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SIMPLE CHAOTIC FLOWS WITH ONE STABLE EQUILIBRIUM

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Using the Routh–Hurwitz stability criterion and a systematic computer search, 23 simple chaotic flows with quadratic nonlinearities were found that have the unusual feature of having a coexisting stable equilibrium point. Such systems belong to a newly introduced category of chaotic systems with hidden attractors that are important and potentially problematic in engineering applications.

Keywords: Chaotic flows; stable equilibrium; hidden attractors; eigenvalue.

1. Introduction

It is widely recognized that mathematically simple systems of nonlinear differential equations can exhibit chaos. With the advent of fast computers, it is now possible to explore the entire parameter space of these systems with the goal of finding parameters that result in some desired characteristics of the system. Recently there has been increasing attention to some unusual examples of such systems such as those having no equilibrium, stable equilibria, or coexisting attractors [Jafari *et al.*, 2013; Wei, 2011a, 2011b; Wang & Chen, 2012, 2013; Wei & Yang, 2010, 2011, 2012; Wang *et al.*, 2012a, 2012b].

Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [Kuznetsov *et al.*, 2010, 2011a, 2011b; Leonov & Kuznetsov, 2011a, 2011b, 2013a, 2013b; Leonov *et al.*, 2011a, 2011b, 2012]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. Thus any dissipative chaotic flow with no equilibrium or with only stable equilibria must have a hidden strange attractor. Only a few such examples have been reported in the literature [Jafari *et al.*, 2013; Wei, 2011a, 2011b;

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Wang & Chen, 2012, 2013; Wei & Yang, 2010, 2011, 2012; Wang *et al.*, 2012a, 2012b]. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

The goal of this paper is to greatly expand the list of known hidden chaotic attractors and to identify the mathematically simplest systems in which they occur. Thus we perform a systematic computer search for chaos in three-dimensional autonomous systems with quadratic nonlinearities and a single equilibrium that is stable according to the Routh–Hurwitz criterion.

2. Simple Chaotic Flows with One Stable Equilibrium

In the search for chaotic flows with a stable equilibrium, we first focus on jerk systems. We consider a general equation with quadratic nonlinearities of the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= f(x,y,z) \\ f &= a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 \\ &\quad + a_6z^2 + a_7xy + a_8xz + a_9yz + a_{10}. \end{aligned} \tag{1}$$

Any equilibrium point of (x^*, y^*, z^*) of system (1) must have $y^* = z^* = 0$ and eigenvalues λ that satisfy

$$\lambda^3 - f_z\lambda^2 - f_y\lambda - f_x \tag{2}$$

in which $f_x = a_1 + 2a_4x^*$, $f_y = a_2 + a_7x^*$, and $f_z = a_3 + a_8x^*$. Using the Routh–Hurwitz stability criterion, we require $f_z < 0$, $f_yf_z + f_x > 0$, and $f_x < 0$ for that equilibrium to be stable.

Table 1. 23 simple chaotic flows with one stable equilibrium.

Model	Equations	Equilibrium	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
SE_1	$\dot{x} = y$	0	−1.9548	0.0377		4
	$\dot{y} = z$	0	−0.0226	0	2.0185	−2
	$\dot{z} = -x - 0.6y - 2z + z^2 - 0.4xy$	0	$\pm 0.7149i$	−2.0377		0
SE_2	$\dot{x} = y$	0	−0.5103	0.0804		−1
	$\dot{y} = z$	0	−0.0198	0	2.1644	0
	$\dot{z} = -0.5x - y - 0.55z - 1.2z^2 - xz - yz$	0	$\pm 0.9896i$	−0.4889		1
SE_3	$\dot{x} = y$	0	−3.9641	0.0711		−2
	$\dot{y} = z$	0	−0.0179	0	2.0175	0
	$\dot{z} = -3.4x - y - 4z + y^2 + xy$	0	$\pm 0.9259i$	−4.0711		2.4
SE_4	$\dot{x} = y$	−1	−1.6942	0.0434		0.5
	$\dot{y} = z$	0	−0.0029	0	2.0249	1
	$\dot{z} = -x - 1.7z + y^2 + 0.6xy - 1$	0	$\pm 0.7683i$	−1.7434		0
SE_5	$\dot{x} = y$	−2.7	−0.9600	0.0136		−6.1
	$\dot{y} = z$	0	−0.0200	0	2.0134	1
	$\dot{z} = -x - z - z^2 + 0.4xy - 2.7$	0	$\pm 1.0204i$	−1.0136		1
SE_6	$\dot{x} = y$	−1	−1.0526	0.0638		−2.2
	$\dot{y} = z$	0	−0.0237	0	2.0600	0.6
	$\dot{z} = -x - 2.9z^2 + xy + 1.1xz - 1$	0	$\pm 0.9744i$	−1.0638		0
SE_7	$\dot{x} = y$	0	−2.0000	0.0360		1
	$\dot{y} = -x + yz$	0	−0.2500	0	2.0014	−0.7
	$\dot{z} = -2z - 8xy + xz - 1$	−0.5	$\pm 0.9682i$	−25.6798		0
SE_8	$\dot{x} = y$	0	−1.0000	0.1412		0
	$\dot{y} = -x + yz$	0	−0.0500	0	2.1034	0.9
	$\dot{z} = -z - 0.7x^2 + y^2 - 0.1$	−0.1	$\pm 0.9987i$	−1.3649		0

Table 1. (*Continued*)

Model	Equations	Equilibrium	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
SE_{9}	$\dot{x} = y$	0	-2.0000	0.0203		0
	$\dot{y} = -x + yz$	0	-0.0750	0	2.0082	0.8
	$\dot{z} = 2x - 2z + y^2 - 0.3$	-0.15	$\pm 0.9972i$	-2.4751		-0.2
SE_{10}	$\dot{x} = y$	0	-2.0000	0.0963		3.9
	$\dot{y} = -x + yz$	0	-0.0250	0	2.0061	0
	$\dot{z} = x - 0.3y - 2z + xz - 0.1$	-0.05	$\pm 0.9997i$	-15.7010		1
SE_{11}	$\dot{x} = y$	0	-12.0000	0.0801		-2
	$\dot{y} = -x + yz$	0	-0.0417	0	2.0056	0
	$\dot{z} = -y - 12z + x^2 + 9xz - 1$	$-1/12$	$\pm 0.9991i$	-14.1917		0.1
SE_{12}	$\dot{x} = y$	0	-66.0000	0.0259		2
	$\dot{y} = -x + yz$	0	-0.0076	0	2.0004	0.6
	$\dot{z} = -66z + y^2 + 35xz - 1$	$-1/66$	$\pm 1.0000i$	-61.6130		0
SE_{13}	$\dot{x} = y$	0	-4.9000	0.0540		0
	$\dot{y} = -x + yz$	0	-0.1020	0	2.0112	-2.2
	$\dot{z} = -4.9z + 0.4y^2 + xy - 1$	$-1/4.9$	$\pm 0.9948i$	-4.8228		0
SE_{14}	$\dot{x} = z$	0	-0.6082	0.0657		0.5
	$\dot{y} = x + z$	-0.7	-0.0459	0	2.0401	-1
	$\dot{z} = -y - 3z^2 + xy + yz - 0.7$	0	$\pm 1.2814i$	-1.6407		0
SE_{15}	$\dot{x} = -z$	0	-1.0618	0.0414		3
	$\dot{y} = x - z$	$-10/9$	-0.0247	0	2.0062	2
	$\dot{z} = 0.9y + 0.2x^2 + xz + yz + 1$	0	$\pm 0.9203i$	-6.6641		0
SE_{16}	$\dot{x} = -z$	0	-1.0549	0.0775		1
	$\dot{y} = -x + z$	0	-0.1726	0	2.0115	6
	$\dot{z} = -7y - 1.4z + x^2 + xz - yz$	0	$\pm 2.5702i$	-6.7190		-6
SE_{17}	$\dot{x} = z$	$0.57/3.1$	-1.0000	0.0832		7.5
	$\dot{y} = x - y$	$0.57/3.1$	-0.0092	0	2.1262	0
	$\dot{z} = -3.1x - 0.3xz + 0.2yz + 0.57$	0	$\pm 1.7607i$	-0.6549		-5
SE_{18}	$\dot{x} = z$	0	-1.0000	0.1469		-28
	$\dot{y} = -y + z$	0	-0.0500	0	2.0383	0
	$\dot{z} = -2.1x - 0.1z - y^2 + 0.11xz + 0.5yz$	0	$\pm 1.4483i$	-3.8348		0
SE_{19}	$\dot{x} = z$	-0.3	-1.3766	0.0241		0.2
	$\dot{y} = -y + z$	0	-0.0667	0	2.0005	6
	$\dot{z} = -x - 2xy + 1.7xz - 0.3$	0	$\pm 0.8497i$	-49.8730		7
SE_{20}	$\dot{x} = z$	0	-0.8543	0.2125		-2.1
	$\dot{y} = -y - z$	0	-0.0728	0	2.1753	0
	$\dot{z} = -11x + 2y - 2y^2 - z^2 - yz$	0	$\pm 3.5875i$	-1.2125		5
SE_{21}	$\dot{x} = z$	0	-0.8875	0.0484		0
	$\dot{y} = -y - z$	0	-0.0563	0	2.0169	-3
	$\dot{z} = -7.1x + y - 2y^2 + xz - yz$	0	$\pm 2.8279i$	-2.8617		8.2
SE_{22}	$\dot{x} = z$	-0.15	-1.0000	0.0557		-6
	$\dot{y} = -y - z$	0	-0.0750	0	2.0194	3.8
	$\dot{z} = -6x - 2y^2 + xz - yz - 0.9$	0	$\pm 2.4483i$	-2.8695		0
SE_{23}	$\dot{x} = -z$	0.5	-0.9060	0.0159		-0.4
	$\dot{y} = -y - z$	0	-0.0470	0	2.0156	1
	$\dot{z} = 4x - 0.2z^2 + xy - 2$	0	$\pm 2.1006i$	-1.0159		-9

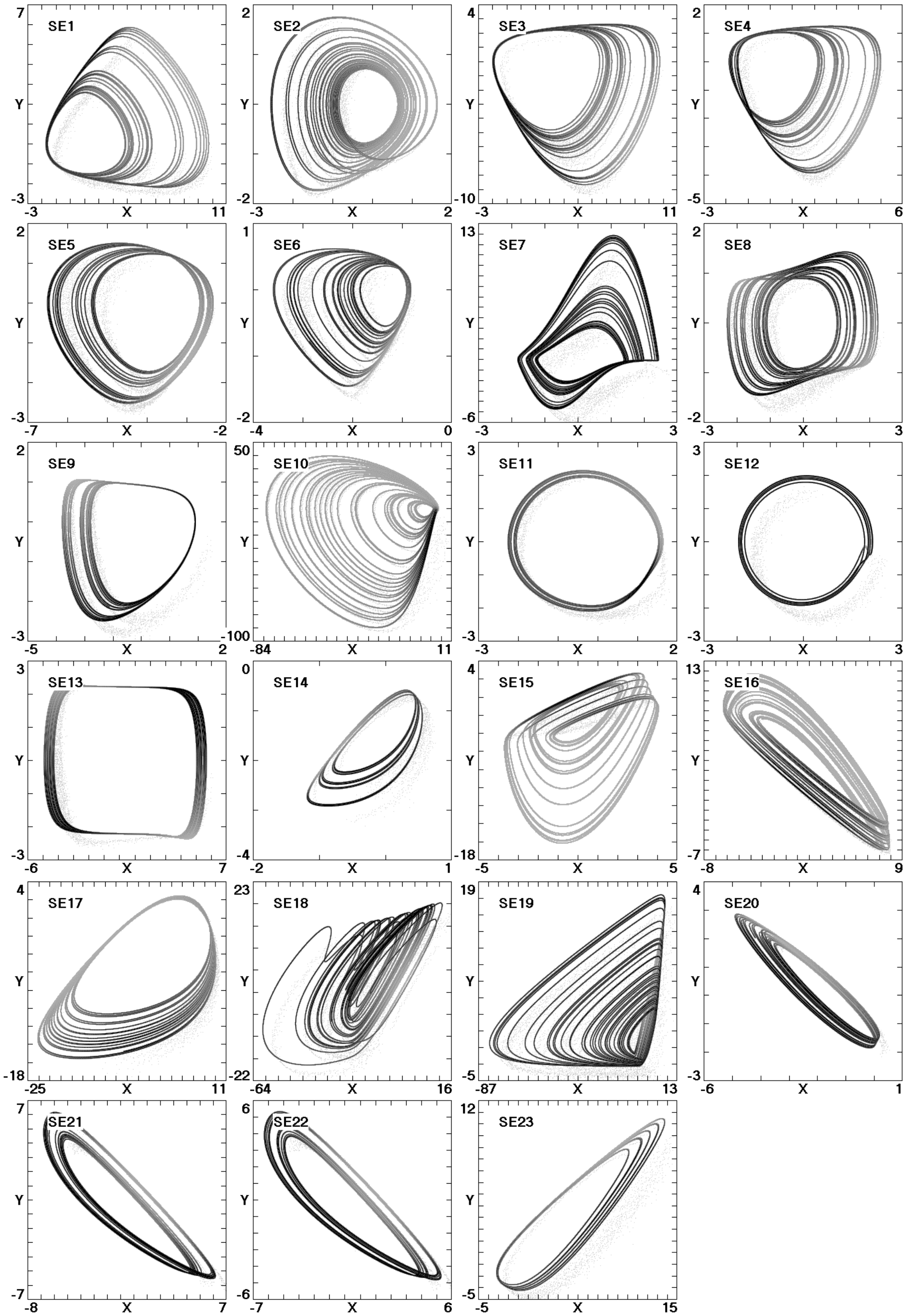

Fig. 1. State space diagram of the cases in Table 1 projected onto the xy-plane.

We can find x^* from $a_1x + a_4x^2 + a_{10} = 0$. For $a_4 \neq 0$, we have $x^*_{1,2} = (-a_1 \pm \sqrt{\Delta})/2a_4$ where $\Delta = a_1^2 - 4a_4a_{10}$. To have an equilibrium, Δ should be greater than or equal to zero, in which case, $x_1^* = (-a_1 + \sqrt{\Delta})/2a_4$ and $x_2^* = (-a_1 - \sqrt{\Delta})/2a_4$. From the stability condition $f_x < 0$ for x_1^*, we have $\sqrt{\Delta} < 0$ which is impossible. Thus a quadratic jerk system cannot have two stable equilibria, and we therefore modify the general case in Eq. (1) to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= a_1x + a_2y + a_3z + a_4y^2 + a_5z^2 \\ &\quad + a_6xy + a_7xz + a_8yz + a_9 \end{aligned} \tag{3}$$

where there is no x^2 term in the $\dot{z}$ equation to ensure that one and only one equilibrium exists. This system has a single equilibrium at $(-a_9/a_1, 0, 0)$ whose stability requires

$$\begin{aligned} &a_1 < 0, \quad \left(a_3 - \frac{a_7a_9}{a_1}\right) < 0, \\ &\left(a_2 - \frac{a_6a_9}{a_1}\right) < \frac{-a_1}{\left(a_3 - \frac{a_7a_9}{a_1}\right)}. \end{aligned} \tag{4}$$

An exhaustive computer search was done considering many thousands of combinations of the coefficients a_1 through a_9 and initial conditions subject to the constraints in Eq. (4), seeking cases for which the largest Lyapunov exponent is greater than 0.001. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" [Sprott, 2010], by which we mean that as many coefficients as possible are set to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest

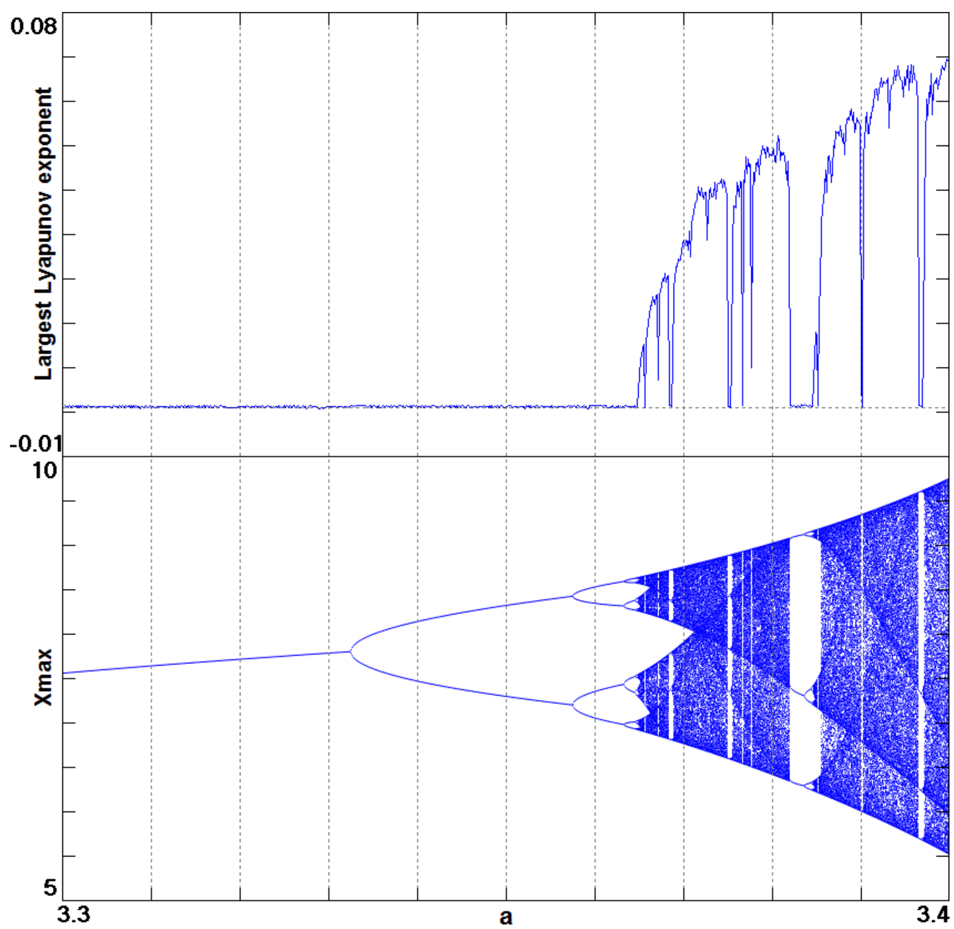

Fig. 2. The largest Lyapunov exponent and bifurcation diagram of SE_3 showing a period-doubling route to chaos ($\dot{z} = -ax - y - 4z + y^2 + xy$).

possible digits. Cases SE_1–SE_6 in Table 1 are six simple cases found in this way. As can be seen, the eigenvalues for the equilibria at the origin have all negative real parts, which means that each equilibrium is stable.

By similar calculations, many other simple structures for chaotic flows were investigated, and 17 other cases (SE_7–SE_{23}) were added to the previous six jerk systems in Table 1. In addition to the cases listed in the table, dozens of additional cases were found, but they were either equivalent to one of the cases listed by some linear transformation of variables, or they were extensions of these cases with additional terms. In this way, we believe we have identified most of the elementary forms of chaotic flows with quadratic nonlinearities that have a single stable equilibrium.

All these cases are dissipative with attractors projected onto the xy-plane as shown in Fig. 1. The Lyapunov spectra and Kaplan–Yorke dimensions are shown in Table 1 along with initial conditions that are close to the attractor. As is usual for strange attractors from three-dimensional autonomous systems, the attractor dimension is only slightly greater than 2.0, the largest of which is SE_{20} with $D_{KY} = 2.1753$, although no effort was made to tune the parameters for maximum chaos. All the cases appear to approach chaos through a succession of period-doubling limit cycles, a typical example of which (SE_3) is shown in Fig. 2. Note that all the equilibria are spiral nodes (one pair of complex conjugate eigenvalues) rather than simple nodes (all eigenvalues real). Another common feature of these systems is the small negative real part in the complex pair of eigenvalues compared to the real eigenvalue.

Since all the cases have a stable equilibrium, a point attractor coexists with a strange attractor for each case. Figure 3 shows a cross-section in the xy-plane at $z = 0$ of the basin of attraction for the two attractors for the typical case SE_3. Note that the cross-section of the strange attractor nearly touches its basin boundary as is typical of low-dimensional chaotic flows.

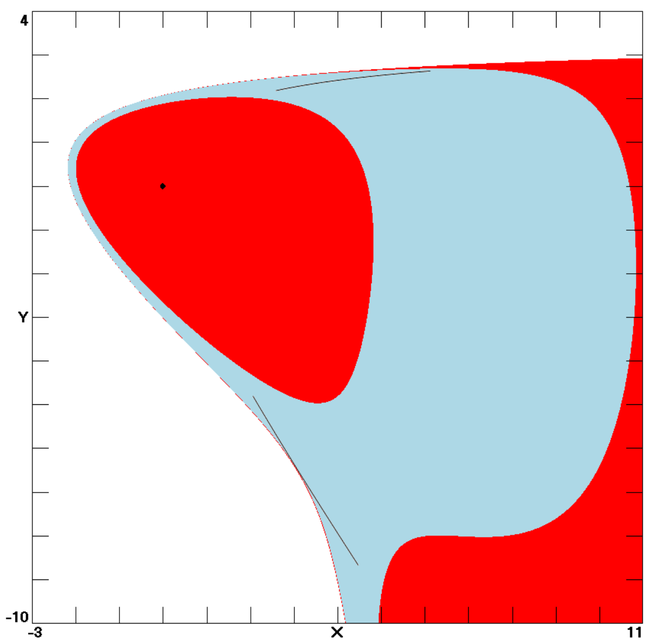

Fig. 3. Cross-section of the basins of attraction of the two attractors in the xy-plane at $z = 0$. Initial conditions in the white region lead to unbounded orbits, those in the red region lead to the point attractor shown as a black dot, and those in the light blue region lead to the strange attractor shown in cross-section as a pair of black lines.

3. Conclusion

In conclusion, it is apparent that simple chaotic systems with a single stable equilibrium that were once thought to be unusual, may in fact, be rather common. These systems belong to the newly introduced class of chaotic systems with hidden attractors. We showed that chaotic jerk systems with quadratic nonlinearities cannot have multiple stable equilibria. Furthermore, it can be proved that in any polynomial jerk system, all the equilibria cannot be simultaneously stable.

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