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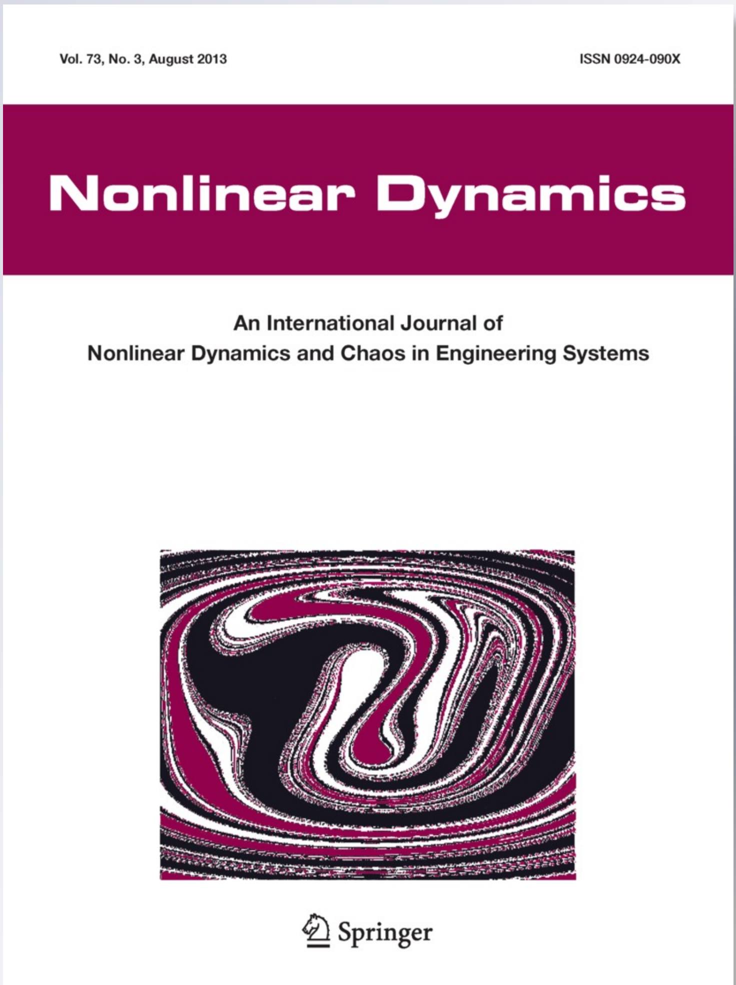

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ORIGINAL PAPER

Application of Takagi–Sugeno fuzzy model to a class of chaotic synchronization and anti-synchronization

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Abstract In this study, we investigate a class of chaotic synchronization and anti-synchronization with stochastic parameters. A controller is composed of a compensation controller and a fuzzy controller which is designed based on fractional stability theory. Three typical examples, including the synchronization between an integer-order Chen system and a fractional-order Lü system, the anti-synchronization of different 4D fractional-order hyperchaotic systems with non-identical orders, and the synchronization between a 3D integer-order chaotic system and a 4D fractional-order hyperchaos system, are presented to illustrate the effectiveness of the controller. The numerical simulation results and theoretical analysis both demonstrate the effectiveness of the proposed approach. Overall, this study presents new insights concerning the concepts of synchronization and anti-synchronization, synchronization and control, the relationship of fractional and integer order nonlinear systems.

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1 Introduction

Synchronization is based on the concept of closeness of the frequencies between different periodic oscillations generated by two systems. Since Pecora and Carroll's pioneering research work [1], chaos synchronization, as an important topic in nonlinear science, has been widely investigated in many fields, such as information science [2], modern management [3], chemistry [4], control engineering [5], physics [6], nervous system [7], secure communication [8, 9], and so on. Therefore, chaos synchronization is unlikely to be a limited or temporary fashion. Many approaches have been proposed for chaos synchronization, such as adaptive control [10, 11], back stepping control [12, 13], sliding mode control [14, 15], feedback control [16, 17], etc. The concept of synchronization has been extended in scope to include phase synchronization [18], generalized synchronization [19], lag synchronization [20], and even anti-phase synchronization (APS) [21–23]. APS can also be interpreted as anti-synchronization (AS), which involves the state vectors of the synchronized system and the master system. Therefore, the sum of two signals is expected to converge to zero when either AS or APS appears.

To the best of our knowledge, chaos synchronization and anti-synchronization has only been conducted between integer-order systems or between fractional-order systems [24]. There has been little information available about the synchronization between a chaotic fractional-order system and an integer-order system or between chaotic systems with non-identical orders.

On the other hand, fuzzy control has also been applied to chaos control and chaos synchronization [24, 25]. An approach to control chaos via linear matrix inequality (LMI) based on fuzzy control system design has been suggested in [26, 27]. Here the key idea is to use the well-known fuzzy model to represent typical chaotic systems and then design a controller for the fuzzy model. Takagi and Sugeno (T–S) type fuzzy models were proved to be a powerful tool for controlling complex nonlinear systems. Time-delayed fuzzy state-feedback controller was presented to reach synchronization of chaotic systems by Lam et al. [28]. Roopaei et al. [29] used sliding mode control integrated with an interval type-2 fuzzy system for synchronization of two different chaotic systems in the presence of system with external disturbances. However, there are few contributions on synchronization of fractional order chaotic systems based on fuzzy control theory [30].

Most of the above studies are realized without any external perturbations. However, noise disturbance is inevitable from a practical point of view. Most of the noise disturbance is stochastic from unknown factors in the environment. Therefore, investigation of chaos synchronization or anti-synchronization under the impact of artificially uncertain parameters has become an important research topic [31].

In light of the above analysis, there are three advantages that make our approach attractive. First, synchronization and anti-synchronization of a class of chaotic systems with stochastic parameters are carried out including three typical examples, which present new insights concerning the concepts of synchronization and anti-synchronization, synchronization and control, the relationship of fractional and integer order nonlinear systems. Second, a new controller composed of a compensation controller and a fuzzy controller is presented based on fractional stability theory, which is suitable for a class of nonlinear systems. Finally, the numerical simulation results are in good agreement with theoretical analysis.

The rest of the paper is organized as follows: Sect. 2 presents stability theorems in the fractional calculus and the generalized T–S fuzzy model. Moreover, controlled stability of fractional order systems based on the T–S fuzzy model is verified. Section 3 introduces the system description and fuzzy controller design. In Sect. 4, three typical examples are presented to demonstrate the validity of the method discussed in this paper. Numerical simulation results are also included. Conclusions close the paper in Sect. 5.

2 Preliminaries

2.1 Stability theorem of fractional-order systems

Theorem 1 [32] *We consider the following linear fractional order system*:

$$D^q x = Ax, \quad x(0) = x_0. \tag{1}$$

Here $A \in R^{n\times n}$, $x \in R^n$, and $q = [q_1, q_2, \dots, q_i, \dots, q_n]$ $(0 < q_i \leq 1)$. System (1) is asymptotically stable if and only if $|\arg(\lambda_i)| > q\pi/2$ is satisfied for all eigenvalues λ_i of the matrix A. Furthermore, this system is stable if and only if $|\arg(\lambda_i)| \geq q\pi/2$ is satisfied for all eigenvalues λ_i of the matrix A and those critical eigenvalues that satisfy the condition $|\arg(\lambda_i)| = q\pi/2$ have geometric multiplicity one. By the way, the geometric multiplicity of an eigenvalue is defined as the dimension of the associated eigenspace, i.e., number of linearly independent eigenvectors with that eigenvalue.

2.2 Generalized T–S fuzzy model

Here we briefly describe the fuzzy logic system. For a continuous nonlinear system, the generalized T–S fuzzy model is shown as follows:

Rule i:

$$\textbf{IF } z_1(t) \text{ is } M_1^i,\ z_2(t) \text{ is } M_2^i, \dots, \text{ and } z_p(t) \text{ is } M_p^i,$$
$$\textbf{THEN } D^\alpha x(t) = A_i x(t),\ i = 1, 2, \dots, r, \tag{2}$$

where $x(t) = [x_1(t) \cdots x_n(t)]^T \in R^n$ is the state vector, $A_i \in R^{n\times n}$, α is the fractional order, and r is the number of fuzzy sets. The $z_j(t)$ $(j = 1, 2, \dots, p)$ are the premise variables, and M_j^i $(j = 1, 2, \dots, p)$ is the input fuzzy set. According to the singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier, the output of the general T–S fuzzy model is inferred in the following form:

$$D^\alpha x(t) = \frac{\sum_{i=1}^r \omega_i A_i x(t)}{\sum_{i=1}^r \omega_i} \tag{3}$$

where $\omega_i = \prod_{j=1}^p M_{ij}(z_j(t))$.

With $h_i(z(t)) = \omega_i / \sum_{i=1}^r \omega_i$, the equation can be rewritten as

$$D^\alpha x(t) = \sum_{i=1}^r h_i\big(z(t)\big) A_i x(t) \tag{4}$$

in which $\sum_{i=1}^r h_i(z(t)) = 1$ and $h_i(z(t))$ can be treated as normalized weights of the IF–THEN rules.

2.3 Fuzzy Control

Suppose that a fractional order system can be represented exactly by the T–S fuzzy model shown in system (2). Assume that the fuzzy controller is chosen as

Rule i:

IF $z_1(t)$ is M_1^i, $z_2(t)$ is $M_2^i, \ldots,$ and $z_p(t)$ is M_p^i,

THEN $U'(t) = -F_i x(t)$. (5)

Therefore, the fuzzy controller can be described as

$$U'(t) = -\sum_{i=1}^r h_i\big(z(t)\big) F_i x(t). \tag{6}$$

Considering Eqs. (6) and (4), the whole controlled system can be described as

$$D^\alpha x(t) = \sum_{i=1}^r h_i\big(z(t)\big)(A_i - BF_i)x(t). \tag{7}$$

Here, the matrix **B** is the matrix of the controller, which is usually an identity matrix.

Theorem 2 *If there exists a feedback gain F_i such that*

$$G = A_1 - BF_1 = A_i - BF_i, \quad i = 1, 2, \ldots, r \tag{8}$$

and the condition $|\arg(\mathrm{eig}(G))| > \alpha\pi/2$ is satisfied as well, then the chaotic fractional order system (2) is asymptotically stable.

Proof If condition (8) holds, that is, $G = A_1 - BF_1 = A_i - BF_i$ $(i = 1, 2, \ldots, r)$, then the fractional order system can be represented as

$$D^\alpha x(t) = Gx(t). \tag{9}$$

Furthermore, the condition $|\arg(\mathrm{eig}(G))| > \alpha\pi/2$ is satisfied. We infer that system (9) is globally stable from Theorem 1. Thus the fractional order chaotic system (2) is asymptotically stable according to the fuzzy controller equation (6). The proof is finished completely. □

Remark 1 If the matrix **B** is nonsingular and the matrix **G** is selected to satisfy the condition $|\arg(eig(G))| > \alpha\pi/2$, then we can obtain the feedback gains $F_i = B^{-1}(A_i - G)$.

3 Generalized synchronization

3.1 System description

We consider a class of fractional order chaotic systems with uncertain system parameters. A drive system and a response system are described, respectively, by

$$D^\alpha x = Ax + f(x) \tag{10}$$

and

$$D^\beta y = Cy + g(y) + U(t), \tag{11}$$

where $x, y \in R^n$ are the n-dimensional state vectors for the drive and response systems, respectively. $f, g : R_n \to R_n$ is a continuous vector function for the system. $U(t)$ is the controller to be designed later. α and β are $n \times 1$ vectors to denote the chaotic order for each state of the drive and response systems. A and C are parameter matrices for the linear part of the above drive and response systems.

Our aim is to design a suitable effective controller $U(t)$ such that the trajectory of the response system asymptotically approaches the drive system and synchronization between the two systems is finally achieved.

3.2 Fuzzy controller design

To obtain the control law, the synchronization error is defined as

$$e = y - \chi x, \tag{12}$$

where χ is an arbitrary constant scaling factor ($\chi \in R$). Here we divide the controller $U(t)$ into two subcontrollers $U_1(t)$ and $U_2(t)$, i.e., $U(t) = U_1(t) +$

$U_2(t)$. The sub-controller $U_1(t)$, a compensation controller, is composed as

$$U_1(t) = D^{\beta}(\chi x). \tag{13}$$

Substituting the sub-controller (13) into system (11) gives the system

$$D^{\beta} e = Cy + g(y) + U_2(t). \tag{14}$$

According to the T–S fuzzy model in Sect. 2.2, the fractional order system (14) without the controller can be exactly represented by the T–S fuzzy model, which is $\sum_{i=1}^{r} h_i(z(t))C_i y(t)$. Furthermore, in view of Sect. 2.3, a fuzzy controller is $U_2(t) = -\sum_{i=1}^{r} h_i(z(t))F_i y(t)$. Therefore, the error dynamical system can be rewritten as

$$\begin{aligned} D^{\beta} e &= \sum_{i=1}^{r} h_i\big(z(t)\big)C_i y(t) - \sum_{i=1}^{r} h_i\big(z(t)\big)BF_i y(t) \\ &= \sum_{i=1}^{r} h_i\big(z(t)\big)(C_i - BF_i)y(t). \end{aligned} \tag{15}$$

Now the total control law can be obtained as

$$\begin{aligned} U(t) &= U_1(t) + U_2(t) \\ &= D^{\beta}\big(\chi x(t)\big) - \sum_{i=1}^{r} h_i\big(z(t)\big)BF_i y(t). \end{aligned} \tag{16}$$

Theorem 3 *When the response system (11) is driven by the controller (16) with appropriately chosen feedback gains $F_i(G = C_1 - BF_1 = C_i - BF_i)$ and $|\arg(\mathrm{eig}(G))| > \beta\pi/2$, the error system (15) will be stable and converge to zero so that synchronization is realized.*

Remark 2 If the chaotic orders in the drive system (10) are $\alpha_i = 1$, that is, $\dot{x} = Ax + f(x)$, then the synchronization between a fractional-order system and an integer-order system can be achieved using the controller (16).

Remark 3 Since there exists a scaling factor χ ($\chi \in R$), we can choose the value of χ arbitrarily to meet our needs. For example, the synchronization is realized when $\chi = 1$, and anti-synchronization is achieved when $\chi = -1$. Synchronization and anti-synchronization can both be achieved by the same control.

Remark 4 Most system parameters change stochastically within a certain range. As we can see, the feedback gains $F_i = B^{-1}(C_i - G)$, which vary with system parameters, that is, when the system parameters change, the controller (16) changes in a certain regular way. For the system with stochastic parameters, the controller is especially effective.

4 Numerical simulation

To illustrate the effectiveness of the proposed synchronization scheme, three examples are considered, and their numerical simulations are performed using the Caputo version and a predictor–corrector algorithm for fractional-order differential equations, which is a generalization of the Adams–Bashforth–Moulton method [33].

Case I Synchronization between an integer-order chaotic system and a fractional-order chaotic system with stochastic parameters.

The integer order drive system [34] is written as

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) \\ \dot{y}_1 = (c-a)x_1 - x_1 z_1 + c y_1 \\ \dot{z}_1 = x_1 y_1 - b z_1 \end{cases} \tag{17}$$

where $(a, b, c) = (35, 3, 28)$, and the initial conditions are $(x_1, y_1, z_1) = (1, 3, 5)$. Assume that $x_1 \in [-d, d]$, $d > 0$. Note that we determine the upper and lower bounds, which is the foundation of the superposition of linear matrices. Then the Chen system is described in the T–S fuzzy model as follows:

$$\begin{aligned} &\text{Rule 1: IF } x_1 \text{ is } M_1(x) \text{ THEN } \dot{X}(t) = A_1 X(t); \\ &\text{Rule 2: IF } x_1 \text{ is } M_2(x) \text{ THEN } \dot{X}(t) = A_2 X(t). \end{aligned} \tag{18}$$

Here,

$$X = [x_1, y_1, z_1]^T, \qquad A_1 = \begin{pmatrix} -a & a & 0 \\ c-a & c & -d \\ 0 & d & -b \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -a & a & 0 \\ c-a & c & d \\ 0 & -d & -b \end{pmatrix},$$

$$M_1(x) = \frac{1}{2}\left(1 + \frac{x}{d}\right), \qquad M_2(x) = \frac{1}{2}\left(1 - \frac{x}{d}\right),$$

$d = 30.$

Therefore, after the above equivalent transformation, the final form of the drive system can be written as

$$\dot{X}(t) = \sum_{i=1}^{2} h_i\big(z(t)\big) A_i x(t). \tag{19}$$

Obviously, the variable x has been replaced by its upper and lower bounds. Therefore, the controller based on the fuzzy model is insensitive to stochastic disturbances.

The response system, a fractional order chaotic system [35], is described as

$$\begin{cases} D^{\beta_1} x_2 = a(y_2 - x_2), \\ D^{\beta_2} y_2 = -x_2 z_2 + c y_2, \\ D^{\beta_3} z_2 = x_2 y_2 - b z_2 \end{cases} \tag{20}$$

where $(a, b, c) = (35, 3, 28)$, with fractional order $\beta = [\beta_1, \beta_2, \beta_3] = [0.90, 0.92, 0.94]$, and initial conditions of the response system are $(x_2, y_2, z_2) = (2, 4, 30)$. Assume that $x_2 \in [-d, d]$, $d > 0$. Then the fractional order Lü system is described in the T–S fuzzy model as follows:

$$\begin{aligned} &\text{Rule 1: IF } x_2 \text{ is } M_1'(x) \text{ THEN } D^{\beta} Y(t) = C_1 Y(t); \\ &\text{Rule 2: IF } x_2 \text{ is } M_2'(x) \text{ THEN } D^{\beta} Y(t) = C_2 Y(t). \end{aligned} \tag{21}$$

Here,

$$Y = [x_2, y_2, z_2]^T, \qquad C_1 = \begin{pmatrix} -a & a & 0 \\ 0 & c & -d \\ 0 & d & -b \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -a & a & 0 \\ 0 & c & d \\ 0 & -d & -b \end{pmatrix},$$

$$M_1'(x) = \frac{1}{2}\left(1 + \frac{x}{d}\right), \qquad M_2'(x) = \frac{1}{2}\left(1 - \frac{x}{d}\right),$$

$d = 30.$

Therefore, after the above equivalent transformation, the final form of the response system, *Lü* system, is inferred as

$$D^{\beta} Y(t) = \sum_{i=1}^{2} h_i'\big(z(t)\big) C_i Y(t). \tag{22}$$

From Remark 4, we select the system parameters to be stochastic, such as $a = 35 + 0.7\,\mathrm{rand}(t)$, $b = 3 + 0.2\,\mathrm{rand}(t)$, $c = 28 + 0.8\,\mathrm{rand}(t)$, where $\mathrm{rand}(t)$ is a mathematical function that is stochastic and bounded, i.e., $|\mathrm{rand}(t)| < 1$.

For simplicity, choose **B** as the identity matrix. According to Remark 1, one selects

$$\mathrm{G} = \begin{pmatrix} -4 & 4 & -1.5 \\ -6 & 2 & -6 \\ -1 & 6 & -4.5 \end{pmatrix},$$

which satisfies $|\arg(\mathrm{eig}(G))| > \beta\pi/2$. Thus, we can obtain $F_1 = C_1 - G$ and $F_2 = C_2 - G$.

The overall control law is given by

$$U(t) = D^{\beta} X - \sum_{i=1}^{2} h_i'\big(z(t)\big) F_i Y. \tag{23}$$

When the controller, Eq. (23), is added to Eq. (22), it works. Trajectories of the states in the drive and response systems with increasing time as shown in Figures 1 and 2 demonstrate that the error gradually converges to zero.

Case II Anti-synchronization between two fractional-order hyper-chaotic systems with uncertain stochastic parameters and non-identical orders.

A new fractional-order hyperchaotic system [36] is regarded as a drive system, described by

$$\begin{cases} D^{\alpha} x_1 = a_1(y_1 - x_1), \\ D^{\alpha} y_1 = d_1 x_1 + c_1 y_1 - x_1 z_1 - w_1, \\ D^{\alpha} z_1 = -b_1 z_1 + x_1 y_1, \\ D^{\alpha} w_1 = x_1 \end{cases} \tag{24}$$

where a_1, b_1, c_1, d_1 are the system parameters. When $a_1 = 36$, $b_1 = 3$, $c_1 = 28$, $d_1 = -16$, and the fractional order is $\alpha = 0.85$, its initial conditions are $(x_1, y_1, z_1, w_1) = (2, 4, 15, -3)$. Assume that $x_1 \in [-d, d]$, $d > 0$. Then the hyperchaotic system is described in the T–S fuzzy model as follows:

$$\begin{aligned} &\text{Rule 1: IF } x_1 \text{ is } M_1(x) \text{ THEN } D^{\alpha} X(t) = A_1 X(t); \\ &\text{Rule 2: IF } x_1 \text{ is } M_2(x) \text{ THEN } D^{\alpha} X(t) = A_2 X(t). \end{aligned} \tag{25}$$

Here,

$$X = [x_1, y_1, z_1, w_1]^T,$$

Fig. 1 Trajectories of the state variables in integer-order chaotic system and fractional-order chaotic system

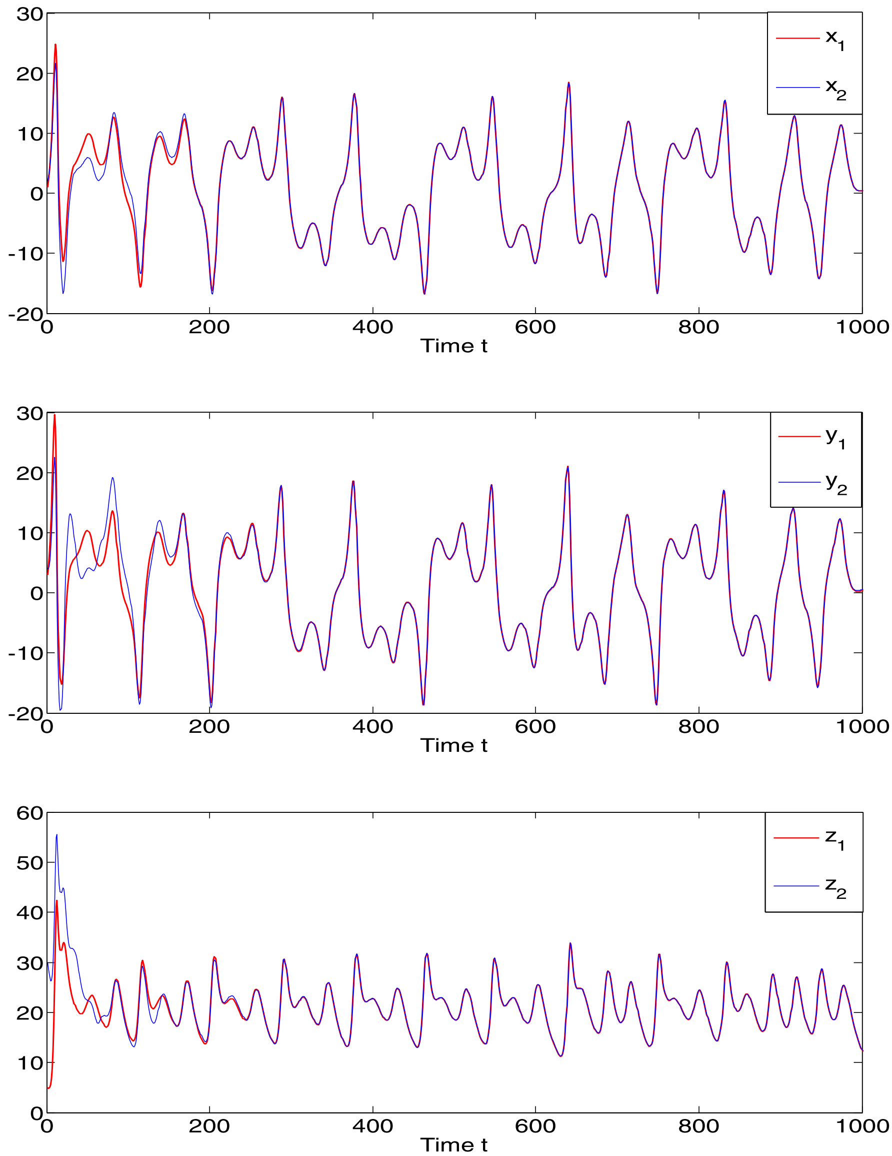
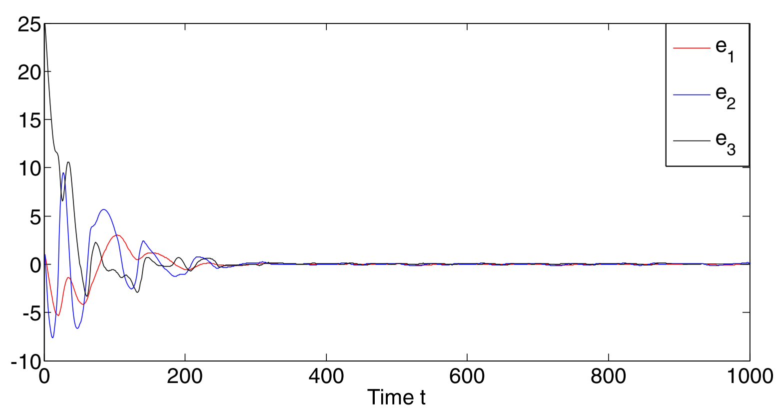


Fig. 2 Synchronization errors between integer-order chaotic system and fractional-order chaotic system ($e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$)



$$A_1 = \begin{pmatrix} -a_1 & a_1 & 0 & 0 \\ d_1 & c_1 & -d & -1 \\ 0 & d & -b_1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -a_1 & a_1 & 0 & 0 \\ d_1 & c_1 & d & -1 \\ 0 & -d & -b_1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M_1(x) = \frac{1}{2}\left(1 + \frac{x}{d}\right), \qquad M_2(x) = \frac{1}{2}\left(1 - \frac{x}{d}\right),$$

$d = 30$.

The final output hyperchaotic fuzzy system is inferred as

$$D^\alpha X(t) = \sum_{i=1}^{2} h_i\big(z(t)\big) A_i X(t). \tag{26}$$

A new fractional-order hyperchaotic Lorenz system [37] is taken as a response system as given by

$$\begin{cases} D^\beta x_2 = a_2(y_2 - x_2) + w_2, \\ D^\beta y_2 = c_2 x_2 - y_2 - x_2 z_2, \\ D^\beta z_2 = -b_2 z_2 + x_2 y_2, \\ D^\beta w_2 = -x_2 z_2 - r_2 w_2 \end{cases} \tag{27}$$

where a_2, b_2, c_2, r_2 are the system parameters. When $a_2 = 10$, $b_2 = 8/3$, $c_2 = 28$, $r_2 = 1$, and the fractional order is $\beta = 0.98$, its initial conditions are $(x_2, y_2, z_2, w_2) = (0, 3, 19, 0)$. Assume that $x_2 \in [-d, d]$, $d > 0$. Then the hyperchaotic Lorenz system is described in the T–S fuzzy model as follows:

$$\begin{aligned} &\text{Rule 1: IF } x_2 \text{ is } M_1'(x) \text{ THEN } D^\beta Y(t) = C_1 Y(t); \\ &\text{Rule 2: IF } x_2 \text{ is } M_2'(x) \text{ THEN } D^\beta Y(t) = C_2 Y(t). \end{aligned} \tag{28}$$

Here,

$Y = [x_2, y_2, z_2, w_2]^T$,

$$C_1 = \begin{pmatrix} -a_2 & a_2 & 0 & 1 \\ c_2 & -1 & -d & 0 \\ 0 & d & -b_2 & 0 \\ 0 & 0 & -d & -r_2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -a_2 & a_2 & 0 & 1 \\ c_2 & -1 & d & 0 \\ 0 & -d & -b_2 & 0 \\ 0 & 0 & d & -r_2 \end{pmatrix},$$

$$M_1'(x) = \frac{1}{2}\left(1 + \frac{x}{d}\right), \qquad M_2'(x) = \frac{1}{2}\left(1 - \frac{x}{d}\right),$$

$d = 30$.

Therefore, after the above equivalent transformation, the final form of the response system can be written as

$$D^\beta Y(t) = \sum_{i=1}^{2} h_i'\big(z(t)\big) C_i Y(t). \tag{29}$$

From Remark 4, we select the system parameters to be stochastic. For the drive system, $a_1 = 36 + 0.4\,\mathrm{rand}(t)$, $b_1 = 3 + 0.2\,\mathrm{rand}(t)$, $c_1 = 28 + 0.6\,\mathrm{rand}(t)$, $d_1 = -16 + 0.2\,\mathrm{rand}(t)$, and for the response system $a_2 = 10 + 0.4\,\mathrm{rand}(t)$, $b_2 = 8/3 + 0.1\,\mathrm{rand}(t)$, $c_2 = 28 + 0.7\,\mathrm{rand}(t)$, $r_2 = 1 + 0.3\,\mathrm{rand}(t)$.

For simplicity, choose **B** as the identity matrix, According to Remark 1, one selects

$$G = \begin{pmatrix} -4 & 4 & -1.5 & 2 \\ -6 & 2 & -6 & 3 \\ -1 & 6 & -4.5 & 3 \\ -2 & 5 & -3 & 1 \end{pmatrix},$$

which should satisfy $|\arg(\mathrm{eig}(G))| > \beta\pi/2$. Thus, we can obtain $F_1 = C_1 - G$ and $F_2 = C_2 - G$.

The overall control law is given by

$$U(t) = D^\beta X - \sum_{i=1}^{2} h_i'\big(z(t)\big) F_i Y. \tag{30}$$

Trajectories of the states of the drive and response systems are shown in Fig. 3, while the synchronization errors are shown in Fig. 4, which shows that the proposed method is successful in anti-synchronizing the two systems.

Case III Synchronization between an integer-order system and a fractional-order system with different dimensions and stochastic parameters.

For a drive system, consider the integer-order Chen system (17) in Case I, and for a response system, consider the fractional-order hyper-chaotic Lorenz system (27) in Case II.

Now let $X = [x_1, y_1, z_1, 0]^T$, so that we can control the state w_2 in the response system to 0, and get the overall control law

$$U(t) = D^\beta X - \sum_{i=1}^{2} h_i'\big(z(t)\big) F_i Y, \tag{31}$$

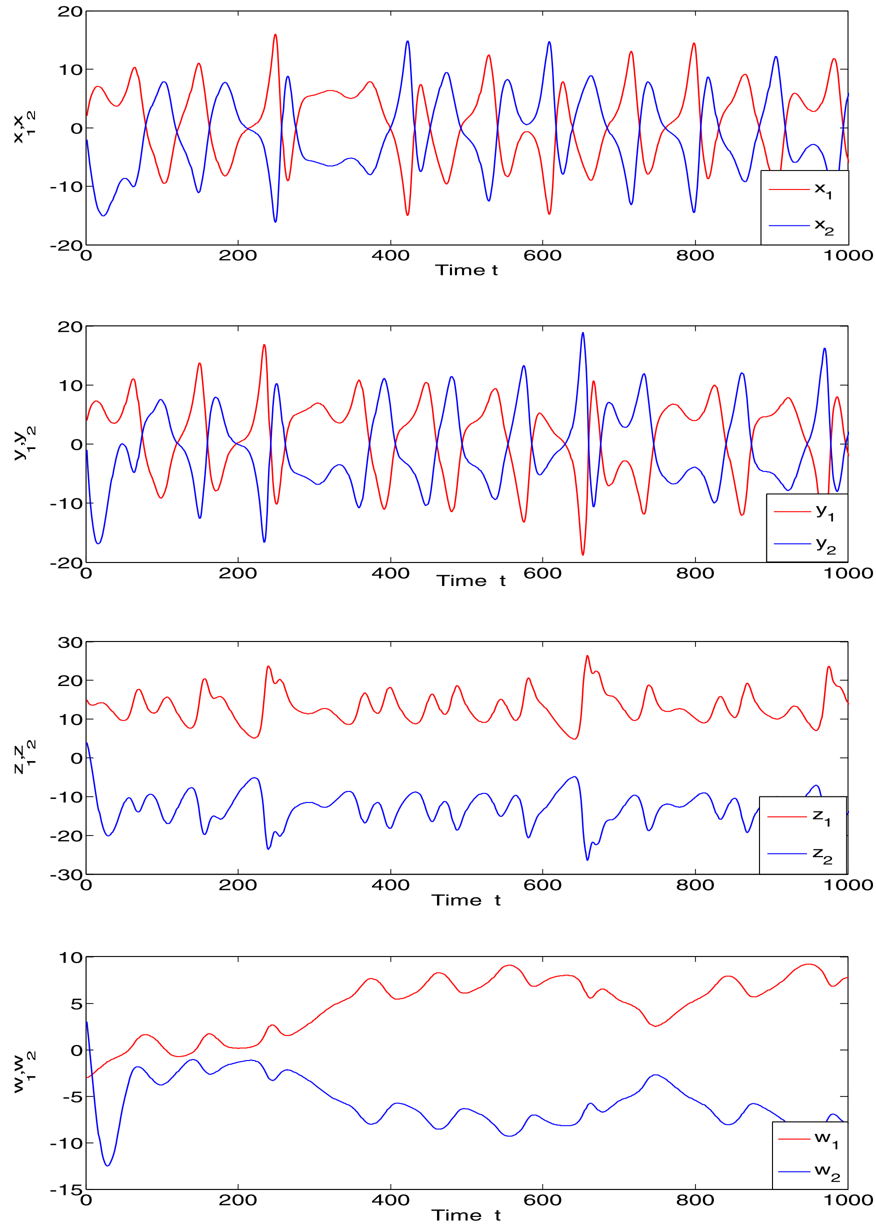


Fig. 3 Trajectories of the state variables in between fractional-order hyper-chaotic system (24) and fractional-order hyperchaotic Lorenz system

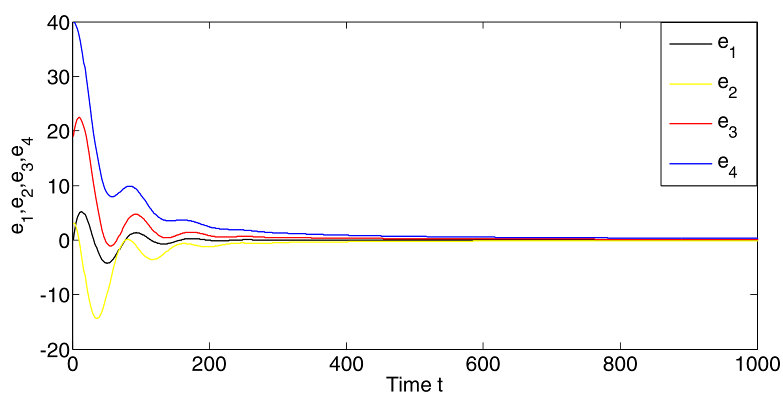


Fig. 4 Anti-synchronization errors between fractional-order hyper-chaotic system (24) and fractional-order hyperchaotic Lorenz system ($e_1 = x_1 + x_2$, $e_2 = y_1 + y_2$, $e_3 = z_1 + z_2$, $e_4 = w_1 + w_2$)

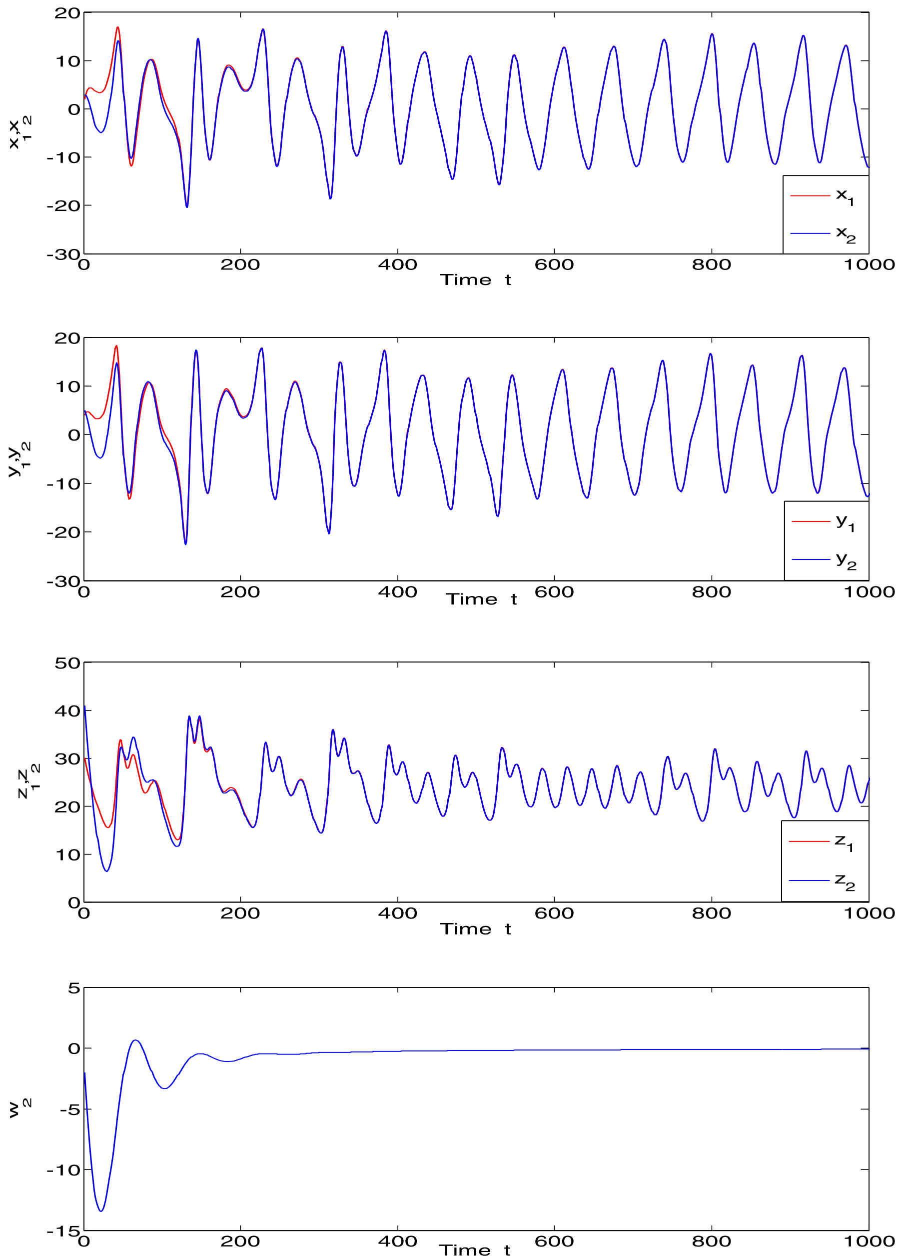


Fig. 5 Trajectories of the state variables in 3D integer-order chaotic system and 4D fractional-order Lorenz system

where the selected parameters are the same as above.

Numerical simulation results are shown in Figs. 5 and 6, which demonstrate the effectiveness of the control proposed in this paper.

From the cases above, it is obvious that the controller designed in this paper is perfect. The numerical simulation results are in good agreement with theoretical analysis. Comparing with prior works on chaotic synchronization with uncertain parameters, the proposed method is suitable for a class of chaotic systems, including integer order chaotic systems and fractional-order chaotic systems, and this type has a wider range of applications. Moreover, this study presents new insights concerning the concepts of synchronization and anti-synchronization, synchronization and control, the relationship of fractional and integer order nonlinear systems. A supplement to this proposed method is that two rules is not a fixed choice. If the selected rules

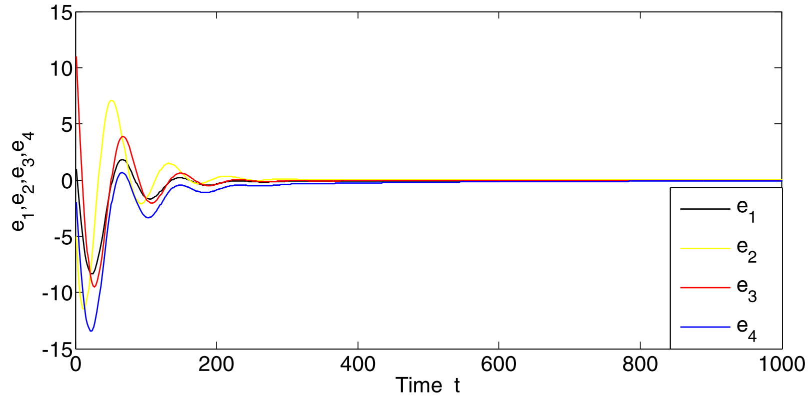


Fig. 6 Synchronization errors between 3D integer-order chaotic system and 4D fractional order 4D Lorenz system ($e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$, $e_4 = w_2$)

meet the requirements in Sect. 2.2, it will work well. The number of the rules, however, is usually even.

5 Conclusions and discussion

In this paper, we have studied synchronization and anti-synchronization of chaotic stochastic systems with uncertain system parameters. Fuzzy control is applied to chaotic systems in the T–S model. Three different kinds of examples are provided to demonstrate the validity of the fuzzy controller proposed in this paper. Case I shows synchronization between an integer-order Chen system and a fractional-order Lü system, which provides a bridge between integer-order chaotic systems and fractional-order chaotic systems, and lends theoretical support for fractional-order chaotic systems. Case II brings attention to the anti-synchronization of different 4D fractional-order hyperchaotic systems with non-identical orders, which not only confirm that the controller is suitable for hyperchaotic systems and non-identical order chaotic systems, but also shows that synchronization and anti-synchronization are essentially the same, which offers the possibility for realizing synchronization and anti-synchronization simultaneously using the same method. Case III shows synchronization between an integer-order system and a fractional-order system with different dimensions and stochastic parameters. It illustrates that different chaotic systems with different dimensions can achieve synchronization. In essence, chaos synchronization is a broad concept of chaos control. In other words, chaos control is a special case of chaos synchronization, and it is considered to achieve synchronization with $O(0, 0, 0)$. Here, the extra dimension w_2 of system (27) is actually controlled to 0. Moreover, the fuzzy controller is insensitive to stochastic disturbances in the chaotic systems for that the variable was replaced by coupling of two matrices with upper and lower bounds in the process of a linear matrix transform, which will be more instructional in practical systems.

More and better methods for the synchronization and anti-synchronization between integer-order chaotic systems and fractional-order chaotic systems or fractional-order chaotic systems should be studied. In particular, it is easier to apply in industry developed synchronization approaches where the number of controls is less than the number of state variables. Moreover, this knowledge should be applied in engineering to fields such as communications, and that will be a subject of our future work.

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