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SIMPLEST 3D CONTINUOUS-TIME QUADRATIC SYSTEMS AS CANDIDATES FOR GENERATING MULTISCROLL CHAOTIC ATTRACTORS

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In this paper, we present a simple approach to produce n-scroll chaotic attractors in 3D quadratic continuous time systems. The method of analysis is based on the number of equilibrium points. Some numerical results are also given and discussed.

Keywords: 3D quadratic continuous time systems; classification; n-scroll chaotic attractors.

1. Introduction

In many situations, the existence of several equilibrium points in a dynamical system makes its dynamics more complex and allows some special structures. Examples include the well-known multiscroll attractors ([Wang, 2009; Qi *et al.*, 2008; Liu & Chen, 2004; Li, 2008; Wang *et al.*, 2009; Lü *et al.*, 2008; Yu *et al.*, 2006; Yu *et al.*, 2008, 2010a; Wang & Chen, 2012] and references therein) such as chaotic attractors with multiple-merged basins of attraction, scroll grid attractors, and $2n$-wing and $n \times m$-wing Lorenz-like attractors. It is remarkable that the vector fields of all these systems are very complex with high dimensions. In some other situations, the number of equilibrium points has no influence on the scroll number of the resulting chaotic attractor. For example, in [Wang & Chen, 2012] the considered system has one equilibrium point, but generates many-petal attractors. So *the existence of a system with only zero, one, or fewer than n equilibrium points that generates n-scroll chaotic attractors* leaves an important open problem whose solution is not evident.

We notice that this type of attractor has several real world applications. For example, the circuit verification of proposed multiscroll chaotic systems is very important for several potential engineering applications such as secure communication and efficient liquid mixing. See [Yu *et al.*, 2010b] for more details. Furthermore, multiscroll chaotic attractors have wide applications in complex networks [Lü & Chen, 2005] and multiagent systems [Zhu *et al.*, 2013].

2. Algebraically Simplest 3D Quadratic Continuous Time Systems with Several Scrolls

It is well known that lower-dimensional systems are the algebraically simplest and appropriate models for generating chaotic attractors. For this reason, it is very interesting to construct such a system with many-scroll chaotic attractors. The natural candidates for this choice are 3D quadratic continuous time systems given by

$$\begin{cases} x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 \\ \qquad + a_6z^2 + a_7xy + a_8xz + a_9yz \\ y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 \\ \qquad + b_6z^2 + b_7xy + b_8xz + b_9yz \\ z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 \\ \qquad + c_6z^2 + c_7xy + c_8xz + c_9yz \end{cases} \tag{1}$$

where $(a_i, b_i, c_i)_{0\le i\le 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Some observations on existing 3D quadratic continuous time systems displaying n-scroll chaotic attractors [Wang, 2009; Qi *et al.*, 2008; Liu & Chen, 2004; Li, 2008; Wang *et al.*, 2009; Yu *et al.*, 2006] suggest that the algebraically simplest form of these systems is given by

$$\begin{cases} x' = a_1x + a_2y + a_3z + a_9yz \\ y' = b_1x + b_2y + b_3z + b_8xz + b_9yz \\ z' = c_1x + c_2y + c_3z + c_7xy + c_8xz + c_9yz. \end{cases} \tag{2}$$

We remark that it is not possible to reduce further Eq. (2) by removing additional terms. The form of the equations in (2) is a generalization of the existing systems in the current literature. The present paper investigates possible new cases giving multiscroll attractors.

System (2) has 15 bifurcation parameters, and its equilibria satisfy

$$\begin{cases} x = \dfrac{\omega_6z^3 + \omega_7z^2 + \omega_8z}{a_1\mu_2}, \quad a_1 \neq 0,\ \ \mu_2 \neq 0 \\ y = \dfrac{-\omega_4z^2 - \omega_5z}{\mu_2}, \quad \mu_2 \neq 0 \\ z(\omega_9z^5 + \omega_{10}z^4 + \omega_{11}z^3 + \omega_{12}z^2 \\ \qquad + \omega_{13}z + \omega_{14}) = 0 \end{cases} \tag{3}$$

where

$$\begin{cases} \mu_2 = \omega_1z^2 + \omega_2z + \omega_3 \\ \mu_3 = \omega_4z^2 + \omega_5z \end{cases} \tag{4}$$

where

$$\begin{cases} \omega_1 = -\dfrac{1}{a_1}a_9b_8, \quad \omega_2 = b_9 - \dfrac{1}{a_1}a_2b_8 - \dfrac{1}{a_1}b_1a_9, \\ \omega_3 = b_2 - \dfrac{1}{a_1}a_2b_1, \quad \omega_4 = -\dfrac{1}{a_1}a_3b_8, \\ \omega_5 = b_3 - \dfrac{1}{a_1}a_3b_1, \quad \omega_6 = \omega_4a_9 - \omega_1a_3, \\ \omega_7 = \omega_4a_2 - \omega_2a_3 + \omega_5a_9, \quad \omega_8 = \omega_5a_2 - \omega_3a_3, \\ \omega_9 = \omega_1\omega_6c_8 = 0 \end{cases} \tag{5}$$

and

$$\begin{cases} \omega_{10} = \delta_1 + \delta_2 \\ \omega_{11} = \delta_3 + \delta_4 \\ \omega_{12} = \delta_5 + \delta_6 + \delta_7 \\ \omega_{13} = \delta_8 + \delta_9 \\ \omega_{14} = a_1c_3\omega_3^2 + \omega_8c_1\omega_3 - \omega_3\omega_5a_1c_2 \end{cases} \tag{6}$$

where

$$\begin{cases} \delta_1 = \omega_1\omega_6c_1 + \omega_1\omega_7c_8 + \omega_2\omega_6c_8 \\ \delta_2 = -\omega_4\omega_6c_7 + \omega_1^2a_1c_3 - \omega_1\omega_4a_1c_9 \\ \delta_3 = 2\omega_1\omega_2a_1c_3 - \omega_1\omega_4a_1c_2 - \omega_1\omega_5a_1c_9 \\ \qquad - \omega_2\omega_4a_1c_9 + \omega_1\omega_7c_1 \\ \delta_4 = \omega_2\omega_6c_1 + \omega_1\omega_8c_8 + \omega_2\omega_7c_8 + \omega_3\omega_6c_8 \\ \qquad - \omega_4\omega_7c_7 - \omega_5\omega_6c_7 \\ \delta_5 = 2\omega_1\omega_3a_1c_3 - \omega_1\omega_5a_1c_2 - \omega_2\omega_4a_1c_2 \\ \qquad - \omega_2\omega_5a_1c_9 \\ \delta_6 = -\omega_3\omega_4a_1c_9 + \omega_2^2a_1c_3 + \omega_1\omega_8c_1 \\ \qquad + \omega_2\omega_7c_1 + \omega_3\omega_6c_1 \\ \delta_7 = \omega_2\omega_8c_8 + \omega_3\omega_7c_8 - \omega_4\omega_8c_7 - \omega_5\omega_7c_7 \\ \delta_8 = \omega_2\omega_8c_1 + \omega_3\omega_7c_1 + \omega_3\omega_8c_8 - \omega_5\omega_8c_7 \\ \delta_9 = 2\omega_2\omega_3a_1c_3 - \omega_2\omega_5a_1c_2 - \omega_3\omega_4a_1c_2 \\ \qquad - \omega_3\omega_5a_1c_9. \end{cases} \tag{7}$$

The condition $\mu_2 \neq 0$ in (3) is equivalent to three assertions: $\omega_2^2 - 4\omega_1\omega_3 < 0$, or $\omega_2^2 - 4\omega_1\omega_3 > 0$ and $z \notin \left\{\frac{-\omega_2+\sqrt{\omega_2^2-4\omega_1\omega_3}}{2\omega_1}, \frac{-\omega_2-\sqrt{\omega_2^2-4\omega_1\omega_3}}{2\omega_1}\right\}$ or $\omega_2^2 - 4\omega_1\omega_3 = 0$ and $z \neq \frac{-\omega_2}{2\omega_1}$, that is,

$$a_1a_2b_8b_9 > \frac{1}{2}((a_1b_9 - b_1a_9)^2 + a_2^2b_8^2 + 2a_9b_8(2a_1b_2 - a_2b_1)) \tag{8}$$

or

$$\begin{cases} a_1a_2b_8b_9 < \frac{1}{2}((a_1b_9 - b_1a_9)^2 + a_2^2b_8^2 \\ \qquad + 2a_9b_8(2a_1b_2 - a_2b_1)) \\ z \notin \left\{\frac{-\omega_2 + \sqrt{\omega_2^2 - 4\omega_1\omega_3}}{2\omega_1}, \right. \\ \qquad \left. \frac{-\omega_2 - \sqrt{\omega_2^2 - 4\omega_1\omega_3}}{2\omega_1}\right\} \end{cases} \tag{9}$$

or

$$\begin{cases} a_1a_2b_8b_9 = \frac{1}{2}((a_1b_9 - b_1a_9)^2 + a_2^2b_8^2 \\ \qquad + 2a_9b_8(2a_1b_2 - a_2b_1)) \\ z \neq \frac{-\omega_2}{2\omega_1}. \end{cases} \tag{10}$$

First, we remark that the point $(0, 0, 0)$ is a solution of the last equation of (3). Furthermore, additional equilibria of system (2) are possible if the quintic equation in (3) is solvable with respect to the variable z. However, since we have that $\omega_9 = 0$, the last equation of (3) becomes

$$z(\omega_{10}z^4 + \omega_{11}z^3 + \omega_{12}z^2 + \omega_{13}z + \omega_{14}) = 0. \tag{11}$$

Solving directly the quartic equation $q(z) = \omega_{10}z^4 + \omega_{11}z^3 + \omega_{12}z^2 + \omega_{13}z + \omega_{14}$ in (11) is a first method to see if the system has several equilibria. Depending on the choice of bifurcation parameters (a_i, b_i, c_i), this equation can have real or complex solutions. We are interested here in finding only real solutions of $q(z) = 0$. Some criteria are available in the current literature about the solvability of quartic equations [Lazard, 2004]. Generally, a quartic equation is solvable by radicals. It is also factorizable into equations of lower degree. However, determining real and different solutions is difficult. Some additional conditions on the bifurcation parameters (a_i, b_i, c_i) can be obtained by assuming that all the equilibria of system (2) are unstable. In this case, possible n-scroll chaotic attractors can be obtained. However, a rigorous formulation for these conditions is not possible for the general case. It is only possible for special cases as shown in [Wang, 2009; Qi *et al.*, 2008; Liu & Chen, 2004; Li, 2008; Wang *et al.*, 2009; Yu *et al.*, 2006].

We are looking for five different equilibria $(x_i, y_i, z_i)_{0\le i\le 4}$ of system (2) with $(z_i)_{1\le i\le 4} \neq 0$ and $z_i \neq z_j$, if $i \neq j$, with $(x_0, y_0, z_0) = (0, 0, 0)$. We can write $q(z) = \omega_{10}(z-z_1)(z-z_2)(z-z_3)(z-z_4)$, where $(z_i)_{1\le i\le 4}$ are given known values. The objective here is to determine the values of the bifurcation parameters (a_i, b_i, c_i) of system (13) in order to get four additional equilibria. By this approach, system (2) has five different equilibria given by

$$P_i = \left(\frac{\omega_6z_i^3 + \omega_7z_i^2 + \omega_8z_i}{a_1(\omega_1z_i^2 + \omega_2z_i + \omega_3)}, \frac{-\omega_4z^2 - \omega_5z}{\omega_1z_i^2 + \omega_2z_i + \omega_3}, z_i\right), \quad i = 0, \ldots, 5, \tag{12}$$

where $z_0 = 0$ and $(z_i)_{1\le i\le 4}$ are supposed to be known values. Equation (12) is possible if the following conditions hold

$$\begin{cases} a_1 \neq 0 \\ \omega_1z_i^2 + \omega_2z_i + \omega_3 \neq 0, \quad i = 1, 2, 3, 4 \\ \frac{\omega_{11}}{\omega_{10}} = -(z_1 + z_2 + z_3 + z_4) \\ \frac{\omega_{12}}{\omega_{10}} = z_3(z_1 + z_2) + z_1z_2 + z_4(z_1 + z_2 + z_3) \\ \frac{\omega_{13}}{\omega_{10}} = -(z_4(z_3(z_1 + z_2) + z_1z_2) + z_1z_2z_3) \\ \frac{\omega_{14}}{\omega_{10}} = z_1z_2z_3z_4 \\ \omega_{10} = -\frac{a_9b_8(a_3b_8c_9 - a_3b_9c_8 + b_3a_9c_8 - c_3a_9b_8)}{a_1} \end{cases} \tag{13}$$

and

$$\begin{gathered} a_9b_8 \neq 0 \quad \text{and} \\ a_3b_8c_9 - a_3b_9c_8 + b_3a_9c_8 - c_3a_9b_8 \neq 0. \end{gathered} \tag{14}$$

This procedure defines a type of pattern in the bifurcation parameter (a_i, b_i, c_i) space, i.e. if (13) holds, then system (2) has five equilibria, and chaotic attractors with several scrolls are possible. A general solution of (13) is hard to find due to the complicated algebraic formulas. However, a simple approach can be used to find some special cases

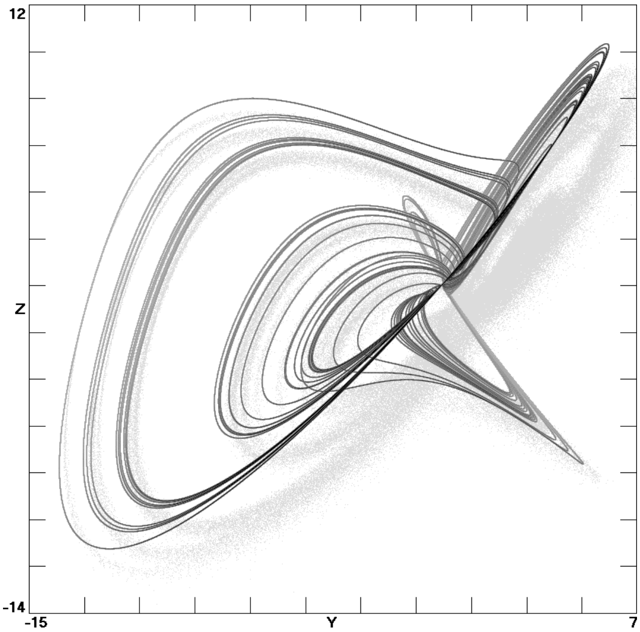

Fig. 1. A three-scroll chaotic attractor of system (16) with $c_8 = 0.23$.

as follows: fix 14 bifurcation parameters of system (2) and vary the last one. In this way, it is possible to solve (13) and determine the corresponding 3D systems of the form (2) with five different equilibria. The set of solutions of the principal equation (13) is not empty since for $(z_i)_{0\le i\le 4} = \{0, -1.607671, -1.318592, 0, 1.278669, 1.575707\}$, we have

$$\begin{cases} a_1 = 1, \quad a_2 = -1, \quad a_3 = 0.5, \quad a_9 = -3, \\ b_1 = -0.1, \quad b_2 = -6, \quad b_3 = 0, \\ b_8 = 1, \quad b_9 = -1, \\ c_1 = 0.06, \quad c_2 = -10, \quad c_3 = -5, \\ c_7 = 2, \quad c_8 = 0.23, \quad c_9 = 0. \end{cases} \tag{15}$$

2.1. *Example*

In this section, we consider the following example to validate the above approach:

$$\begin{cases} x' = x - y + 0.5z - 3yz \\ y' = -0.1x - 6y + xz - yz \\ z' = 0.06x - 10y - 5z + 2xy + c_8 xz. \end{cases} \tag{16}$$

System (16) is not symmetric under the natural coordinate transforms $(x, y, z) \to (-x, -y, -z)$ and $(x, y, z) \to (-x, -y, z)$. For $c_8 = 0.23$, i.e. $c_8 \neq 0$, system (16) produces a three-scroll strange attractor as shown in Fig. 1. Initial conditions of $(1, 0, 0)$

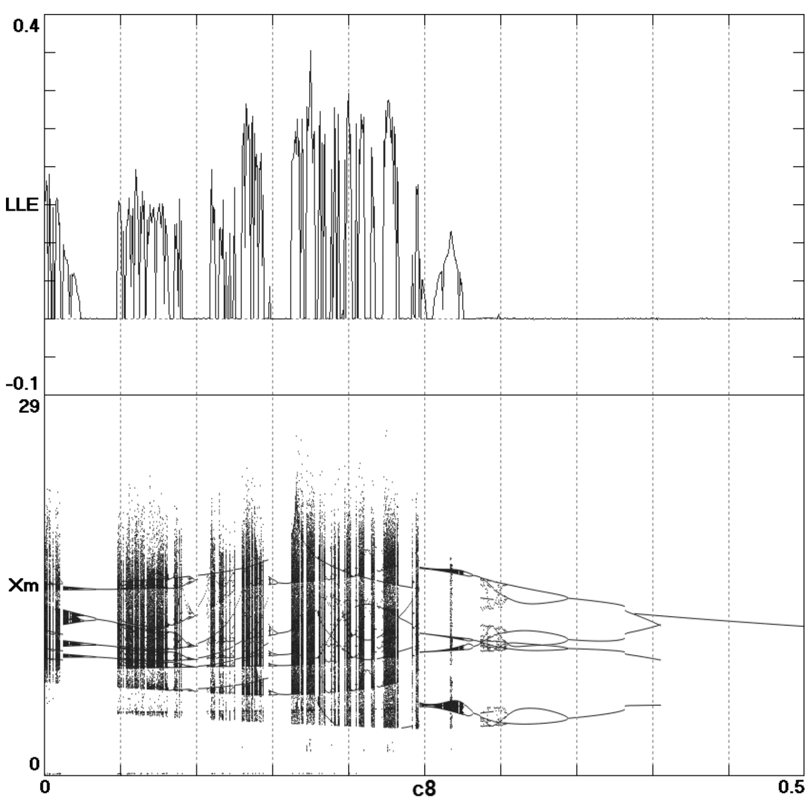

Fig. 2. Bifurcation diagram of system (16) for $c_8 < 0.5$.

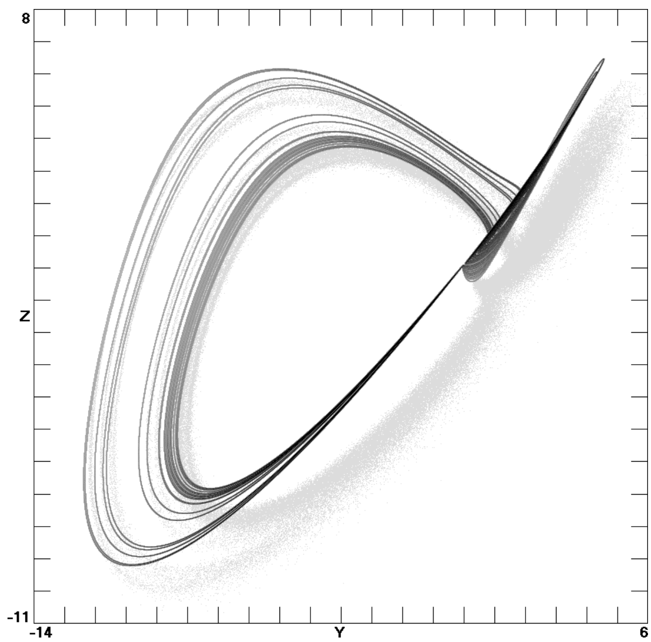

Fig. 3. A three-scroll chaotic attractor of system (16) with $c_8 = 0.0$.

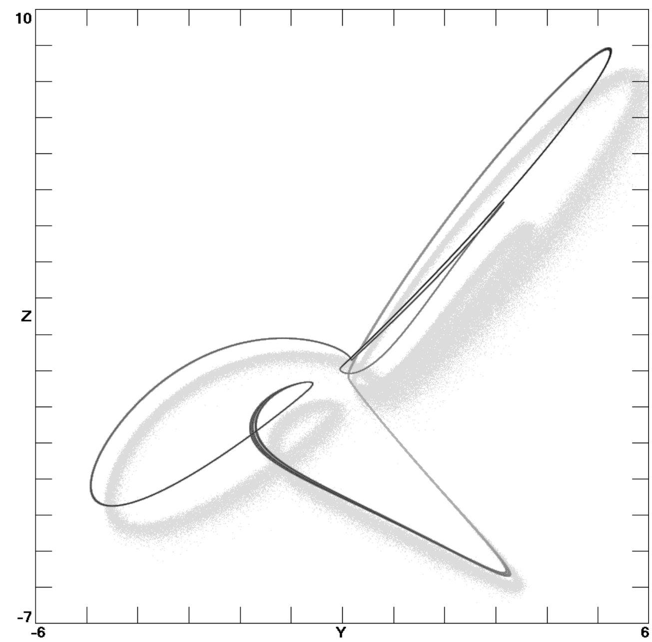

Fig. 5. A three-scroll chaotic attractor of system (16) with $c_8 = 0.25$.

start the orbit close to the attractor. The Lyapunov exponent spectrum is $(0.2553, 0, -9.4545)$, giving a Kaplan–Yorke dimension of 2.0270. We notice that system (16) is dissipative since the sum of the LEs is negative. Numerical calculations verify that the system (16) has five equilibria with $z \in \{-1.607671, -1.318592, 0, 1.278669, 1.575707\}$. All the interesting behavior occurs for $c_8 < 0.5$ as shown in the bifurcation diagram in Fig. 2.

Other chaotic attractors of system (16) are shown in Figs. 3 ($c_8 = 0$), 4 ($c_8 = 0.07$), 5 ($c_8 = 0.25$), and 6 ($c_8 = 0.27$).

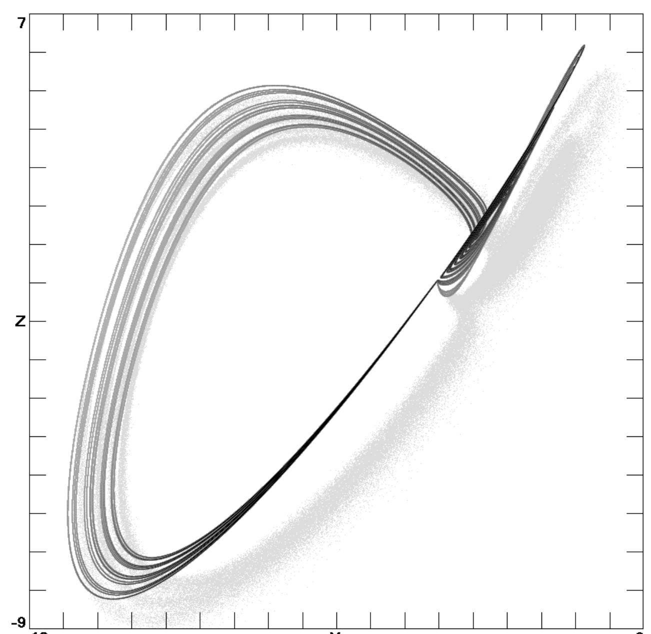

Fig. 4. A three-scroll chaotic attractor of system (16) with $c_8 = 0.07$.

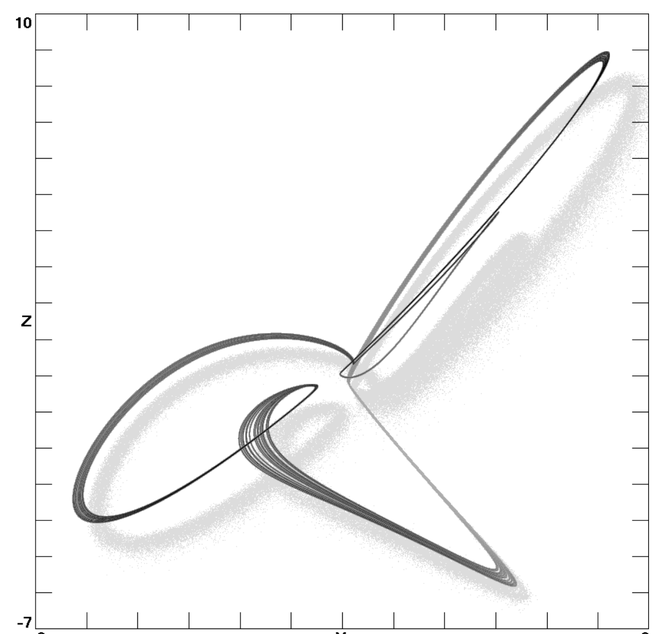

Fig. 6. A three-scroll chaotic attractor of system (16) with $c_8 = 0.27$.

3. Conclusion

In this paper, we have presented an approach based on finding the number of equilibrium points to determine the algebraically simplest forms of 3D quadratic continuous time systems with n-scroll chaotic attractors. A numerical example is also given and discussed to validate this approach.

References

Lazard, D. [2004] "Solving quintics in radicals," in *Olav Arnfinn Laudal, Ragni Piene. The Legacy of Niels Henrik Abel* (Berlin), pp. 207–225.

Li, D. Q. [2008] "A three-scroll chaotic attractor," *Phys. Lett. A* **372**, 387–393.

Liu, W. B. & Chen, G. R. [2004] "Can a three-dimensional smooth autonomous quadratic chaotic system generate a single four scroll attractor?" *Int. J. Bifurcation and Chaos* **14**, 1395–1403.

Lü, J. & Chen, G. [2005] "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Autom. Contr.* **50**, 841–846.

Lü, J., Murali, K., Sinha, S., Leung, H. & Aziz-Alaoui, M. A. [2008] "Generating multi-scroll chaotic attractors by thresholding," *Phys. Lett. A* **372**, 3234–3239.

Qi, G. Y., Chen, G. R., van Wyk, M. A., van Wyk, B. J. & Zhang, Y. H. [2008] "A four-wing chaotic attractor generated from a new 3-D quadratic autonomous system," *Chaos Solit. Fract.* **38**, 705–721.

Wang, L. [2009] "3-scroll and 4-scroll chaotic attractors generated from a new 3-D quadratic autonomous system," *Nonlin. Dyn.* **56**, 453–462.

Wang, Z., Sun, Y., van Wyk, B. J., Qi, G. & van Wyk, M. A. [2009] "A 3-D four-wing attractor and its analysis," *Brazilian J. Phys.* **39**, 547–553.

Wang, X. & Chen, G. [2012] "Constructing a chaotic system with any number of equilibria," *Nonlin. Dyn.* **71**, 429–436.

Yu, S. M., Lu, J. H., Tang, W. K. S. & Chen, G. [2006] "A general multiscroll Lorenz system family and its DSP realization," *Chaos* **16**, 033126-1–10.

Yu, S. M., Tang, W. K. S., Lü, J. & Chen, G. [2008] "Generation of $n \times m$-wing Lorenz-like attractors from a modified Shimizu–Morioka model," *IEEE Trans. Circuits Syst.-II* **55**, 1168–1172.

Yu, S. M., Tang, W. K. S., Lü, J. & Chen, G. [2010a] "Generating $2n$-wing attractors from Lorenz-like systems," *Int. J. Circuit Th. Appl.* **38**, 243–258.

Yu, S., Lü, J., Chen, G. & Yu, X. [2010b] "Design and implementation of grid multiwing butterfly chaotic attractors from a piecewise Lorenz system," *IEEE Trans. Circuits Syst.-II*: *Exp. Briefs* **57**, 803–807.

Zhu, J., Lü, J. & Yu, X. [2013] "Flocking of multi-agent non-holonomic systems with proximity graphs," *IEEE Trans. Circuits Syst.-I* **60**, 199–210.