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Elementary quadratic chaotic flows with no equilibria

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ABSTRACT

Three methods are used to produce a catalog of seventeen elementary three-dimensional chaotic flows with quadratic nonlinearities that have the unusual feature of lacking any equilibrium points. It is likely that most if not all the elementary examples of such systems have now been identified.

1. Introduction

Recently there has been interest in finding and studying rare examples of simple chaotic flows such as those in which there are no equilibria or in which any existing equilibria are stable [1–4]. There is little knowledge about the characteristics of such systems. Here we consider chaotic flows with no equilibria. Such systems can have neither homoclinic nor heteroclinic orbits [5], and thus the Shilnikov method [6,7] cannot be used to verify the chaos.

We are aware of only three such examples that have been previously reported, although there may be additional cases for which the lack of an equilibrium was not specifically noted. The oldest and best-known example is the conservative Sprott A system [8] listed as NE_1 in Table 1. This is an important system since it is a special case of the Nose–Hoover oscillator [9] which describes many natural phenomena [10], and thus it suggests that such systems may have practical as well as theoretical importance. Recently, two other dissipative examples have been reported, which we call the Wei system [1] listed as NE_2 in Table 1 and the Wang–Chen system [2], a simplified version of which with one fewer term than previously published is listed as NE_3 in Table 1.

2. Main results

We performed a systematic search to find additional three-dimensional chaotic systems with quadratic nonlinearities and no equilibria. Our search was based on the methods proposed in [11] and used our own custom software. Our objective was to find the algebraically simplest cases which cannot be further reduced by the removal of terms without destroying the chaos. The search was inspired by the observation that each of the known examples contains a constant term, and that if the constant is set to zero, the resulting system is nonhyperbolic (the equilibria have eigenvalues with a real part equal to zero). Two of them (Wei and Wang–Chen) have a pair of imaginary eigenvalues. It is a general requirement that chaotic systems of this type include such a constant term since there would otherwise be at least one equilibrium point at the origin $(0, 0, 0)$. Thus we proceeded to search for additional examples using three basic methods:

(1) We added a constant term a to other nonhyperbolic systems. For example, the system

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -x + z \\
\dot{z} &= k_1 x^2 + k_2 z^2 + k_3 y^2 + a \qquad (1)
\end{aligned}
$$

with $a = 0$ has an equilibrium at $(0, 0, 0)$ whose eigenvalues are zero and pure imaginary. Adjusting and simplifying the parameters k_1, k_2, k_3, and a gives the chaotic system listed as NE_7 system in Table 1.

(2) We looked at cases where we could show algebraically that the equilibrium points are imaginary. For example, any chaotic solution of a parametric system such as

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z
\end{aligned}
$$

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Table 1
Seventeen simple chaotic systems with no equilibria.

Model	Equations	a	LEs	D_{KY}	(x_0, y_0, z_0)
NE_1 Sprott A (Nose–Hoover)	$\dot{x} = y$ $\dot{y} = -x - zy$ $\dot{z} = y^2 - a$	1.0	0.0138, 0, −0.0138	3.0000	(0, 5, 0)
NE_2 Wei	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = 2y^2 + xz - a$	0.35	0.0776, 0, −1.5008	2.0517	(0, 0.4, 1)
NE_3 Simplified Wang–Chen	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y + 0.1x^2 + 1.1xz + a$	1.0	0.0522, 0, −2.6585	2.0196	(1, 1, −1)
NE_4	$\dot{x} = -0.1y + a$ $\dot{y} = x + z$ $\dot{z} = xz - 3y$	1.0	0.0235, 0, −8.480	2.0028	(−8.2, 0, −5)
NE_5	$\dot{x} = 2y$ $\dot{y} = -2x - z$ $\dot{z} = -y^2 + z^2 + a$	2.0	0.0168, 0, −0.3622	2.0465	(0.98, 1.8, −0.7)
NE_6	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y - xz - yz - a$	0.75	0.0280, 0, −3.4341	2.0082	(0, 3, −0.1)
NE_7	$\dot{x} = y$ $\dot{y} = -x + z$ $\dot{z} = -0.8x^2 + z^2 + a$	2.0	0.0252, 0, −6.8524	2.0037	(0, 2.3, 0)
NE_8	$\dot{x} = y$ $\dot{y} = -x - yz$ $\dot{z} = xy + 0.5x^2 - a$	1.3	0.0314, 0, −10.2108	2.0031	(0, 0.1, 0)
NE_9	$\dot{x} = y$ $\dot{y} = -x - yz$ $\dot{z} = -xz + 7x^2 - a$	0.55	0.0504, 0, −0.3264	2.1544	(0.5, 0, 0)
NE_{10}	$\dot{x} = z$ $\dot{y} = z - y$ $\dot{z} = -0.9y - xy + xz + a$	0.6	0.0061, 0, −1.3002	2.0047	(1, 0.7, 0.8)
NE_{11}	$\dot{x} = y$ $\dot{y} = -x + z$ $\dot{z} = z - 2xy - 1.8xz - a$	1.0	0.0706, 0, −0.6456	2.1094	(0, 1.6, 3)
NE_{12}	$\dot{x} = z$ $\dot{y} = x - y$ $\dot{z} = -4x^2 + 8xy + yz + a$	0.1	0.0654, 0, −2.0398	2.0321	(0.5, 0, −1)
NE_{13}	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = xy + xz + 0.2yz - a$	0.4	0.1028, 0, −2.1282	2.0483	(2.5, 0, 0)
NE_{14}	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = x^2 - y^2 + 2xz + yz + a$	1.0	0.0532, 0, −11.8580	2.0045	(1, 0, −4)
NE_{15}	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = x^2 - y^2 + xy + 0.4xz + a$	1.0	0.1101, 0, −1.3879	2.0793	(0, 1, −4.9)
NE_{16}	$\dot{x} = -0.8x - 0.5y^2 + xz + a$ $\dot{y} = -0.8y - 0.5z^2 + yx + a$ $\dot{z} = -0.8z - 0.5x^2 + zy + a$	1.0	0.0607, 0, −0.1883	2.3224	(0, 1, −1)
NE_{17}	$\dot{x} = -y - z^2 + 2.3xy + a$ $\dot{y} = -z - x^2 + 2.3yz + a$ $\dot{z} = -x - y^2 + 2.3zx + a$	2.0	0.2257, 0, −1.7477	2.1292	(1, −1, 0)

$$\dot{z} = k_1 x + k_2 y + k_3 z + k_4 x^2 + k_5 y^2 + k_6 z^2 + k_7 xy + k_8 xz + k_9 yz + a$$

$$k_1^2 - 4k_4 a \leqslant 0 \tag{2}$$

is a candidate. Adjusting the parameters $k_1, \ldots, k_9$, and a gives the system listed as NE_{14} in Table 1.

(3) We added a constant to each of the derivatives in known chaotic systems and looked for solutions where the numerically calculated equilibria do not exist. For example, Case O of reference [8] with added constants a_1, a_2, and a_3

$$\dot{x} = k_1 y + a_1$$
$$\dot{y} = k_2 x + k_3 z + a_2$$
$$\dot{z} = k_4 x + k_5 xz + k_6 y + a_3 \tag{3}$$

gives the system listed as NE_4 in Table 1.

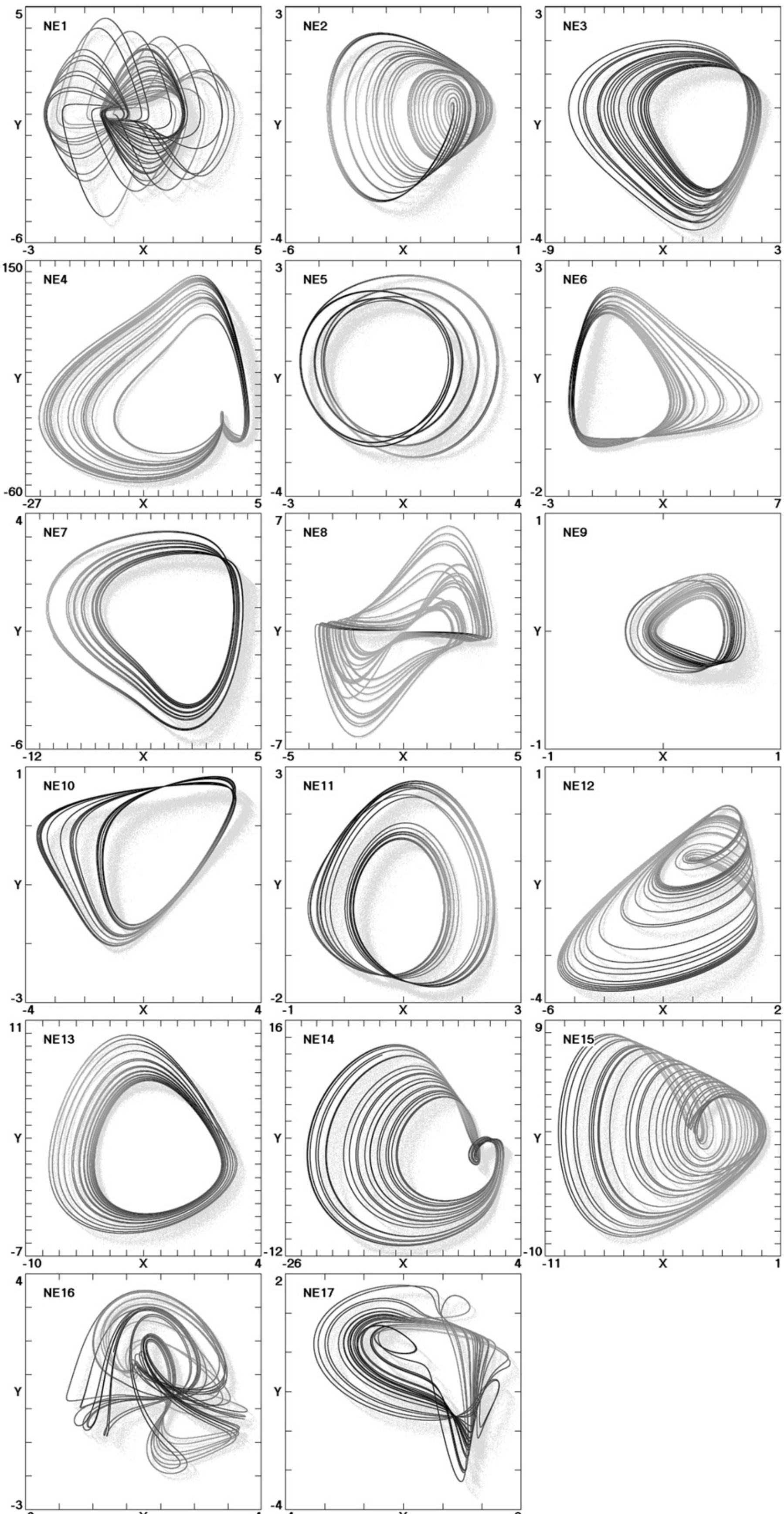

Fig. 1. State space diagram of the cases in Table 1 projected onto the xy-plane.

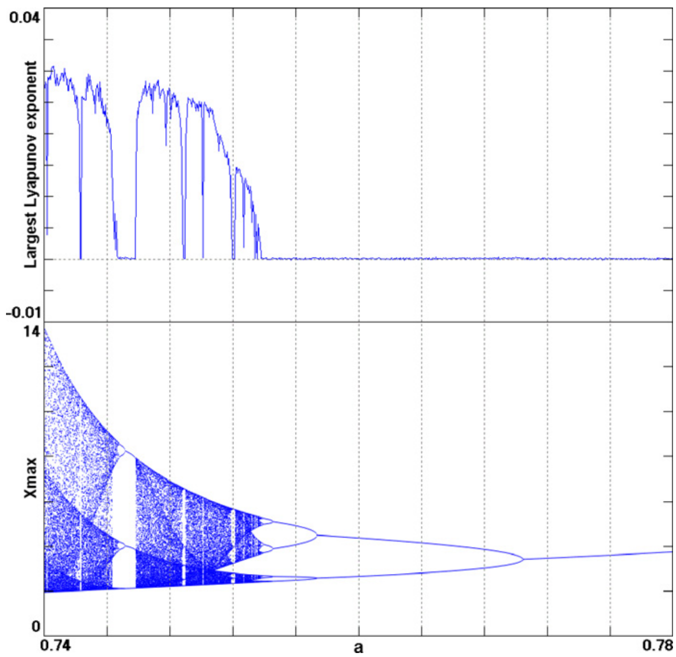

Fig. 2. The largest Lyapunov exponent and bifurcation diagram of NE_6 showing a period-doubling route to chaos.

All nineteen of the simple cases in reference [8] were examined in this way, and only Case A (NE_1) and Case D (NE_2) have solutions in addition to Case O (NE_4). No variants of the Lorenz [12] or Rössler [13] systems with added constant terms appear to admit chaotic solutions when the equilibria are removed. However, there are two examples of circulant systems (systems symmetric with respect to a cyclic rotation of the variables x, y, and z) [11] listed as NE_{16} and NE_{17} in Table 1.

In addition to the seventeen cases listed in the table, dozens of additional cases were found that are extensions of these cases with additional terms. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" [11], by which we mean that as many coefficients as possible are set to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits. In this way, we believe we have identified most if not all of the elementary forms of three-dimensional chaotic systems with quadratic nonlinearities that have no equilibrium points.

3. Conclusion

Except for NE_1, all these cases are dissipative with attractors projected onto the xy-plane as shown in Fig. 1. The Lyapunov spectra and Kaplan–Yorke dimension are shown in Table 1 along with initial conditions that are close to the attractor. As is usual for strange attractors from three-dimensional autonomous systems with only a few quadratic nonlinearities, the attractor dimension is only slightly greater than 2.0, the largest of which is NE_{16} with $D_{KY} = 2.3224$. All the cases appear to approach chaos through a succession of period-doubling limit cycles, a typical example of which (NE_6) is shown in Fig. 2. Case NE_4 is the simplest dissipative flow with no equilibrium in the sense that it has only six terms and a single quadratic nonlinearity. Along with the other cases in the table, it is worthy of further study.

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