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A New Cost Function for Parameter Estimation of Chaotic Systems Using Return Maps as Fingerprints

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Estimating parameters of a model system using observed chaotic scalar time series data is a topic of active interest. To estimate these parameters requires a suitable similarity indicator between the observed and model systems. Many works have considered a similarity measure in the time domain, which has limitations because of sensitive dependence on initial conditions. On the other hand, there are features of chaotic systems that are not sensitive to initial conditions such as the topology of the strange attractor. We have used this feature to propose a new cost function for parameter estimation of chaotic models, and we show its efficacy for several simple chaotic systems.

Keywords: Chaotic system; parameter estimation; cost function; return maps; butterfly effect.

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1. Introduction

Chaos is a common feature in nonlinear dynamical systems. Many systems in fields such as biology and economics exhibit chaos, and the study of such systems and their signals has progressed in recent decades. It has been claimed that many biological systems, including the brain (both in microscopic and macroscopic aspects) [Rabinovich *et al.*, 2006; Korn & Faure, 2003; Gong *et al.*, 1999] and the heart [Signorini *et al.*, 1997; Kantz & Schreiber, 1997], have chaotic properties, as well as the atmosphere [Patil *et al.*, 2001] and electronic circuits [D'Humieres *et al.*, 1892; Cumming & Linsay, 1988].

Control and synchronization of chaotic systems have been investigated in a variety of fields during the past two decades [Ott *et al.*, 1990; Hubler, 1989]. Most existing approaches require adjusting the parameters of the model. These parameters provide insight into their complex behavior. However, direct measurement of experimental system parameters is often difficult, although the time series of a chaotic experiment can usually be recorded. Therefore, estimating model system parameters from an observed chaotic scalar time series has become an active area of research [Wang & Xu, 2011; Wang & Li, 2010; Tao *et al.*, 2007; Konnur, 2005; Mukhopadhyay & Banerjee, 2012; Tien & Li, 2012; Sun *et al.*, 2010; Tang & Guan, 2009; Yuan & Yang, 2012; Modares *et al.*, 2010; Wang *et al.*, 2011; Tang *et al.*, 2012; Li *et al.*, 2006b, 2012; Yang *et al.*, 2009; Coelho & Bernert, 2010; Marino & Miguez, 2006; Li *et al.*, 2006a; Li & Yin, 2012; Li *et al.*, 2011; Chang *et al.*, 2008; Chang, 2007].

A basic method for achieving this goal involves optimization [Wang & Xu, 2011; Wang & Li, 2010; Tao *et al.*, 2007; Konnur, 2005; Mukhopadhyay & Banerjee, 2012; Tien & Li, 2012; Sun *et al.*, 2010; Tang & Guan, 2009; Yuan & Yang, 2012; Modares *et al.*, 2010; Wang *et al.*, 2011; Tang *et al.*, 2012; Li *et al.*, 2006b, 2012; Yang *et al.*, 2009; Coelho & Bernert, 2010; Marino & Miguez, 2006; Li *et al.*, 2006a; Li & Yin, 2012; Li *et al.*, 2011; Chang *et al.*, 2008; Chang, 2007] in which the model parameters are chosen to minimize some cost function. Although many optimization approaches are used for this problem in the mentioned works (see Table 1), they have one thing in common. They define their cost functions based on similarity between a time series obtained from the real system and ones obtained from a model. They consider this similarity in the time domain. In other words, they regard the time correlation between two chaotic time series as the similarity indicator.

On the other hand, we believe this method has limitations and lacks consistency when the model has sensitive dependence on initial conditions [Aguirre & Billings, 1994; Wigdorowitz & Petrick, 1991] as does any chaotic system [Hilborn, 2001]. Thus there can be two completely identical (both in structure and parameters) chaotic systems that produce time series with essentially no correlation due to a tiny difference in their initial conditions

Table 1. Different optimization methods which have been used for parameter estimation of chaotic systems.

Method	Reference
Hybrid biogeography-based optimization algorithm	[Wang & Xu, 2011]
Hybrid quantum-inspired evolutionary algorithm	[Wang & Li, 2010]
Differential evolution algorithm	[Tang *et al.*, 2012; Li *et al.*, 2011; Chang, 2007]
Genetic algorithm	[Tao *et al.*, 2007]
Chaotic multiswarm particle swarm optimization	[Mukhopadhyay & Banerjee, 2012]
Hybrid Taguchi-chaos of multilevel immune and the artificial bee colony	[Tien & Li, 2012]
Drift particle swarm optimization	[Sun *et al.*, 2010]
Particle swarm optimization	[Tang & Guan, 2009; Modares *et al.*, 2010]
Hybrid Nelder–Mead simplex search and differential evolution algorithm	[Wang *et al.*, 2011]
Chaotic ant swarm	[Li *et al.*, 2006b]
Chaotic gravitational search algorithm	[Li *et al.*, 2012]
Quantum-behaved particle swarm optimization	[Yang *et al.*, 2009]
Modified ant colony optimization algorithm based on differential evolution	[Coelho & Bernert, 2010]
Gradient-descent method	[Marino & Miguez, 2006]
Hybrid PSOSA strategy	[Li *et al.*, 2006a]
Cuckoo search algorithm with an orthogonal learning method	[Li & Yin, 2012]
Evolutionary programming	[Chang *et al.*, 2008]

[Jafari *et al.*, 2012b; Jafari *et al.*, 2013b]. We will discuss this problem in detail in the next part. In the third section we propose a new cost function based on the similarity in return maps between data from the real system and its model. In Sec. 4 we use this idea for chaotic flows using return maps (or Poincaré maps), and finally Sec. 5 is the conclusion.

2. Time Domain Versus State Space

Consider an iterated map (a set of difference equations) known to be chaotic:

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \boldsymbol{\theta}) \tag{1}$$

in which $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the state vector of the system and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ is a set of parameters. In all the examples that follow, it is assumed that we know the form of the system map but not the parameters. Thus we have a model of the form

$$\mathbf{y}_{k+1} = \mathbf{f}(\mathbf{y}_k, \tilde{\boldsymbol{\theta}}) \tag{2}$$

in which $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is the state vector of the model and $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_m)$ is the set of estimated parameters. Our goal is to find values of $\tilde{\theta}$ that are close to θ when we do not know θ but only have access to a measured scalar time series from the system.

A simple conventional method is to define the following cost function (or something conceptually similar) [Wang & Xu, 2011; Wang & Li, 2010; Tao *et al.*, 2007; Konnur, 2005; Mukhopadhyay & Banerjee, 2012; Tien & Li, 2012; Sun *et al.*, 2010; Tang & Guan, 2009; Yuan & Yang, 2012; Modares *et al.*, 2010; Wang *et al.*, 2011; Tang *et al.*, 2012; Li *et al.*, 2006b, 2012; Yang *et al.*, 2009; Coelho & Bernert, 2010; Marino & Miguez, 2006; Li *et al.*, 2006a; Li & Yin, 2012; Li *et al.*, 2011; Chang *et al.*, 2008; Chang, 2007]:

$$\text{Cost Function} = \sum_{k=1}^{N} \|\bar{\mathbf{y}}_k - \bar{\mathbf{x}}_k\| \tag{3}$$

where N denotes the length of the time series used for parameter estimation, $\|\cdot\|$ is the Euclidean norm, and $\bar{\mathbf{x}}_k$ is those elements of $\mathbf{x}$ to which we have experimental access. Because of limitations in the measurement instruments and the environment, all experimental data are mixed with noise to some extent [Kantz & Schreiber, 1997]. Thus we can never have the exact initial conditions of the system (the error may be small, but not zero). Here, we deal with chaotic systems whose main characteristic is their sensitive dependence on initial conditions and thus for which errors in the time domain are not a good indicator as clarified by the following simple

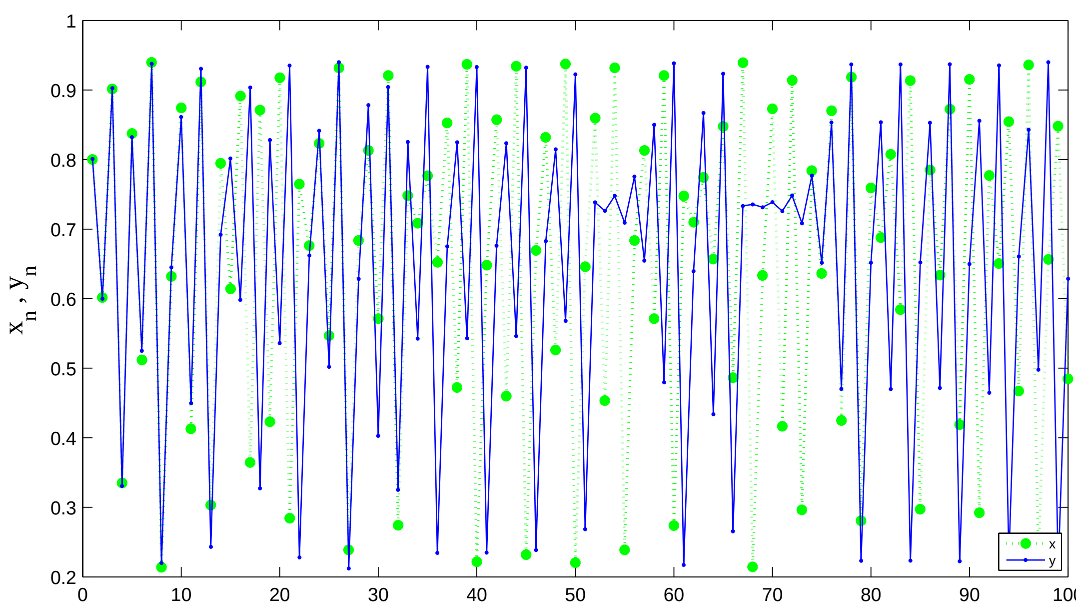


Fig. 1. Chaotic evolution of the logistic map with $A = 3.76$ showing the effect of a difference in the initial conditions of 0.1%. The initial conditions are $x_1 = 0.8$ and $y_1 = 0.8008$.

example. Consider the logistic map,

$$x_{k+1} = Ax_k(1 - x_k). \tag{4}$$

Suppose that the value of A in the real system is 3.76 (for which the map is chaotic) and we have a model exactly like our system:

$$y_{k+1} = Ay_k(1 - y_k). \tag{5}$$

Figure 1 shows the effect of a difference in the initial conditions of 0.1%.

If we calculate the cost function based on Eq. (4) with that 0.1% difference in initial conditions, we see that this method does not give a proper cost function (Fig. 2) because it does not have a global minimum at 3.76, and it is not monotonic on both sides of that minimum (it is not convex). Furthermore, we expect discontinuties in the cost function due to the bifurcations that typically occur in chaotic systems. Especially in the periodic windows, we may see local maxima in the cost function since the behavior will be very different from the chaotic case with $A = 3.76$. As can be seen, the global minimum occurs near $A = 3$, which is far from the correct value.

One way to solve this problem is by controlling the chaotic system to take it out of the chaotic regime. Then using the same controller on the model, one can estimate the parameters in the new situation where there is no sensitivity to initial conditions. Such a method has limitations since many chaotic systems such as the atmosphere, brain, or heart, are not controllable. Or in some cases the data may have been gathered previously and are no longer accessible.

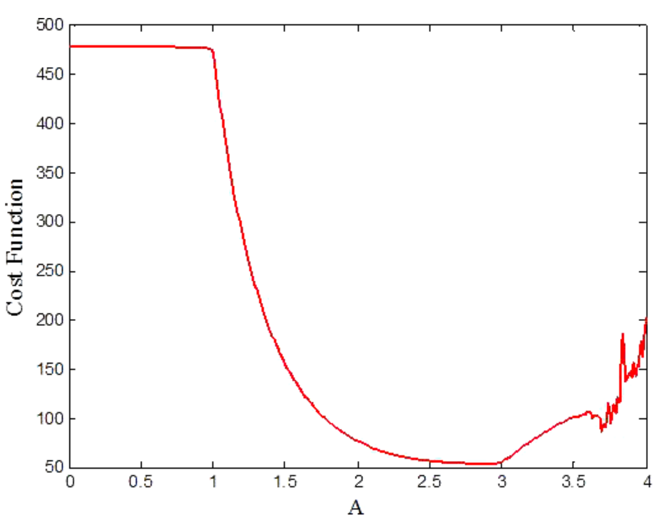


Fig. 2. Cost function obtained from Eq. (4). The initial conditions are $x_1 = 0.8$ and $y_1 = 0.8008$ with $N = 1000$.

Another reasonable but not always possible way to overcome this problem uses near-term correlation and frequent reinitialization of the model. However, this method has limitations which are discussed in detail in [Rowlands & Sprott, 1992]. Furthermore, all the states of the system may not be observable or measurable, and thus it may not be possible to repeatedly reinitialize the model. One may only have access to one of the variables in a high-dimensional system, or the observed variable may have been transformed by a complicated or unknown observer function. Such examples will be shown in Secs. 3.2 and 3.4.

In chaotic systems, there are some features that are not sensitive to initial conditions (invariants) such as the Lyapunov exponents and attractor dimension [Hilborn, 2001] (note that while this fact is widely used and is correct for most systems, in the general case it may be not true, e.g. because of so-called Perron effects of Lyapunov exponents sign reversal [Leonov & Kuznetsov, 2007; Kuznetsov *et al.*, 2014]). In fact, the need for more general criteria has motivated researchers to investigate geometrical and statistical invariants which would provide a means for characterizing nonlinear systems possessing low-dimensional chaotic dynamics. Thus embedded trajectories [Broomhead & King, 1986; Moon, 1987], bifurcation diagrams [Haynes & Billings, 1993], Lyapunov exponents [Wolf *et al.*, 1985; Abarbanel *et al.*, 1989, 1990], and correlation dimension [Grassberger & Procaccia, 1983; Wolf & Bessoir, 1991] have been used to characterize and compare reconstructed attractors and identified models.

Although it may seem that such quantities could be used as similarity indicators, they also have limitations. For example, many very different parameters give the same Lyapunov exponent for a chaotic system such as the logistic map [Hilborn, 2001], although it may be possible to find a weighted combination of such quantities that provides a proper cost function.

Although chaotic systems have random-like behavior in the time domain, they are ordered in state space and have a specific topology. In other words, there exists an attractor from which the system trajectories do not escape, even though the initial conditions change (of course, initial conditions should be in the basin of attraction of that

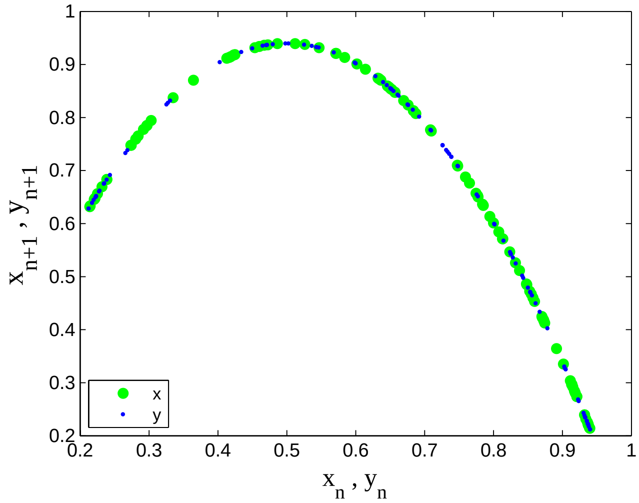


Fig. 3. Return maps for the logistic map with $A = 3.76$ showing independence of initial conditions. The initial conditions are $x_1 = 0.8$ and $y_1 = 0.8008$.

attractor, and there may be some difficulties due to e.g. riddled basins or Wada basins [Breban & Nusse, 2005; Alexander *et al.*, 1992]). For example, if we plot those random-like signals in Fig. 1 in return maps, we obtain ordered patterns (attractors) which are the same geometrically (Fig. 3). Even two completely different initial conditions result in similar patterns (Fig. 4). In this work we propose using the similarity between these attractors as the objective function for parameter estimation. In the next section we describe the algorithm used.

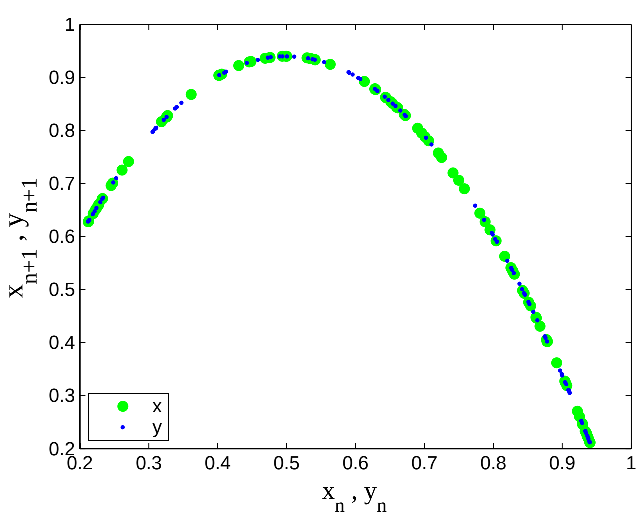


Fig. 4. Return maps for the logistic map with $A = 3.76$ showing independence of initial conditions. The initial conditions are $x_1 = 0.5$ and $y_1 = 0.9$.

3. Proposed Cost Function

To clarify the method, we describe the proposed algorithm with several specific examples. As mentioned in Sec. 2, we are dealing with nonlinear chaotic systems whose behavior can vary greatly and suddenly with changes in the parameters. Thus we expect rugged surfaces with many local optima. As the number of parameters increases, the curse of dimensionality appears. Therefore, we present four examples in which the dimension of the parameter space increases from one to four.

3.1. *Logistic map example* (*one parameter*)

The attractors of the logistic map for nine different values of A are shown in Fig. 5 after any transients have decayed. As can be seen, all values of A produce similar parabolas that monotonically change their amplitude. The only exception to this trend is at $A = 3.835$ where a period-3 window occurs and the pattern contains only three points on a parabola. We need a computational tool to quantify this qualitative discussion. There are many possible algorithms based on pattern recognition and image processing, but we here use a very simple algorithm.

Consider a time series of 300 samples from the logistic map with $A = 3.76$ after discarding 100 samples to remove any transient. Then for each value of A, we produce a time series from the model, which in this case is the same logistic map. The model time series need not be of the same length as the original data, and is here taken as 600 points. From the resulting two patterns in a 2-D embedding, we perform the following two steps:

(a) For each point of the real pattern we find its nearest neighbor in the model set of points and calculate its Euclidean distance separation.
(b) For each point of the model pattern we find its nearest neighbor in the real system set of points and calculate its Euclidean distance separation.

Then we take the cost function for each value of A to be the average of those distances over the whole data set.

It is very important to note that "a" and "b" steps do not work exactly the same and have different roles. Their difference is most evident in periodic windows. For example, consider the situation in Fig. 6. Although the two patterns are very different, if we only use only "a" to calculate the cost

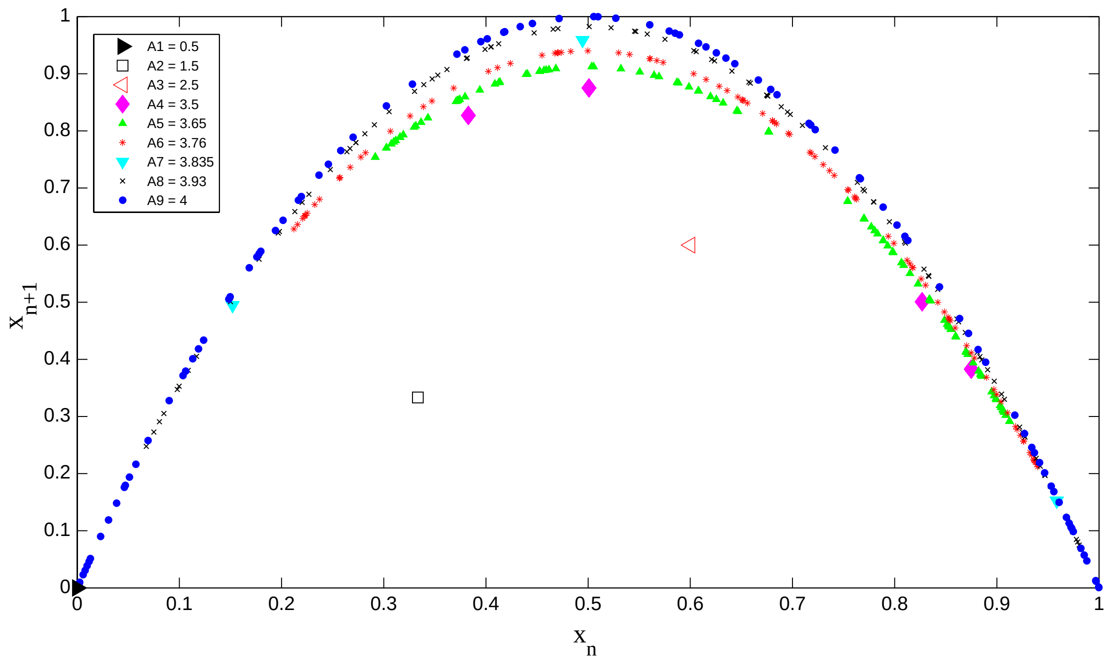


Fig. 5. Return maps for the logistic map with nine different values of A. The initial conditions are random numbers between 0 and 1. Transient parts of the data are omitted.

function, it results in a spuriously low cost, whereas also applying "b" solves this problem. Details can be found in the flowchart in Fig. 7 which shows the abovementioned procedure.

An important issue is determining the number of data points required for a proper comparison and quantifying the error that results from having a paucity of data. The answer depends on the case under consideration and always involves a trade-off. Fewer data samples may not cover the attractor, while having more data increases the simulation time and complexity. If there is no limitation in the runtime (as opposed to real-time applications which impose constraints on the runtime), one can bias the tradeoff in favor of accuracy (considering more samples). However, there are additional limitations. For example, one may have access to a limited sample of an experimental time series, whereas arbitrarily many samples are usually available from computer models.

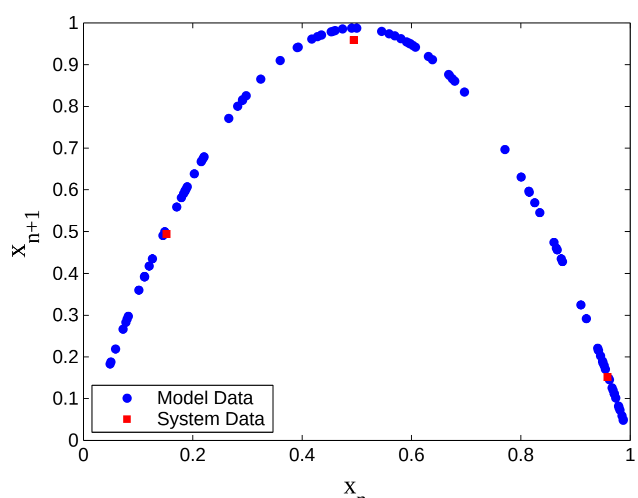


Fig. 6. Return maps for the logistic map with $A = 3.835$ (red squares) and $A = 3.95$ (blue circles). The initial conditions are random numbers between 0 and 1. Transient parts of the data are omitted.

The cost function for this case is shown in Fig. 8 along with a bifurcation diagram for the logistic map. As can be seen, this cost function has the desired ideal properties. It shows the effect of changing the parameter of the model, including the bifurcations and the monotonic trend along with a global minimum at exactly the right value of A (here $A = 3.76$).

3.2. *Logistic map with delayed observer function (one parameter)*

Although we chose a simple example in which the data and model are from the same equation (the logistic map) to demonstrate the method,

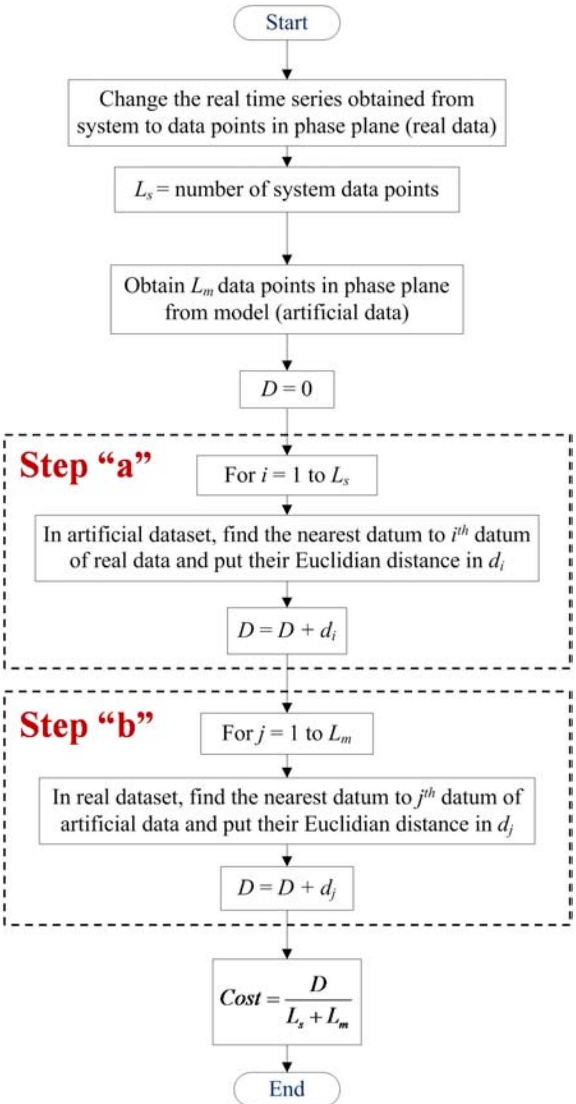


Fig. 7. Flowchart of the proposed method. L_s is the number of real data and L_m is the number of model data.

the method can be applied to more complicated situations. For example, consider the following system which is the logistic map with a delayed observer function:

$$\begin{aligned} x_{k+1} &= Ax_k(1 - x_k) \\ y_{k+1} &= x_{k+1} + x_k. \end{aligned} \tag{6}$$

If we only have access to the time series of y, we cannot use near-term correlation since we do not know the values of x and thus cannot use them to reinitialize the model.

However, our method can easily be applied to this system. For this observer function, the time series must be embedded in a 3-D space. Figure 9 shows the result for that system with $A = 3.9$.

3.3. *Hénon map example (two parameters)*

An example with more than one parameter is the Hénon map:

$$x_{n+1} = 1 - Cx_n^2 + Bx_{n-1}. \tag{7}$$

Assume that the data come from the system with $B = 0.2$ and $C = 1.4$. Since the Hénon map is two-dimensional, the return map will be embedded in a 3-D space (each iterate depends on two previous iterates). The cost function as seen in Fig. 10 is an approximately convex surface with a global minimum in the right place. For some values of B, C, and initial conditions, the solutions of the model are unbounded, resulting in infinite cost function solutions, and these cases have been fixed at a value of 0.1.

3.4. *Tinkerbell map example (four parameters)*

Consider the Tinkerbell map as a two-dimensional map with four parameters:

$$\begin{aligned} x_{n+1} &= x_n^2 - y_n^2 + ax_n + by_n \\ y_{n+1} &= 2x_ny_n + cx_n + dy_n. \end{aligned} \tag{8}$$

The parameters $a = 0.9$, $b = -0.6013$, $c = 1.8$ and $d = 0.5$ give a chaotic solution and are the values to be predicted.

Like the logistic map with the delayed observer function, this map is good for demonstrating limitations of any cost function in the time domain. If one has only access to the x or y time series in the experimental data, one cannot use near-term correlation because it is impossible to reinitialize the map every few samples (because initial conditions for both x and y are unknown; having only one of them is not enough). On the other hand, having four parameters makes it a difficult benchmark for illustration. We performed the method in the 4-D space of parameters to obtain the cost function. The boundaries of the search space were selected in such a way that the time series remains bounded. For visualization purposes, a section of the cost function with $b = -0.6013$ and $d = 0.5$ is shown in Fig. 11(a) [Fig. 11(b) shows its contour plot]. As can be seen, although the surface is rugged (caused by period windows) with many local minima, the global minimum is in the right place [white square in Fig. 11(b)], and the trend of the cost function

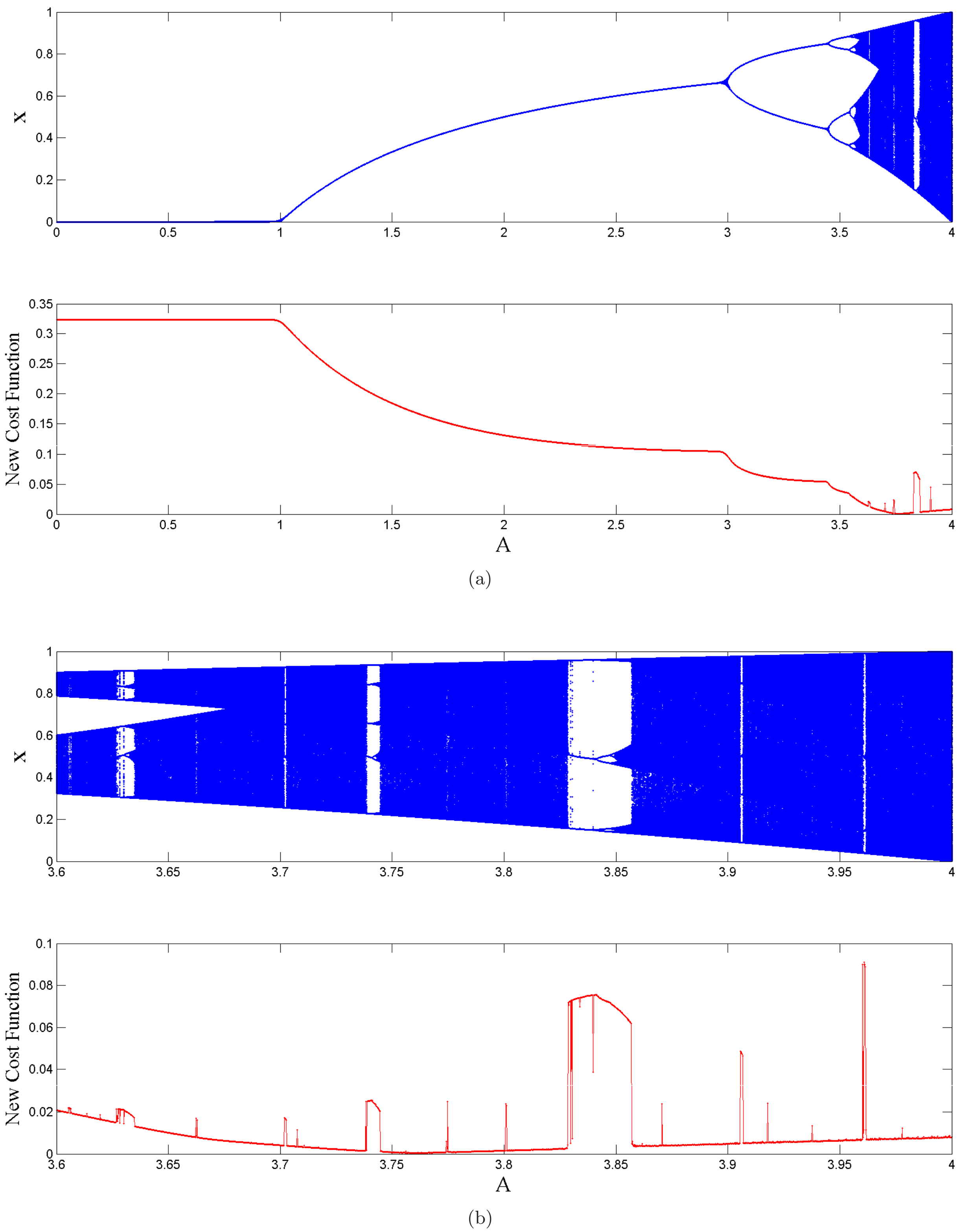


Fig. 8. New cost function obtained by the proposed algorithm. The initial conditions are random numbers between 0 and 1. Transient parts of data are omitted.

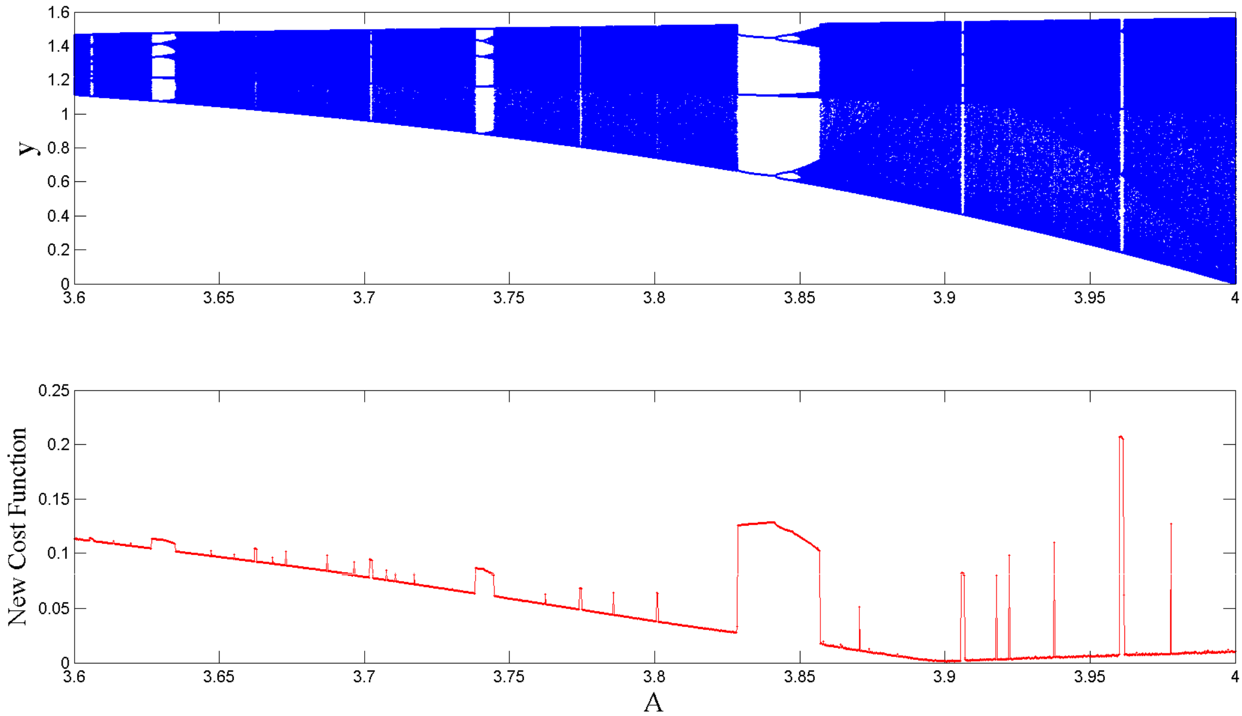


Fig. 9. New cost function obtained by the proposed algorithm for the system described by Eq. (6) with $A = 3.9$. The initial conditions are random numbers between 0 and 1. Transient parts of data are omitted.

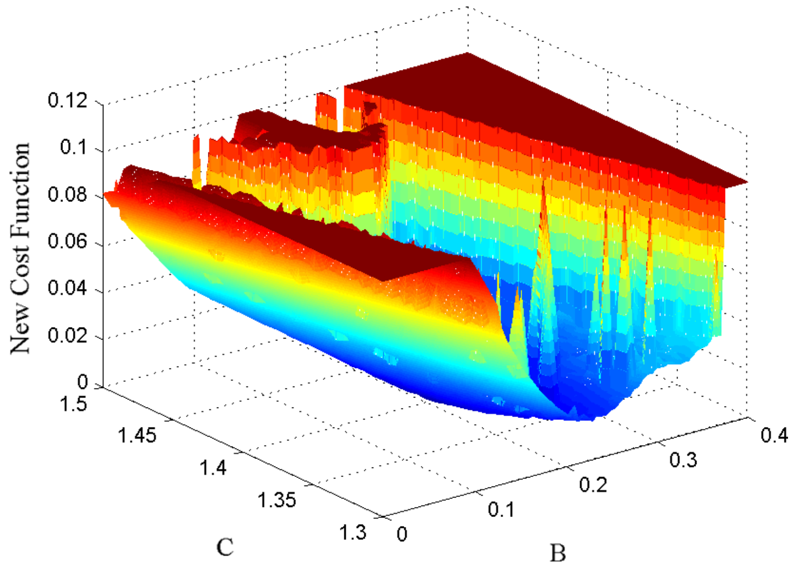
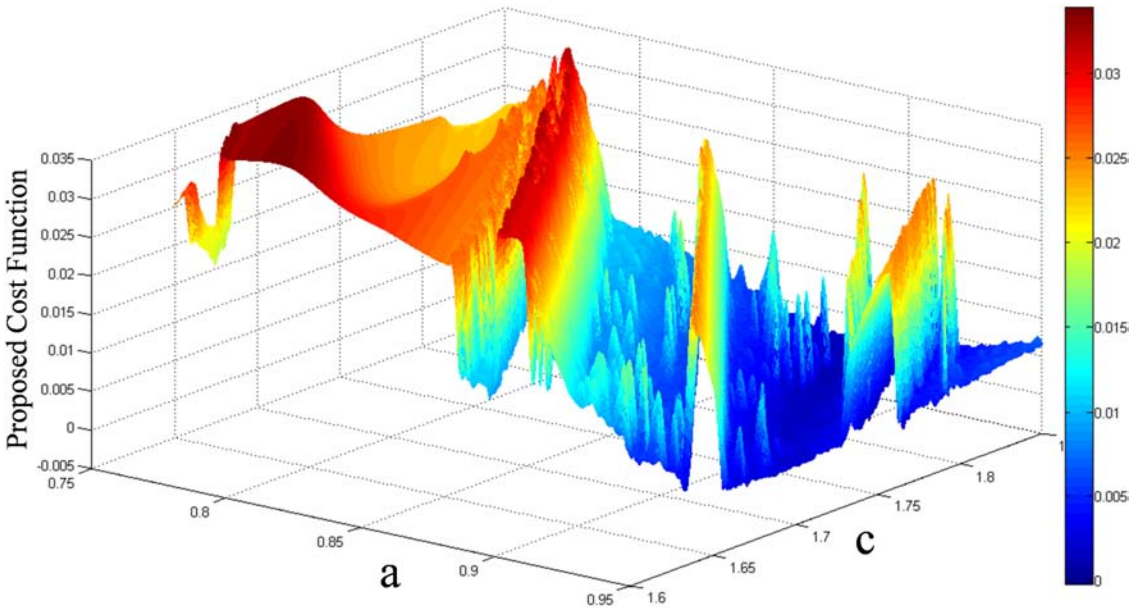
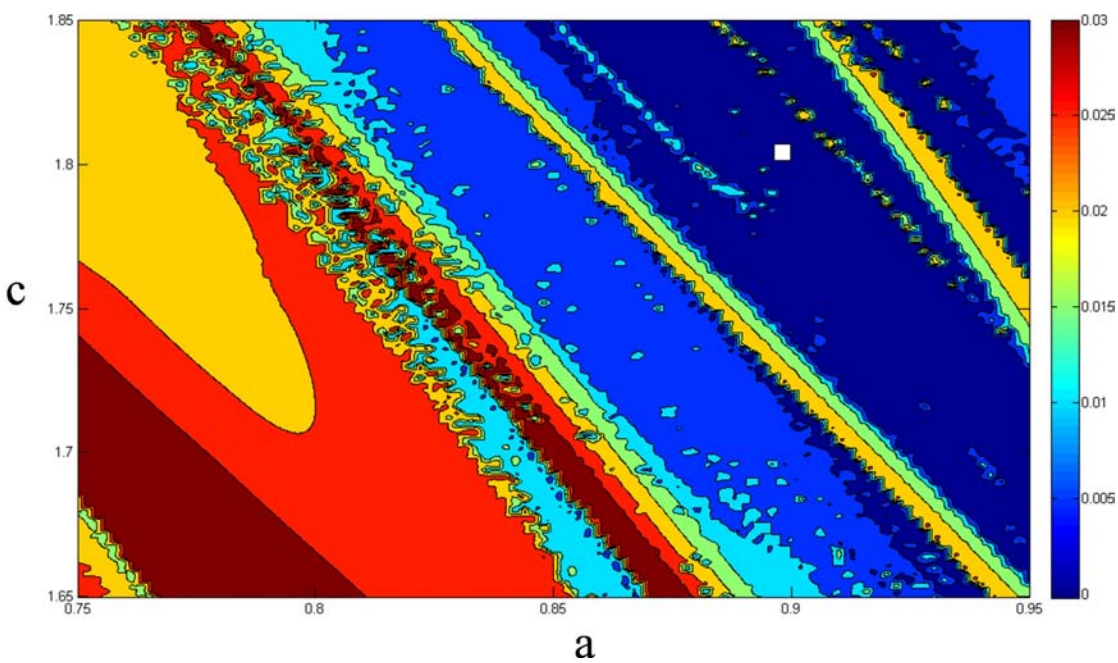


Fig. 10. New cost function obtained by the proposed algorithm for the Hénon map.

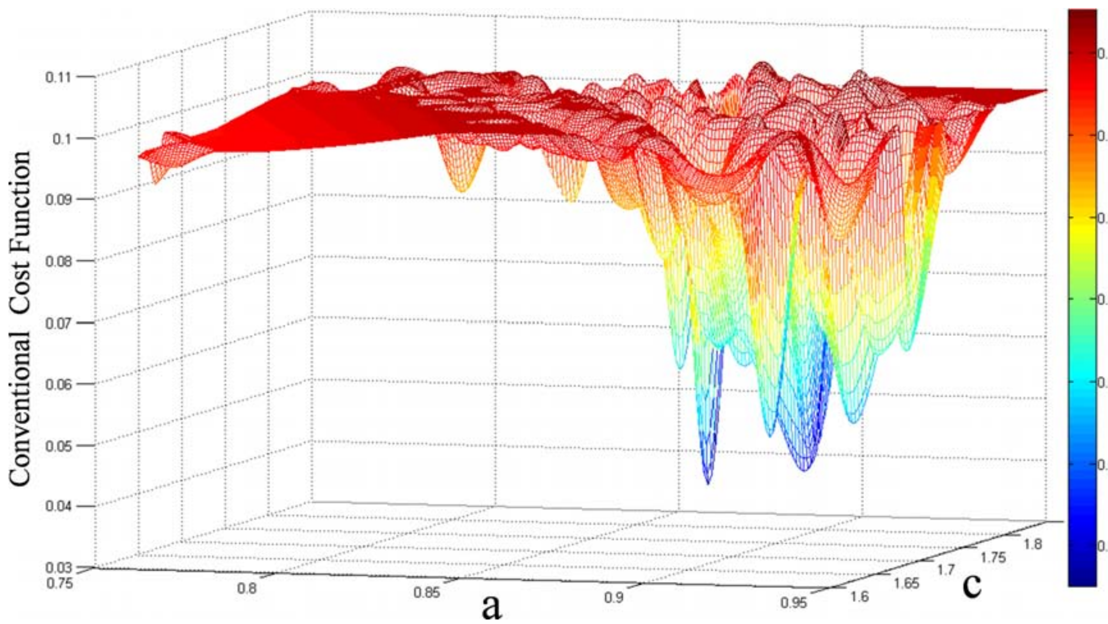


(a)

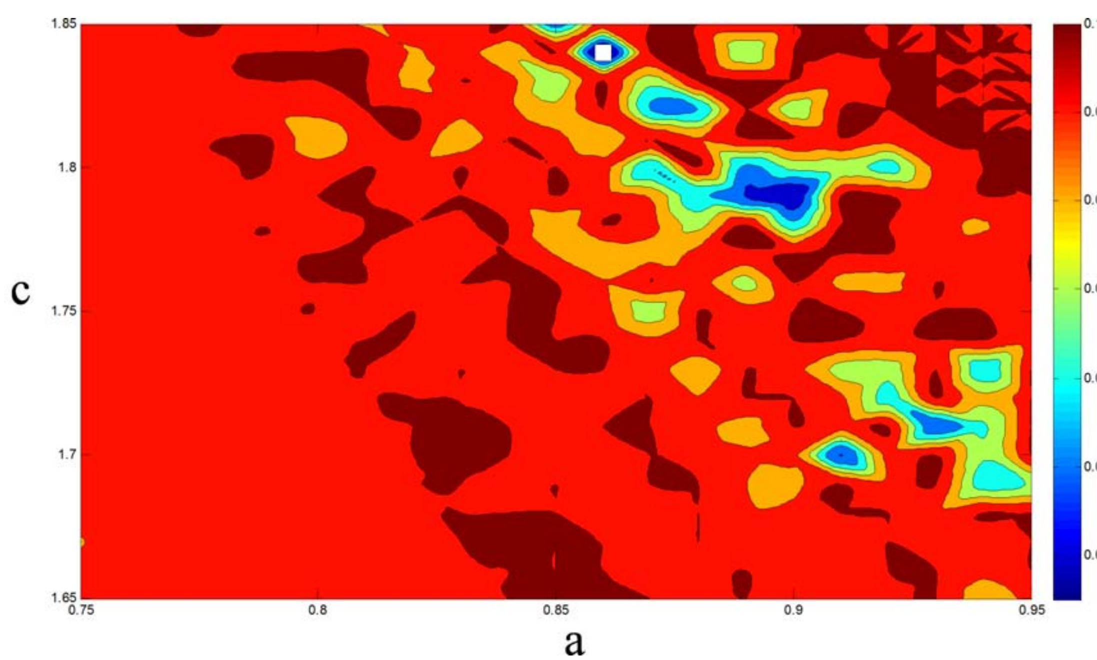


(b)

Fig. 11. New cost function obtained by the proposed algorithm for the Tinkerbell map: (a) the a–c section with $b = -0.6013$ and $d = 0.5$ and (b) the contour plot of the a–c section. The white square is the global minimum of the cost function.



(a)



(b)

Fig. 12. Conventional cost function for the Tinkerbell map: (a) the a–c section with $b = -0.6013$ and $d = 0.5$ and (b) the contour plot of the a–c section. The white square is the global minimum of the cost function.

can guide optimization methods. For a better judgment, a similar section for the conventional method (Eq. (3) with $N = 100$ and only 1% difference in initial conditions) has been provided in Fig. 12. As can be seen, the cost function surface is also rugged, but its global minimum is not in the right place. Furthermore, there is no guiding trend to the optimum.

3.5. *The effect of noise*

As previously mentioned, all experimental data have some amount of noise. There are many methods that can be employed for removing such noise, usually at the expense of distortion of the signal [Kostelich & Schreiber, 1993; Han & Liu, 2009; Matassini & Kantz, 2002; Jafari *et al.*, 2012a]. However, our method is robust to small amounts of noise since it involves averaging over the entire data set. Figure 13 shows the cost function for the logistic map with $A = 3.76$ and additive Gaussian noise. With a signal-to-noise ratio (SNR) of 30 dB or even 20 dB, the cost functions are still acceptable. However, when the SNR is 10 dB, the cost function no longer has its minimum at the correct value of A.

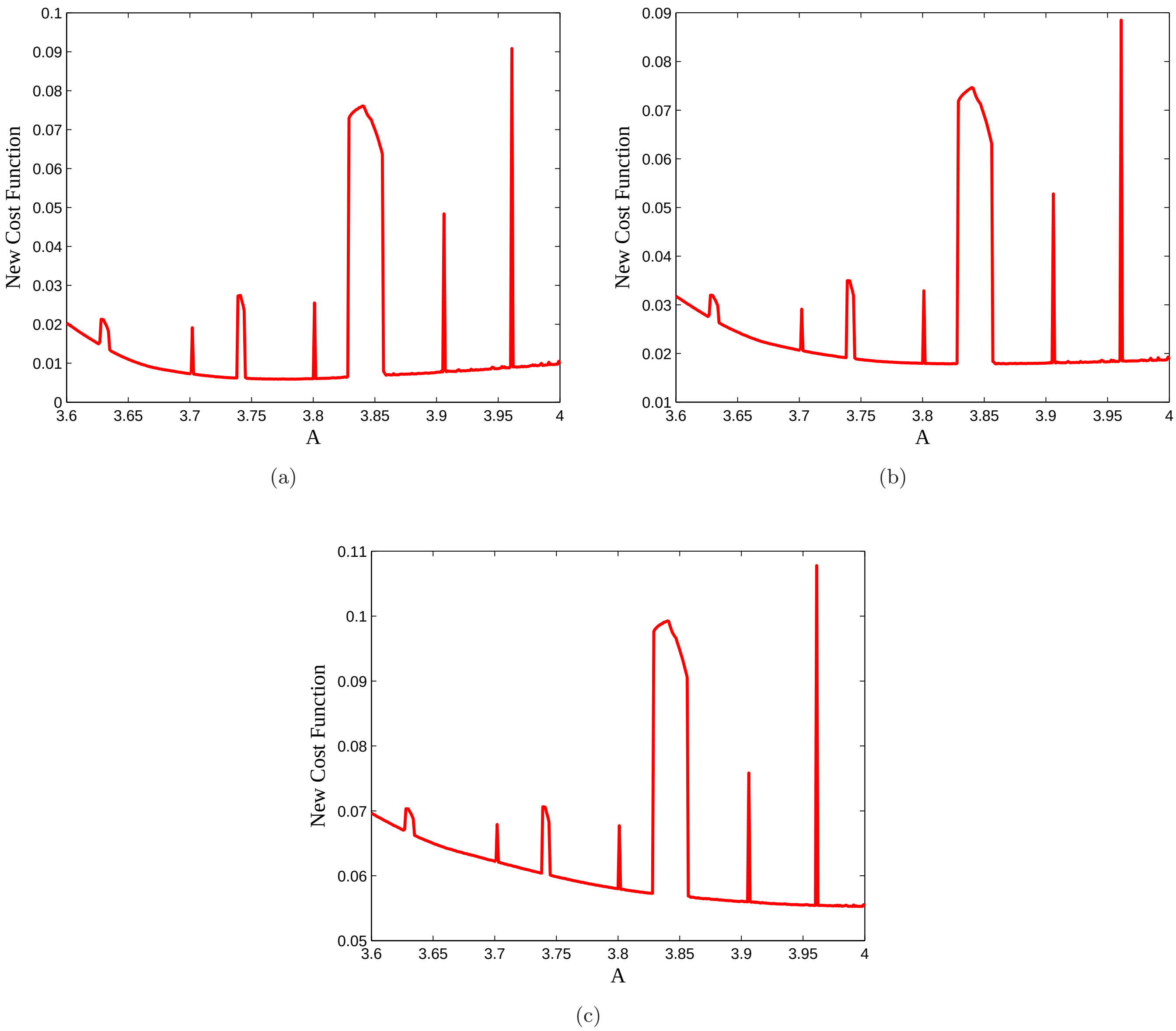


Fig. 13. New cost function obtained by proposed algorithm in the presence of noise: (a) SNR = 30 dB, (b) SNR = 20 dB and (c) SNR = 10 dB.

4. Applying Proposed Cost Function on Chaotic Flows

4.1. *Lorenz example*

Consider a chaotic flow like the well-known Lorenz system [Eq. (9)] and a model of it with two unknown parameters [Eq. (10)]:

$$\begin{aligned} \dot{x} &= P(y-x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \\ r &= 32, \quad P = 10, \quad b = \frac{8}{3} \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{x} &= P(y-x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \\ r &= ?, \quad P = ?, \quad b = \frac{8}{3}. \end{aligned} \tag{10}$$

First, construct a return map based on one real observed time series, for example by recording successive local maxima of one of the time variables (e.g. $x(t)$). These samples constitute a new time series $P(n)$. The return map is a plot of $P(n+1)$ as a function of $P(n)$ as shown in Fig. 14. Thereafter, the procedure is the same as for the previous iterated map example as shown by the flowchart in Fig. 7. The cost function obtained by this method is shown in Fig. 15.

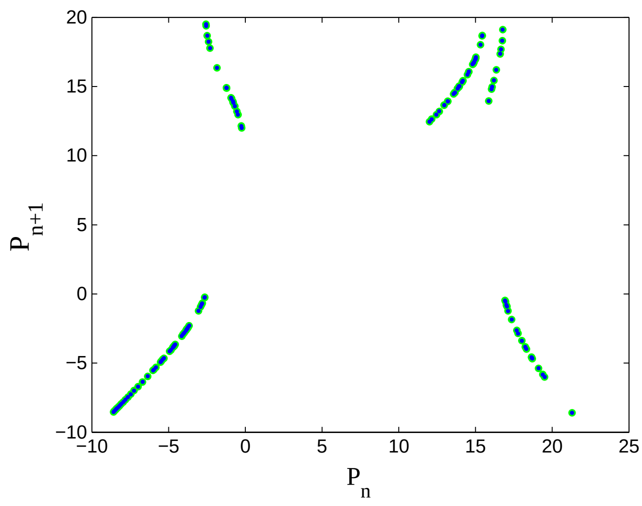


Fig. 14. Return map for the Lorenz system with $r = 32$, $P = 10$ and $b = 8/3$ obtained from local maxima of the x-variable.

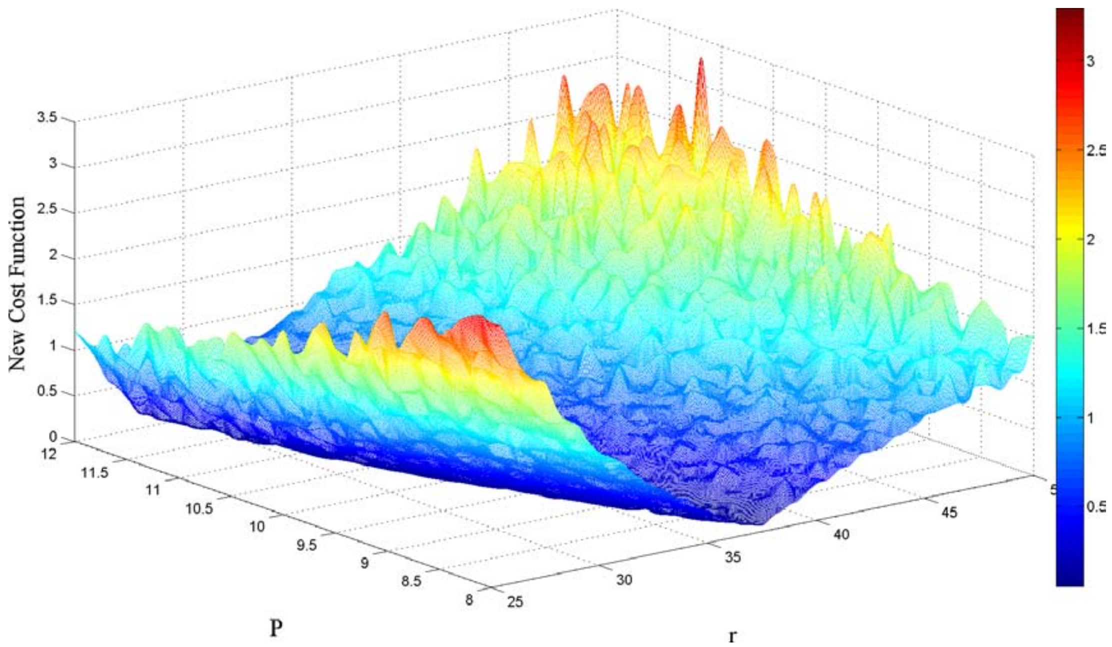
4.2. *Hidden attractors*

Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [Kiseleva *et al.*, 2014; Kuznetsov *et al.*, 2010, 2011a; Kuznetsov *et al.*, 2011b; Leonov & Kuznetsov, 2011a, 2011b, 2013b; Leonov *et al.*, 2014; Leonov & Kuznetsov, 2013a; Leonov *et al.*, 2011a; Leonov *et al.*, 2011b, 2012; Bragin *et al.*, 2014]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. The classical attractors of Lorenz, Rössler, Chua, Chen, Sprott (cases B to S), and other widely-known attractors are those excited from unstable equilibria. From a computational point of view this allows one to use a numerical method in which a trajectory that started from a point on the unstable manifold in the neighborhood of an unstable equilibrium, reaches an attractor and identifies it [Leonov *et al.*, 2011b]. Hidden attractors cannot be found by this method but are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

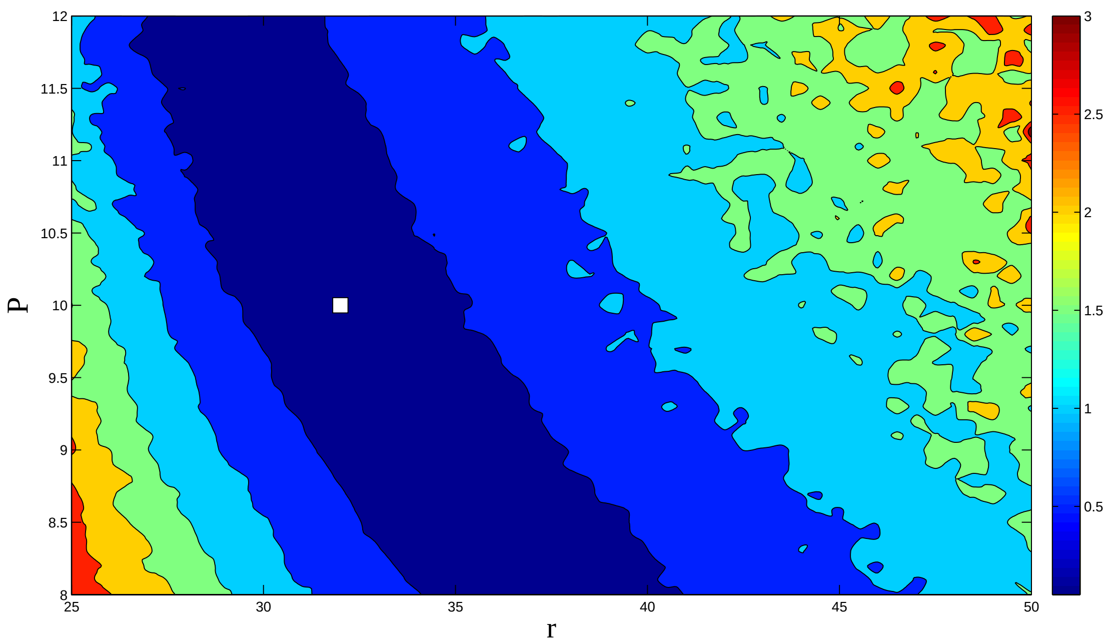
The chaotic attractors in dynamical systems without any equilibrium points, with only stable equilibria, or with lines of equilibria are hidden attractors. That is the reason such systems are rarely found, and only a few such examples have been reported in the literature [Jafari & Sprott, 2013; Jafari *et al.*, 2013a; Molaei *et al.*, 2013; Lao *et al.*, 2014; Pham *et al.*, 2014; Wang & Chen, 2012, 2013; Wei & Yang, 2012; Wang *et al.*, 2012a; Wei, 2011; Wang *et al.*, 2012b; Kingni *et al.*, 2014]. In this part we apply our method to one such system in which the accurate parameter estimation may be of even more importance, since even with the right value of parameter one may fail to find the attractor. We have chosen the system NE_8 from [Jafari *et al.*, 2013a] [Eq. (11)] and a model of it with one parameter [Eq. (12)]:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - yz \\ \dot{z} &= xy + 0.5x^2 - 1.3 \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - yz \\ \dot{z} &= xy + 0.5x^2 - a \end{aligned} \tag{12}$$



(a)



(b)

Fig. 15. New cost function obtained by the proposed algorithm for the Lorenz system: (a) the cost function surface and (b) the contour plot of the cost function surface. The white square is the global minimum of the cost function which is in the right place.

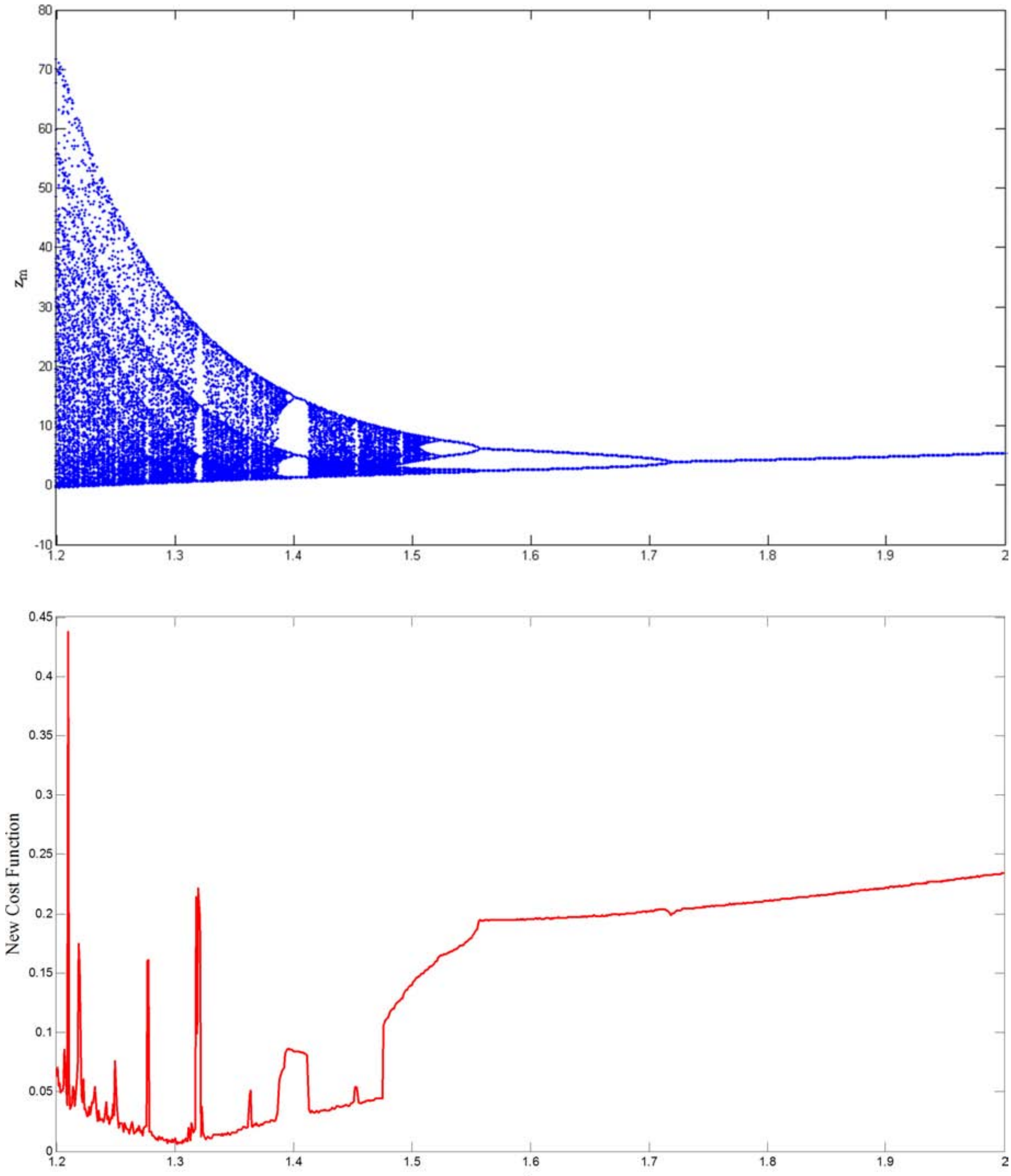


Fig. 16. New cost function obtained by the proposed algorithm for a hidden attractor. The initial conditions are random numbers and transient parts of the data are omitted.

The cost function obtained by this method can be seen in Fig. 16 along with a bifurcation diagram. As can be seen, this cost function too has the desired properties. It shows the effect of changing the parameter of the model, including the bifurcations and the monotonic trend along with a global minimum at the right value of a (here $a = 1.3$).

5. Conclusions

To estimate the parameters of a chaotic system that produced a given time series requires proper indicators of similarity between the experimental time series and the time series generated by the model. Sensitive dependence on initial conditions makes this difficult in the time domain unless the system is one-dimensional with a simple observer function. For parameter identification of chaotic systems, it is generally better to use state space instead of the time domain since the topology of a strange attractor does not depend on the initial conditions (within its basin of attraction). Therefore a computational method to obtain a new cost function based on the similarity between attractors is proposed. This new cost function works well for the examples shown and it is easily extended to chaotic flows using Poincaré section(s) or return maps to extract an appropriate time series. We suggest that other geometry-based cost functions can be designed, for example, using the density of points in the different regions of the attractor as a criterion for comparison of the data with the model.

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References

Abarbanel, H. D. L., Brown, R. & Kadtke, J. B. [1989] "Prediction and system identification in chaotic nonlinear systems: Time series with broadband spectra," *Phys. Lett.* **138**, 401–408.

Abarbanel, H. D. L., Brown, R. & Kadtke, J. B. [1990] "Prediction in chaotic nonlinear systems: Methods for time series with broadband Fourier spectra," *Phys. Rev. A* **41**, 1782–1807.

Aguirre, L. A. & Billings, S. A. [1994] "Validating identified nonlinear models with chaotic dynamics," *Int. J. Bifurcation and Chaos* **4**, 109–125.

Alexander, J. C., Yorke, J. A., You, Z. & Kan, I. [1992] "Riddled basins," *Int. J. Bifurcation and Chaos* **2**, 795.

Bragin, V. O., Vagaitsev, V. I., Kuznetsov, N. V. & Leonov, G. A. [2014] "Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits," *J. Comp. Syst. Sci. Int.* **50**, 511–543.

Breban, R. & Nusse, H. E. [2005] "On the creation of Wada basins in interval maps through fixed point tangent bifurcation," *Physica D* **207**, 52–63.

Broomhead, D. S. & King, G. P. [1986] "Extracting qualitative dynamics from experimental data," *Physica D* **20**, 217–236.

Chang, W. D. [2007] "Parameter identification of Chen and Lu systems: A differential evolution approach," *Chaos Solit. Fract.* **32**, 1469–1476.

Chang, J. F., Yang, Y. S., Liao, T. L. & Yan, J. J. [2008] "Parameter identification of chaotic systems using evolutionary programming approach," *Exp. Syst. Appl.* **35**, 2074–2079.

Coelho, L. S. & Bernert, D. L. A. [2010] "A modified ant colony optimization algorithm based on differential evolution for chaotic synchronization," *Exp. Syst. Appl.* **37**, 4198–4203.

Cumming, A. & Linsay, P. S. [1988] "Quasiperiodicity and chaos in a system with three competing frequencies," *Phys. Rev. Lett.* **60**, 2719–2722.

D'Humieres, D., Beasley, M. R., Huberman, B. A. & Libchaber, A. [1892] "Chaotic states and routes to chaos in the forced pendulum," *Phys. Rev. A* **26**, 3483–3496.

Gong, Y. F., Ren, W., Shi, X. Z., Xu, J. X. & Hu, S. J. [1999] "Recovering strange attractors from noisy interspike intervals of neuronal firings," *Phys. Lett. A* **258**, 253–256.

Grassberger, P. & Procaccia, I. [1983] "Measuring the strangeness of strange attractors," *Physica D* **9**, 189–208.

Han, M. & Liu, Y. [2009] "Noise reduction method for chaotic signals based on dual-wavelet and spatial correlation," *Exp. Syst. Appl.* **36**, 10060–10067.

Haynes, B. R. & Billings, S. A. [1993] "Global analysis and model validation in nonlinear system identification," *Nonlin. Dyn.* **5**, 93–130.

Hilborn, R. C. [2001] *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers* (Oxford University Press, NY).

Hubler, A. W. [1989] "Adaptive control of chaotic system," *Helv. Phys. Acta.* **62**, 343–346.

Jafari, S., Golpayegani, S. M. R. H. & Jafari, A. H. [2012a] "A novel noise reduction method based on geometrical properties of continuous chaotic signals," *Sci. Iran.* **19**, 1837–1842.

Jafari, S., Golpayegani, S. M. R. H., Jafari, A. H. & Gharibzadeh, S. [2012b] "Some remarks on chaotic systems," *Int. J. Gen. Syst.* **41**, 329–330.

Jafari, S. & Sprott, J. C. [2013] "Simple chaotic flows with a line equilibrium," *Chaos Solit. Fract.* **57**, 79–84.

Jafari, S., Sprott, J. C. & Golpayegani, S. M. R. H. [2013a] "Elementary quadratic chaotic flows with no equilibria," *Phys. Lett. A* **377**, 699–702.

Jafari, S., Golpayegani, S. M. R. H. & Darabad, M. R. [2013b] "Comment on parameter identification and synchronization of fractional-order chaotic systems," *Commun. Nonlin. Sci. Numer. Simulat.* **18**, 811–814.

Kantz, H. & Schreiber, T. [1997] *Nonlinear Time Series Analysis* (Cambridge University Press, Cambridge).

Kingni, S. T., Jafari, S., Simo, H. & Woafo, P. [2014] "Three-dimensional chaotic autonomous system with only one stable equilibrium: Analysis, circuit design, parameter estimation, control, synchronization and its fractional-order form," *Eur. Phys. J. Plus* **129**, 76.

Kiseleva, M. A., Kuznetsov, N. V., Leonov, G. A. & Neittaanmaki, P. [2014] "Drilling systems: Stability and hidden oscillations," *Discontinuity and Complexity in Nonlinear Physical Systems*, Nonlinear Systems and Complexity, Vol. 6 (Springer International Publishing), pp. 287–304.

Konnur, R. [2005] "Estimation of all model parameters of chaotic systems from discrete scalar time series measurements," *Phys. Lett. A* **346**, 275–280.

Korn, H. & Faure, P. [2003] "Is there chaos in the brain? II. Experimental evidence and related models," *C. R. Biol.* **326**, 787–840.

Kostelich, E. & Schreiber, T. [1993] "Noise reduction in chaotic time-series data: A survey of common methods," *Phys. Rev. E* **48**, 1752–1763.

Kuznetsov, N. V., Leonov, G. A. & Vagaitsev, V. I. [2010] "Analytical-numerical method for attractor localization of generalized Chua's system," *IFAC. Proc.* **4**, 29–33.

Kuznetsov, N. V., Kuznetsova, O. A., Leonov, G. A. & Vagaitsev, V. I. [2011a] "Hidden attractor in Chua's circuits," *Proc. 8th Int. Conf. Informatics in Control, Automation and Robotics* **4**, 279–283.

Kuznetsov, N. V., Leonov, G. A. & Seledzhi, S. M. [2011b] "Hidden oscillations in nonlinear control systems," *IFAC. Proc.* **18**, 2506–2510.

Kuznetsov, N. V., Mokaev, T. N. & Vasilyev, P. A. [2014] "Numerical justification of Leonov conjecture on Lyapunov dimension of Rossler attractor," *Commun. Nonlin. Sci. Numer. Simulat.* **19**, 1027–1034.

Lao, S.-K., Shekofteh, Y., Jafari, S. & Sprott, J. C. [2014] "Cost function based on Gaussian mixture model for parameter estimation of a chaotic circuit with a hidden attractor," *Int. J. Bifurcation and Chaos* **24**, 1450010-1–11.

Leonov, G. A. & Kuznetsov, N. V. [2007] "Time-varying linearization and the Perron effects," *Int. J. Bifurcation and Chaos* **17**, 1079–1107.

Leonov, G. A. & Kuznetsov, N. V. [2011a] "Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems," *Dokl. Math.* **84**, 475–481.

Leonov, G. A. & Kuznetsov, N. V. [2011b] "Analytical-numerical methods for investigation of hidden oscillations in nonlinear control systems," *IFAC. Proc.* **18**, 2494–2505.

Leonov, G. A., Kuznetsov, N. V., Kuznetsova, O. A., Seldedzhi, S. M. & Vagaitsev, V. I. [2011a] "Hidden oscillations in dynamical systems," *Trans. Syst. Contr.* **6**, 54–67.

Leonov, G. A., Kuznetsov, N. V. & Vagaitsev, V. I. [2011b] "Localization of hidden Chua's attractors," *Phys. Lett. A* **375**, 2230–2233.

Leonov, G. A., Kuznetsov, N. V. & Vagaitsev, V. I. [2012] "Hidden attractor in smooth Chua system," *Physica D* **241**, 1482–1486.

Leonov, G. A. & Kuznetsov, N. V. [2013a] "Hidden attractors in dynamical systems: From hidden oscillation in Hilbert–Kolmogorov, Aizerman and Kalman problems to hidden chaotic attractor in Chua circuits," *Int. J. Bifurcation and Chaos* **23**, 1330002-1–69.

Leonov, G. A. & Kuznetsov, N. V. [2013b] "Analytical-numerical methods for hidden attractors' localization: The 16th Hilbert problem, Aizerman and Kalman conjectures, and Chua circuits," *Numerical Methods for Differential Equations, Optimization, and Technological Problems*, Computational Methods in Applied Sciences, Vol. 27 (Springer International Publishing), pp. 41–64.

Leonov, G. A., Kuznetsov, N. V., Kiseleva, M. A., Solovyeva, E. P. & Zaretskiy, A. M. [2014] "Hidden oscillations in mathematical model of drilling system actuated by induction motor with a wound rotor," *Nonlin. Dyn.* **77**, 277–288.

Li, L., Yang, L. & Liu, L. [2006a] "An effective hybrid PSOSA strategy for optimization and its application to parameter estimation," *Appl. Math. Comput.* **179**, 135–146.

Li, L., Yang, Y., Peng, H. & Wang, X. [2006b] "Parameters identification of chaotic systems via chaotic ant swarm," *Chaos Solit. Fract.* **28**, 1204–1211.

Li, N., Pan, W., Yan, L., Luo, B., Xu, M. & Jiang, N. [2011] "Parameter estimation for chaotic systems with and without noise using differential evolution-based method," *Chin. Phys. B* **20**, 060502.

Li, X. & Yin, M. [2012] "Parameter estimation for chaotic systems using the cuckoo search algorithm with an orthogonal learning method," *Chin. Phys. B* **21**, 050507.

Li, C., Zhou, J., Xiao, J. & Xiao, H. [2012] "Parameters identification of chaotic system by chaotic gravitational search algorithm," *Chaos Solit. Fract.* **45**, 539–547.

Marino, I. P. & Miguez, J. [2006] "An approximate gradient-descent method for joint parameter estimation and synchronization of coupled chaotic systems," *Phys. Lett. A* **351**, 262–267.

Matassini, L. & Kantz, H. [2002] "Optimizing of recurrence plots for noise reduction," *Phys. Rev. E* **65**, 1–6.

Modares, H., Alfi, A. & Fateh, M. [2010] "Parameter identification of chaotic dynamic systems through an improved particle swarm optimization," *Exp. Syst. Appl.* **37**, 3714–3720.

Molaei, M., Jafari, S., Sprott, J. C. & Golpayegani, S. M. R. H. [2013] "Simple chaotic flows with one stable equilibrium," *Int. J. Bifurcation and Chaos* **23**, 1350188-1–7.

Moon, F. [1987] *Chaotic Vibrations — An Introduction for Applied Scientists and Engineers* (John Wiley and Sons, NY).

Mukhopadhyay, S. & Banerjee, S. [2012] "Global optimization of an optical chaotic system by chaotic multi swarm particle swarm optimization," *Exp. Syst. Appl.* **39**, 917–924.

Ott, E., Grebogi, C. & Yorke, J. A. [1990] "Controlling chaos," *Phys. Rev. Lett.* **64**, 1196–1199.

Patil, D. J., Kalnay, E., Yorke, J. A., Ott, E. & Hunt, B. R. [2001] "Local low dimensionality of atmospheric dynamics," *Phys. Rev. Lett.* **86**, 5878–5881.

Pham, V.-T., Volos, C. K., Jafari, S., Wei, Z. & Wang, X. [2014] "Constructing a novel no-equilibrium chaotic system," *Int. J. Bifurcation and Chaos* **24**, 1450073-1–6.

Rabinovich, M. I., Varona, P., Selverston, A. L. & Abarbanel, H. D. I. [2006] "Dynamical principles in neuroscience," *Rev. Mod. Phys.* **78**, 1213–1265.

Rowlands, G. & Sprott, J. C. [1992] "Extraction of dynamical equations from chaotic data," *Physica D* **58**, 251–259.

Signorini, M. G., Marchetti, F., Cirigioni, A. & Cerutti, S. [1997] "Nonlinear noise reduction for the analysis of heart rate variability signals in normal and heart transplanted subjects," *Proc IEEE Int. Conf. Engineering in Medicine and Biology Society* (*EMBC '97*) (Chicago, USA), pp. 1402–1405.

Sun, J., Zhao, J., Wu, X., Fang, W., Cai, Y. & Xu, W. [2010] "Parameter estimation for chaotic systems with a drift particle swarm optimization method," *Phys. Lett. A* **374**, 2816–2822.

Tang, Y. & Guan, X. [2009] "Parameter estimation for time-delay chaotic system by particle swarm optimization," *Phys. Lett. A* **40**, 1391–1398.

Tang, Y., Zhang, X., Hua, C., Li, L. & Yang, Y. [2012] "Parameter identification of commensurate fractional-order chaotic system via differential evolution," *Phys. Lett. A* **376**, 457–464.

Tao, C., Yu, Z. & Jiang, J. J. [2007] "Estimating system parameters from chaotic time series with synchronization optimized by a genetic algorithm," *Phys. Rev. E* **76**, 016209.

Tien, J. & Li, T. S. [2012] "Hybrid Taguchi-chaos of multilevel immune and the artificial bee colony algorithm for parameter identification of chaotic systems," *Comput. Math. Appl.* **64**, 1108–1119.

Wang, L. & Li, L. [2010] "An effective hybrid quantum-inspired evolutionary algorithm for parameter estimation of chaotic systems," *Exp. Syst. Appl.* **37**, 1279–1285.

Wang, L. & Xu, Y. [2011] "An effective hybrid biogeography-based optimization algorithm for parameter estimation of chaotic systems," *Exp. Syst. Appl.* **38**, 15103–15109.

Wang, L., Xu, Y. & Li, L. [2011] "Parameter identification of chaotic systems by hybrid Nelder–Mead simplex search and differential evolution algorithm," *Exp. Syst. Appl.* **38**, 3238–3245.

Wang, X. & Chen, G. [2012] "A chaotic system with only one stable equilibrium," *Commun. Nonlin. Sci. Numer. Simulat.* **17**, 1264–1272.

Wang, X., Chen, J., Lu, J. A. & Chen, G. [2012a] "A simple yet complex one-parameter family of generalized Lorenz-like systems," *Int. J. Bifurcation and Chaos* **22**, 1250116-1–16.

Wang, Z., Cang, S., Ochola, E. & Sun, Y. [2012b] "A hyperchaotic system without equilibrium," *Nonlin. Dyn.* **69**, 531–537.

Wang, X. & Chen, G. [2013] "Constructing a chaotic system with any number of equilibria," *Nonlin. Dyn.* **71**, 429–436.

Wei, Z. [2011] "Dynamical behaviors of a chaotic system with no equilibria," *Phys. Lett. A* **376**, 102–108.

Wei, Z. & Yang, Q. [2012] "Dynamical analysis of the generalized Sprott C system with only two stable equilibria," *Nonlin. Dyn.* **68**, 543–554.

Wigdorowitz, B. & Petrick, M. H. [1991] "Modelling concepts arising from an investigation into a chaotic system," *Math. Comput. Model.* **15**, 1–16.

Wolf, A., Swift, J. B., Swinney, H. L. & Vastano, J. A. [1985] "Determining Lyapunov exponents from a time series," *Physica D* **16**, 285–317.

Wolf, A. & Bessoir, T. [1991] "Diagnosing chaos in the space circle," *Physica D* **50**, 239–258.

Yang, K., Maginu, K. & Nomura, H. [2009] "Parameters identification of chaotic systems by quantum-behaved particle swarm optimization," *Int. J. Comput. Math.* **86**, 2225–2235.

Yuan, L. & Yang, Q. [2012] "Parameter identification and synchronization of fractional-order chaotic systems," *Commun. Nonlin. Sci. Numer. Simulat.* **17**, 305–316.