ for discrete maps and $x' = U(\alpha) f(x) + W(\alpha) g(x) + V(\alpha) h(x)$ for continuous-time systems. The matrices $U(\alpha), W(\alpha)$, and $V(\alpha)$ are defined by the functions $u(\alpha), w(\alpha)$, and $v(\alpha)$ in their diagonals and zeros in all other entries. Again, it is possible to construct any type of chaos at any point $\alpha_1 \in (0,1)$ or (a,b).

4 Conclusion

In this paper, we present a new method to generate a unified mathematical model displaying rigorously the three main types of chaotic attractors known in the current literature. This result is the first step toward a global unification of processes in nature.

Finally, we introduce the following first open problem:

(1) Is a unifying chaotic dynamical system possible?

and the second one is about the persistence property of each type of chaos in the model. Indeed, dynamical persistence means that a behavior type, *i.e.*, equilibrium, oscillation, or chaos does not change with functional perturbation or parameter variation. Mathematically, *persistent chaos* (p-chaos) of degree p for a dynamical system can be defined as follows: Assume a map $f_\xi : X \to X \subset \mathbb{R}^d$ depends on a parameter $\xi \in \mathbb{R}^k$. The map f_ξ has chaos of degree-p on an open set $\mathcal{O} \subset X$ that is persistent for $\xi \in \mathcal{U} \subset \mathbb{R}^k$ if there is a neighborhood $\mathcal{N}$ of $\mathcal{U}$ such that $\forall\, \xi \in \mathcal{N}$, the map f_ξ retains at least $p \geq 1$ positive Lyapunov characteristic exponents (LCEs) for Lebesgue almost every X in $\mathcal{O}$.

In the previous analysis, the proposed mathematical system guaranteed the nature property of the attractor only in a point in the bifurcation parameters space, thus, we have the following question:

(2) Is there is a proposed model in which the persistence property is verified by the resulting chaotic attractor in a specific region of space of bifurcation parameters?

This problem add another view to the global properties of possible models unifying chaotic phenomena.

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