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On a conjecture about monomial Hénon mappings

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Abstract: ***A monomial Hénon mapping is defined as the well-known two-dimensional Hénon map with the quadratic term replaced by a monomial. This paper introduces a conjecture about monomial Hénon mappings: Even Hénon mappings are chaotic and odd Hénon mappings are not chaotic in the first quadrant of the bifurcation parameter space. This conjecture is based on numerical simulations of this type of map.***

Keywords: *Monomial Hénon mappings, Chaos, No chaos.*
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1 Introduction

Many papers have described 2-D discrete chaotic maps, the most famous of which is the one proposed by Hénon in [1]

$$f_2 : \begin{cases} x_{n+1} = 1 - ax_n^2 + by_n \\ y_{n+1} = x_n \end{cases} \tag{1}$$

and studied in detail by others [2-9 and references therein]. Bounded chaotic attractors exist for the Hénon map (1) over a range of positive values of a and b. In the current literature the form of this map has been modified to obtain other chaotic attractors [3-4-5-6] with special properties and real-world applications [7-8].

2 Monomial Hénon mappings and chaos

We define a *monomial Hénon mapping* as the two-dimensional map obtained by replacing the quadratic term in (1) by a monomial. In this paper, we investigate the dynamics of the monomial Hénon mappings given by

$$f_m : \begin{cases} x_{n+1} = 1 - ax_n^m + by_n \\ y_{n+1} = x_n \end{cases} \tag{2}$$

where the quadratic term x^2 in the Hénon map (1) is replaced by the monomial x^m, with m is a positive integer. Maps of the form (2) are the simplest discrete systems with a polynomial of degree m.

In this paper, numerical calculations are performed to find regions of different types of solutions in the ab-plane for the map (2). The regions of different solutions of the map f_m are marked as follows: unbounded solutions (white), fixed points (gray), periodic orbits (blue), quasi-periodic orbits (green), and chaotic attractors (red). We have used $|LE| < 0.0001$ as the criterion for quasi-periodic orbits with 10^6 iterations for each point and initial conditions at $(x, y) = (0, 0)$. The result is surprising and shows that mappings (2) with m even are chaotic for some positive values of a and b while mappings (2) with m odd are not chaotic for any positive values of a and b. Indeed, for $m = 1$, the map is linear and hence cannot be chaotic. For $m = 2$, the result is well known: The original Hénon map is chaotic as shown in Fig. 1. For $m = 3$, the regions of dynamical behaviors in ab-space are shown in Fig. 2. Apparently, the map f_3 is not chaotic. It displays only fixed points and periodic orbits. The map f_4 displays chaos as shown in Fig. 3. We continue observing this phenomenon alternatively: The maps f_{2m} are chaotic for positive values of a and b just like f_2, and the maps f_{2m+1} are not chaotic just like the linear map f_1.

3 Open problems

We propose the following conjecture:

Conjecture 1 *Even monomial Hénon mappings can be chaotic and odd monomial Hénon mappings cannot be chaotic in the first quadrant of the bifurcation parameter space.*

This conjecture leads to the following questions:

Problem 1: What are the principale consequences (related to dynamics of these maps) if this conjecture is true?

Problem 2: Is numerical calculations are the only way to verify this conjecture?

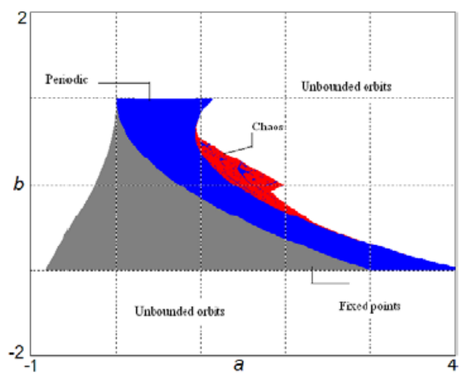

Figure 1: Regions of dynamical behavior in the ab-plane for the Hénon map f_2.

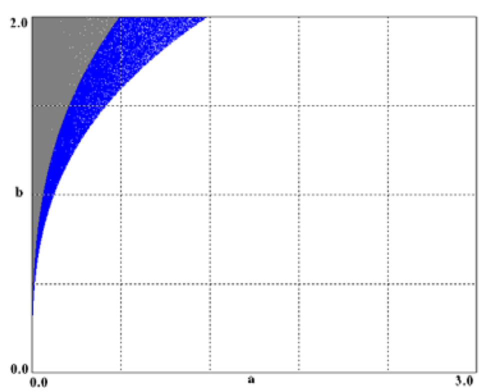

Figure 2: Regions of dynamical behaviors in the ab-plane for the monomial Hénon map f_3.

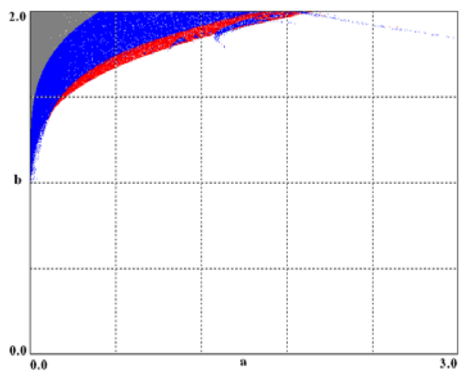

Figure 3: Regions of dynamical behaviors in the ab-plane for the monomial Hénon map f_4.

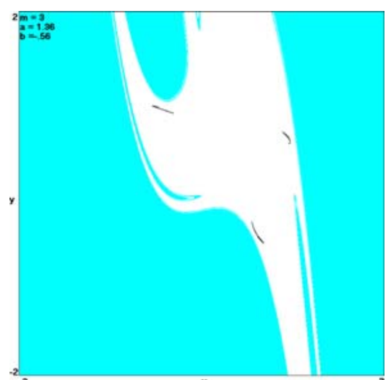

Figure 4: A chaotic attractor obtained from the map f_3 for $a = 1.36$ and $b = -0.56$.

One approach to proving this conjecture is to show the equivalence between even monomial Hénon mappings $(f_{2m+1})_{m\in\mathbb{N}}$ and the linear map f_1 which is not chaotic and the equivalence between odd monomial Hénon mappings $(f_{2m})_{m\in\mathbb{N}}$ and the map f_2 which is chaotic. However, this approach seems to be wrong since numerical calculations suggest that even and odd monomial Hénon mappings have different numbers of fixed points.

The conjecture is not true for negative b. Indeed, Figs. 4-5 show such cases in which the chaotic attractors appear to be one-dimensional, i.e., in each case, the attractor is apparently three short line segments (shown in black) located near the boundary of its basin of attraction (the white region surrounded by cyan).

A second open problem is concerned with the boundedness of the monomial Hénon mappings given by (2) since it is well known the Hénon mapping (1)

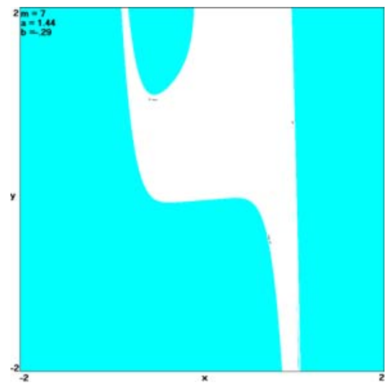

Figure 5: A chaotic attractor obtained from the map f_7 for $a = 1.44$ and $b = -0.29$.

is bounded for some values of a and b as shown in [2]. hence, we state the following problem:

Problem 3: Find values of a, b and m such the monomial Hénon mappings given by (2) is bounded.

A fisrt idea to solve this problem is to follow the arguments used in [2]. However, this approach seems to be valid only for $m = 2$ in (2).

4 Conclusion

In this paper we have introduced a new conjecture about monomial Hénon mappings, claiming that even Hénon mappings can be chaotic and odd Hénon mappings cannot be chaotic in the first quadrant of the bifurcation parameter space. This conjecture is based on numerical simulations, which appears to be the only technique available to check the result. A confirmation of this conjecture implies that there is a well-defined order (maybe a polarization) within this family of mappings.

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