Broken symmetry in modified Lorenz model

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Abstract: The Lorenz model is of interest because of its abundant bifurcations and dynamical phenomena, due largely to the presence of critical sets or non-definition sets. The model is investigated as a three-parameter quadratic family. This article further develops and refines a study of its basins of attraction, and it is explained by using two types of nonclassical singularity sets. This has an important impact on the number of preimages and shows the essential role played by the vanishing denominator in the inverses. A deeper analysis of the global dynamic properties of the model in the parameter ranges where three steady states exist, reveals the role of symmetry with an interesting and complex dynamic structure.

Keywords: bifurcation, symmetry, critical sets, vanishing denominator.

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1 Introduction

The Burgers mapping expressed by

$$T^1_{\varepsilon,\mu,\nu} : \begin{cases} x' = (1-\varepsilon\nu)x - \varepsilon y^2 \\ y' = (1+\varepsilon\mu+\varepsilon x)y \end{cases} \tag{1}$$

provides an extraordinarily rich repertoire of mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, and obtained by a drastic change of the Navier-Stokes equations. The mapping (1) is smooth and noninvertible. For the parameter values considered in Burgers (1939), the behavior of the solutions is mainly determined by the three fixed points, $(0,0)$ and $(-\mu, \pm\sqrt{\mu\nu})$, and the singular line $x = -\mu - 1/\varepsilon$, such that every point maps onto the x-axis, which is the stable manifold of the hyperbolic point $(0,0)$ so long as $\varepsilon\nu$ is less than 2. The other fixed points are stable for ε less than $1/2\mu$.

Identical behavior is observed in certain other mappings, such as the Lorenz model given by

$$T_{a,b,c} : \begin{cases} x' = (1+ab)x - bxy \\ y' = (1-b)y + cx + bx^2 \end{cases} \tag{2}$$

Lorenz (1989) analyzed this system with $c = 0$, which was obtained as an approximation to an ordinary differential equation via Euler's forward differencing scheme. He was interested in the chaotic behavior when b is extremely large. He illustrated the pertinence of the concept of computational chaos. The authors considered these same mappings for detecting fractal sets in Djellit et al. (2010). The fractal structure was revealed by fractal basin boundaries and by the patterns of self-similarity.

Moreover, Whitehead and MacDonald (1984) considered a mapping derived from a differential equation model of turbulence given by

$$T^2_{\varepsilon,\mu,\nu} : \begin{cases} x' = (1-\varepsilon\nu)x - \varepsilon xy \\ y' = (1+\varepsilon\mu+\varepsilon x)y \end{cases} \tag{3}$$

Like the other models, this mapping exhibits chaos. Elabbasy et al. (2007) gave the theoretical analysis of (3) for $\varepsilon = 1$, and complex behavior including chaos was observed using a numerical method.

The map (2) is not symmetric in the equation, but the attractors are symmetric about $x = 0$ for $c = 0$. The standard analysis of local stability and bifurcations suggests that oscillations occur in the presence of only one unstable equilibrium, whereas the coexistence of three equilibria is characterized by bi-stability, the central equilibrium being on the boundary which separates the basins of the two stable ones.

All these maps give identical behavior and display homoclinic structure associated with the basin bifurcation bounded-nonbounded.

The main topic of this paper is to investigate the basic patterns of complex non-uniqueness of the dynamic behavior of a class of Lorenz mappings proposed in Lorenz (1989) and given by Eq. (2). It is mainly focused on a new research area to identify and verify some properties of focal points on such maps.

The existence of invariant sets and weak attractors in the sense of Tsybulin and Yudovich is rigourously proved in Djellit and Hachemi (2011). We conclude that non-unique dynamics, associated with extremely complex structures of the basin boundaries, can have a profound effect on our understanding of the dynamical behavior. The model is first investigated as a two-parameter quadratic family, and its analysis is explained by using two types of nonclassical singularity sets. The first one, the critical curve, separates the plane into two regions having a different number of real inverses (here one and three). The second one is a curve of non-definition for two of the three inverses of the map, i.e., these inverses have a vanishing denominator on this line.

We also present interesting results and typical scenarios between the map and its inverses. Several papers have shown the importance of the critical curves in the bifurcations associated with invariant chaotic areas in noninvertible maps, and we consider this paper as an interaction between analytical methods and numerical methods.

The rest of the paper is organized as follows. Section 2 describes some peculiar properties of the Lorenz map. We discuss some cases where bifurcation can cause qualitative changes in the structure of the domain as some parameters are varied. Also, critical and prefocal curves are considered and have been used to examine the structure of the basins. In Section 3, the peculiar case of the embedding parameter $b = 1$ is considered, the inverses are determined and analyzed, and the essential role played by the vanishing denominator of these inverses is evidenced. Finally, we consider the modified Lorenz map and the effects of this perturbation on the symmetry in Section 4.

2 The Lorenz model

Consider the dynamical system generated by a family of two-dimensional continuous noninvertible maps $T_{a,b}$ defined by

$$T_{a,b} : \begin{cases} x' = (1 + ab)x - bxy \\ y' = x^2 \end{cases} \tag{4}$$

where a, b are real parameters, and the functions $f\,(x, y) = (1 + ab)x - bxy$ and $g\,(x, y) = (1 - b)y + bx^2$ are continuous and differentiable. The map $T_{a,b}$ is noninvertible. There are regions in the phase plane where we can have different numbers of preimages. Indeed, a point $(x', y') \in \mathbb{R}^2$ may have up to three rank-one preimages that can be computed by solving the third-degree algebraic system (4) with respect to x and y.

The locus, where the number of preimages changes, is made up by the critical curves. The critical curves are computed as follows: LC_{-1} coincides with the set of points in which the Jacobian determinant vanishes, *i.e.*, $\det DT_{a,b} = 0$, and the critical curve $LC = T_{a,b}(LC_{-1})$ divides the plane in two zones Z_1 and Z_3. Then $T_{a,b}$ is of type $Z_1 < Z_3$ if we adopt the notation of Mira et al. (1996), where points on one side of the critical curve LC have one preimage, and points on the other side of LC have three preimages.

The map (4), which exhibits pitchfork and Neimark-Sacker bifurcations, has an important feature in that their inverses have denominators that vanish along a line in the plane for $b = 1$. This has a great consequence on the chaotic attractor structure. This is due to the fact that the attractor contains the focal point of the inverse. It is evident that the origin $O = (0, 0)$ is a fixed point of $T_{a,b}$ (not unique), and that the whole line $x = 0$ is mapped into the origin in one iteration. This kind of noninvertible map of the plane is structurally well-known. Its “embedding” into a larger structurally stable family $T_{a,b}$, for $b < 1$ and $b > 1$, gives regions Z_k; $(k = 1, 3)$ being the number of real preimages) which explains the complex nature when the embedding parameter takes as value 1, leading to a structurally different map.

Recent works dealing with cases of multiple attractors in noninvertible maps have highlighted how noninvertibility can be a source of bifurcations and complex structures of the basins of attraction (see references cited in Djellit et al. (2010), Djellit and Hachemi (2011), and Elabbasy et al. (2007)). However, the global phenomena and the complex structures of the basins shown here are due to the “dual situation” which occurs when the inverse map has two prefocal curves and is itself undefined in the whole plane.

2.1 Fixed points and prefocal curves

The fixed points of $T_{a,b}$ for the system (4) are solutions obtained by a trivial manipulation of (4) with $x' = x$ and $y' = y$. Besides the solution $(0,0)$, two additional fixed points are guaranteed to exist if $a > 0$, and we can easily verify that if $a < 0$, then the origin $(0,0)$ is the unique fixed point of the map $T_{a,b}$ defined by (4). If $a > 0$, then two further fixed points, P_1 and P_2, exist, symmetric with respect to the y-axis, with $x = \pm\sqrt{a}; y = a$.

From the Jacobian matrix, we can see that $J = \det DT_{a,b}(x,y) = (1-b)(1+ab-by) + 2b^2x^2$ vanishes on two curves given by $y = \frac{1}{b}\,(1+ab) + \frac{2bx^2}{(1-b)}$ for $b \neq 0$ and $b \neq 1$ or $x = 0$ for $b = 1$. The latter curve is focused by $T_{a,b}$ into a single point (following the terminology used in Mira et al. (1996) and in Ferchichi and Djellit (2009)).

Since $T_{a,b}(x = 0) = (0,0)$, we expect that the map $T_{a,b}$ has at least one inverse with the denominator, and such that $O(0,0)$ is a focal point with respect to the y-axis as a corresponding prefocal curve δ_O. In fact, the inverses

$$T_1^{-1} : \begin{cases} x = \sqrt{y'} \\ y = \frac{-x'+(1+a)\sqrt{y'}}{\sqrt{y'}} \end{cases} \tag{5}$$

and

$$T_2^{-1} : \begin{cases} x = -\sqrt{y'} \\ y = \frac{x'+(1+a)\sqrt{y'}}{\sqrt{y'}} \end{cases} \tag{6}$$

are such that they have a focal point in $O(0,0)$ with $x = 0$ as a prefocal curve, where $y' > 0$. The non-definition set δ_s is given by

$$\delta_s = \left\{(x,y) \in \mathbb{R}^2 \mid \sqrt{y'} = 0\right\}$$

and δ_s is a smooth curve in the phase plane. The two-dimensional dynamical system obtained by successive iteration of $T_{1,2}^{-1}$ is well defined, provided that the initial conditions belong to E, given by

$$E = \mathbb{R}^2 \setminus \bigcup_{k=0}^{\infty} T_{a,b,0}^{-k}(\delta_s)$$

The second curve given by $LC_{-1} : y = \frac{1}{b}\,(1+ab) + \frac{2bx^2}{(1-b)}$ (obtained by setting $J = 0$) is a curve of merging two preimages.

For $a > 0, b > 0$ and $b \neq 1$, the phase plane includes a region of noninvertibility of the map (4). The noninvertibility region is an unbounded set defined by $27\,(1-b)^2x^2 - 4b(y-(1+ab)(1-b)/b)^3 < 0$. This curve has a pointing cusp on the y-axis where $x = 0$.

The antecedents have coordinates (x,y), such that from (4), x satisfies $b^2x^3 - bx(y' - (1+ab)(1-b))/b) - (1-b)x' = 0$, and $y = ((1+ab)x - x')/bx$.

Moreover, the map $T^1_{a,b}$ is of type $Z_1 < Z_3$ whose critical curve $LC\ :\ 27((1-b)^2x^2 - 4b(y-(1+ab)(1-b)/b)^3 = 0$, image of LC_{-1}, separates the plane into two areas Z_1 and Z_3 where there exists one antecedent and three antecedents, respectively.

To understand the behavior of the map when $b \longrightarrow 1$, we follow the evolution of the phase plane when we vary the parameters a and b. We made a survey in the phase plane and plotted the different singularities, their basins of attraction, and the critical curves for the system. By analyzing the figures, we show how the particular feature is involved in explaining some properties of the dynamic behaviors of the map associated with the basin boundaries and attracting sets.

Consider the case $a = 0.1$, and vary the parameter b. There are two stable nodes. Both of the basins are simply connected sets and are located symmetrically with respect to the origin.

For $b = 1.80$, the points above the curve formed by LC with a pointing cusp given by $27(1-b)^2x^2 - 4b(y-(1+ab)(1-b)/b)^3 < 0$ have three preimages, one in the area R_1, the second in R_2, and the third in R_3. The map $T_{a,b}$ is a $Z_1 < Z_3$ type according to Mira et al. (1996) (see Figure 1).

For $b = 1.0005$, the points above the curve formed by LC, which loses its pointing cusp and is given for this b-value by $y = 0$, have two preimages, one in the area R_1, and the other in R_2. Here, we are concerned with the question of how the number of preimages is reduced as $b \to 1$. The area R_3 disappears, and we obtain instead the line $x = 0$. This line corresponds to the prefocal curve, representing a transition curve as seen in Figure 2. The map $T_{a,b}$ changes, losing one root on each side of the plane, and becoming a map with either two real preimages or none.

3 Peculiar case b=1

We consider the case $b = 1$ where the map $T_{a,1}$ admits two inverses given in Eqs. (5–6). One of these components is expressed as a quotient whose denominator depends on y. If we fix $b = 1$ then the Eq. (4) becomes

$$T_{a,1} : \begin{cases} x' = (1+a)x - xy \\ y' = x^2 \end{cases} \tag{7}$$

Then if $y\prime > 0$, we have two solutions, and no solutions if $y' < 0$. So, $T_{a,1}$ is a $Z_0 - Z_2$ noninvertible map, where Z_0 (the region whose points have no preimages) is the half plane $\{(x\,,\,y)/y < 0\}$, and Z_2 (the region whose points have two distinct rank-1 preimages) is the half plane $\{(x\,,\,y)/y > 0\}$.

The Jacobian vanishes on the line $x = 0$. The image of this line by $T_{a,1}$ is reduced to a point $(0, 0)$. Therefore, according to the definition, this point is a focal point of $T_{a,1}^{-1}$, and the line $x = 0$ is associated with the prefocal curve.

A specific class of map, with at least one of the components defined by a fractional rational function, has been studied in Ferchichi and Djellit (2009) and has very interesting properties. Some particular dynamical properties have been observed in iterated maps (or in one of the inverses) where the denominator vanishes, or where a component has the indeterminate form $0/0$ at a point $T_{a,1}^{-1}$. This characteristic has revealed new types of singularities in the phase plane, such as focal points and prefocal curves. The presence of these sets may cause new kinds of bifurcations generated by contact between them and other singularities, which gives rise to new dynamical phenomena and new basin structures and invariant sets.

In the present case, the inverse $T_{a,1}^{-1} = T_1^{-1} \cup T_2^{-1}$ of $T_{a,1}$ has a vanishing denominator. Indeed, we consider a smooth arc γ transverse to δ_s, and we study the shape of its image under $T_{1,2}^{-1}$, *i.e.*, $T_{1,2}^{-1}(\gamma)$ with the condition that $y \geq 0$. We assume that γ is deprived of the point where it crosses δ_s.

First we consider arcs $\gamma(t)$ in the positive half-plane with the following parametric form:

$$\gamma(t) : \begin{cases} x(t) = \xi_1 t + \xi_2 t^2 + \xi_3 t^3 \\ y(t) = \eta_1 t + \eta_2 t^2 + \eta_3 t^3 \end{cases}$$

where η_1, η_2, η_3 are such that $y(t) = \eta_1 t + \eta_2 t^2 + \eta_3 t^3 \geq 0$ when $t \to 0^+$ and $y(t) \geq 0$ when $t \to 0^-$. Proceeding in this fashion we can (in principle) generate the desired limit.

However, the use of a truncated power series does not effect calculations when we take $y(t)$ of degree less than 3 with respect to t, since we can have all possible cases of $\lim\limits_{t->0^+}$ $T_i^{-1}(\gamma(t))$, $i = 1, 2$:

$$\lim_{t\to 0^+} T_{1,2}^{-1}(\gamma(t)) = \lim_{t\to 0^+} (\sqrt{\eta_1 t + \eta_2 t^2 + \eta_3 t^3}, \frac{(1+a)\sqrt{\eta_1 t + \eta_2 t^2 + \eta_3 t^3} - (\xi_1 t + \xi_2 t^2 + \xi_3 t^3)}{\sqrt{\eta_1 t + \eta_2 t^2 + \eta_3 t^3}})$$

$$= (0, 1 + a) \text{ if } \eta_1 \neq 0,$$

the slope of the half-tangent of $y(t)$ in $t = 0$ is $m = \frac{\eta_1}{\xi_1}$, then we have a transversal contact with δ_s . If $\xi_1 = 0$ the slope is vertical, but the contact remains transversal.

$$\lim_{t \to 0^+} T^{-1}_{1,2}(\gamma(t)) = (0, \infty) \text{ if } \eta_1 = \eta_2 = 0 \,(m = \frac{0}{\xi_1}, \text{ then a tangential contact with } \delta_s).$$
$$= (0, 1 + a) \text{ if } \eta_1 = \xi_1 = 0 \,(m = \frac{0}{\xi_1}, \text{ with a transversal contact with } \delta_s).$$
$$= (0, 1 + a) \text{ if } \xi_1 = \eta_1 = \eta 2 = 0 \,(\text{the slope } m = \frac{0}{\xi_2},$$
$$\text{with a tangential contact with } \delta_s).$$
$$= (0, 1 + a - \frac{\xi_1}{\sqrt{\eta_2}}) \text{ if } \eta_1 = 0 \,(m = \frac{0}{\xi_1}, \text{ and a tangential contact with } \delta_s).$$
$$= (0, 1 + a + \frac{\xi_1}{\sqrt{\eta_2}}).$$

Thus varying ξ_1 and η_2, the curvature of such arcs tangent to $y = 0$ at the origin can be obtained, which satisfies the definition of prefocal sets of the inverse (see Whitehead and MacDonald (1984), Mira et al. (1996)).

This property clarifies the geometric structure of the attracting set occurring for particular choices of the parameters.

Arcs crossing tangentially through the origin have two distinct rank-one preimages crossing through δ_O, and all transverse directions tend to the single point $(0, 1 + a)$.

Besides the elements seen up to now, there is also another particularity in the dynamics of $T_{a,1}$. The second iterate of the line $y = 1 + a$ is the fixed point O, $T^2_{a,1}(y = 1 + a) = (0, 0)$, and the significance of this computation lies in the requirement that this line be the prefocal curve of $T^{-2}_{a,1}$ associated with the origin (see Figure 3).

Note that the existence of two curves, the lines $x = 0$ and $y = 1 + a$, mapped into the origin make the iteration properties of $T_{a,1}$ very different from that of maps with a unique inverse. Any arc crossing these lines twice is mapped into an arc with a loop at the origin as seen in Ferchichi and Djellit (2009). Since the origin is a saddle fixed point, this requires that these curves belong to the stable set of the point. Their role is important in understanding the homoclinic bifurcation of the point $(0, 0)$, giving rise to a unique chaotic attractor which intersects the line $y = 1 + a$ in infinitely many arcs with self-similar structure and loops issuing from the origin.

4 Broken symmetry in the modified Lorenz model

Now consider the map $T_{a,b,c}$ given in Eq.(2).

Given x' and y', if we try to solve the algebraic system with respect to the unknowns x and y, we get three solutions from (2) in which x satisfies $b^2x^3 + cbx^2 - bx(y' - (1 + ab)(1 - b))/b) - (1 - b)x' = 0$ and $y = ((1 + ab)x - x')/bx$.

The Jacobian determinant is given by $\det DT_{a,b,c}(x, y) = (1 - b)[(1 + ab) - by] + 2bx^2 + cbx$. It vanishes on the curve $LC_{-1} : y = \frac{1}{b}\ (1 + ab) + \frac{cx+2bx^2}{(1-b)}$. The noninvertibility region is an unbounded set defined by $27[(1 - b)^2x^2 + 2c^3/27b + c(y - (1 + ab)(1 - b)/b)] - 4[b(y - (1 + ab)(1 - b)/b + c/3)]^3 < 0$. This curve also possesses a pointing cusp.

When $c = 0$, the boundary that separates the basin of the two fixed points, symmetric with respect to the saddle fixed point $O = (0, 0)$, is still formed by the whole stable manifold $W^s(O)$.

When $b = 1$, for $c = 0$, $(0, 0)$ is a focal point for the two inverse determinations.

$T_1^{-1}(x, y) = ((-c + \sqrt{(c^2 + 4y)})/2, ((1 + a)(c - \sqrt{c^2 + 4y} + 2x)/(c - \sqrt{c^2 + 4y}))$

$T_2^{-1}(x, y) = ((-c - \sqrt{c^2 + 4y})/2, ((1 + a)(c + \sqrt{c^2 + 4y} + 2x)/(c + \sqrt{c^2 + 4y}))$

- For $c > 0$, $(0, 0)$ is a focal point for the inverse determination T_1^{-1}.
- For $c < 0$, $(0, 0)$ is a focal point for the inverse determination T_2^{-1}.

Figure 4a, obtained with $a = 3.00$, $b = 0.1$, and $c = 0.18$, shows the basins of the two fixed points $P_1 = (\frac{-c+\sqrt{c^2+4b^2a}}{2b}, a)$ and $P_2 = (\frac{-c-\sqrt{c^2+4b^2a}}{2b}, a)$, which are stable foci. For $c = 0.25$, P_2 becomes unstable with the appearance of a closed curve via a Neimark-Sacker bifurcation in Figure 4b, where the basins of the two fixed points P_1 and P_2 are represented by blue and orange respectively, and they are simply connected sets. The basins do not maintain the same qualitative structure, and the symmetry has broken. This asymmetry has a real effect on the local stability properties of the equilibria, and it results in an evident asymmetry in the basins of attraction. As shown in Figure 4b, when $c \neq 0$, the extension of the basin of the fixed point $B(P_2)$ located on the left side of the origin is less than the extension of $B(P_1)$. The origin O is still a saddle point without any homoclinic orbit.

We observe that small changes in the parameter b have some effects on the properties of the attractors, and they may cause remarkable asymmetries in the structure of the basins, which can only be detected from the global properties of the studied model. In Figures 4b, 4c, and Figure 5, we show that the attracting sets are quite close to their boundary, and as b is increased, the homoclinic bifurcation of the saddle point O occurs when the two attractors contact their boundaries. In Figure 5, the attracting set is uniquely tangent to LC. In Figure 5, we show in blue region the unstable manifold of the saddle point O and the points that visit the attractor and the critical curve.

For the special case $b = 1$ (see Figure 6), we find a similar situation using the two branches of the unstable manifold of the saddle point O which converge to this homoclinic closed curve. The focus point is internal to this closed structure which remains bounded and tangential to both branches of the invariant manifold at $(0, 0)$. In Figure 7, the very complicated unstable manifold has degenerated from a simple closed curve to a chaotic complex curve with loops associated with the focal point $(0, 0)$. This chaotic behavior is created by the unstable manifold.

When the parameter c is increased, the structure of the homoclinic closed curve expands gradually with loops created from the focus O around the focus P_2.

5 Conclusion

Lorenz mappings have an extraordinarily rich repertoire of behaviors, giving rise to examples of many kinds of regular and irregular features. They are smooth and noninvertible with nonconstant Jacobian. For the time-increment parameter value $b = 1$ and $c = 0$, the behavior of the solutions is mainly determined by the inverse maps having a vanishing denominator and undefined in the whole plane but only in the positive half plane with respect to y. The structure of the chaotic attractor is strongly influenced by the fact that the focus (which can also be a saddle point) is contained in the attractor. When $c \neq 0$, complex and degenerate behaviors can be seen in the phase plane.

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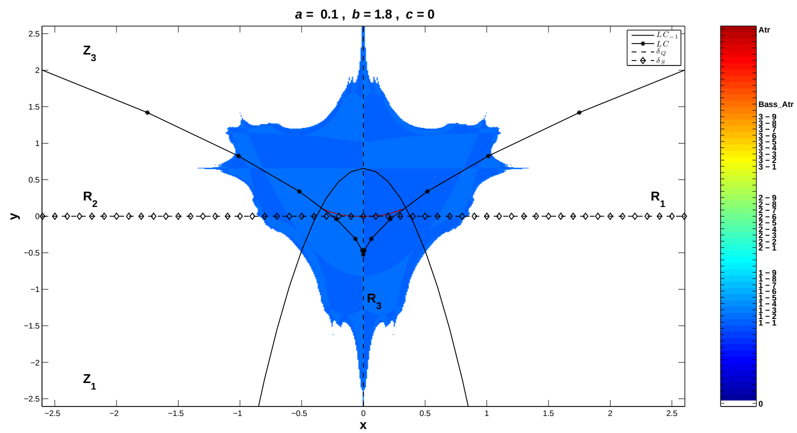

Figure 1: Critical curves of the map which is of type $Z_1 < Z_3$

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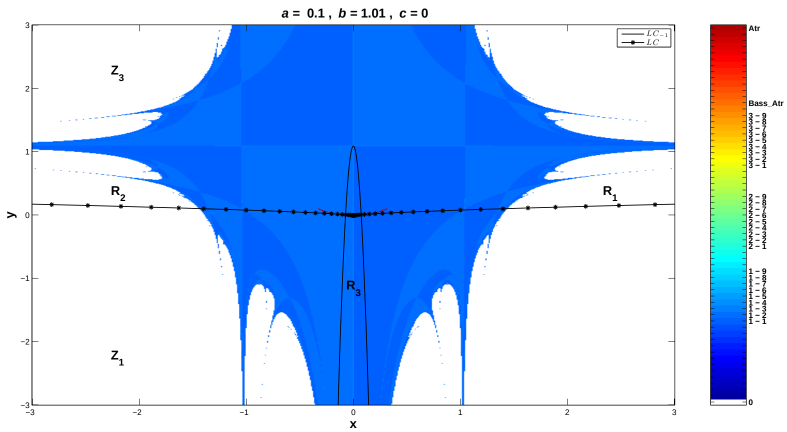


Figure 2: When the number of preimages changes, the map becomes of type $Z_0 - Z_2$

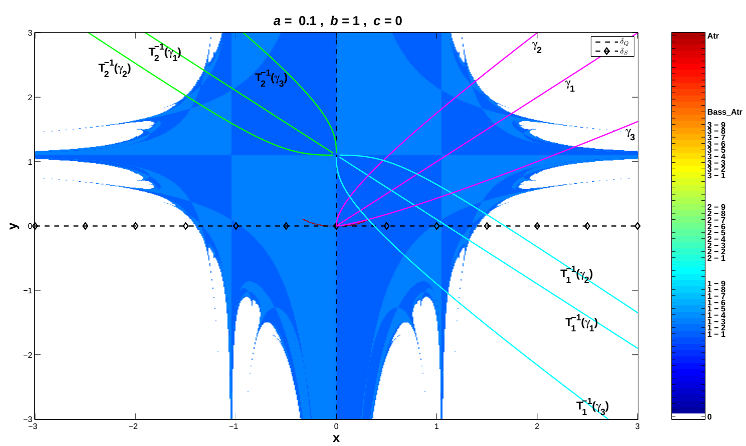
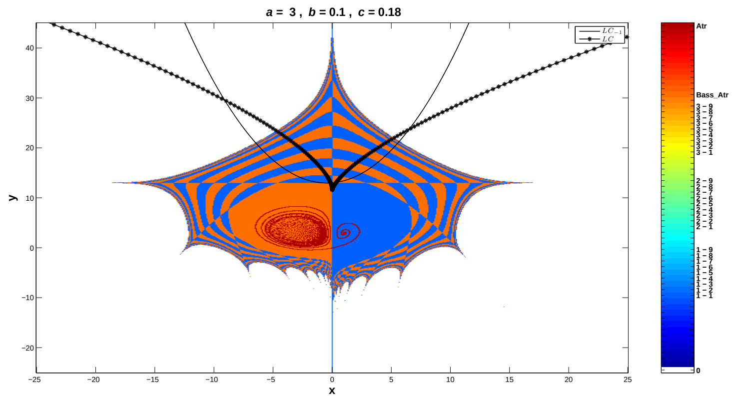
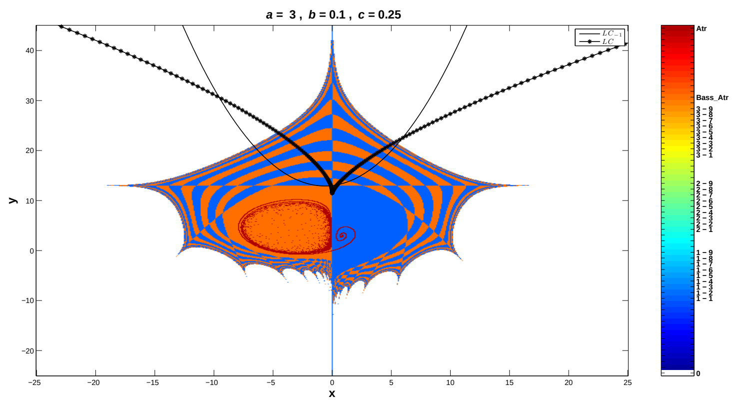


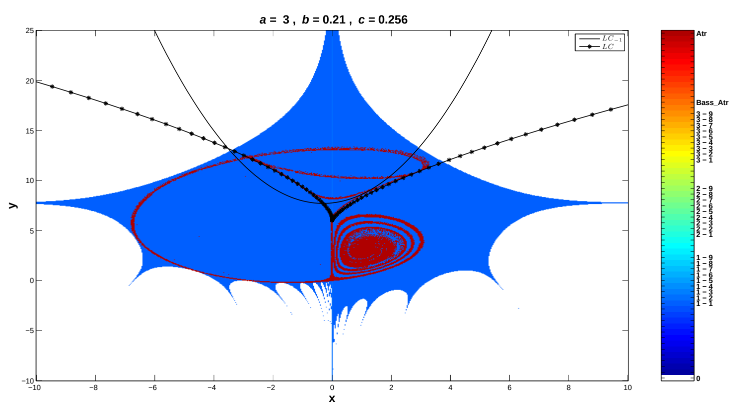
Figure 3: Preimages of arcs issuing from the origin and belonging to the positive half-plane with respect to y.



(a)



(b)



(c)

Figure 4: The blue region and orange points denote the basins of P_1 and P_2 for the map $T_{a,b,c}$

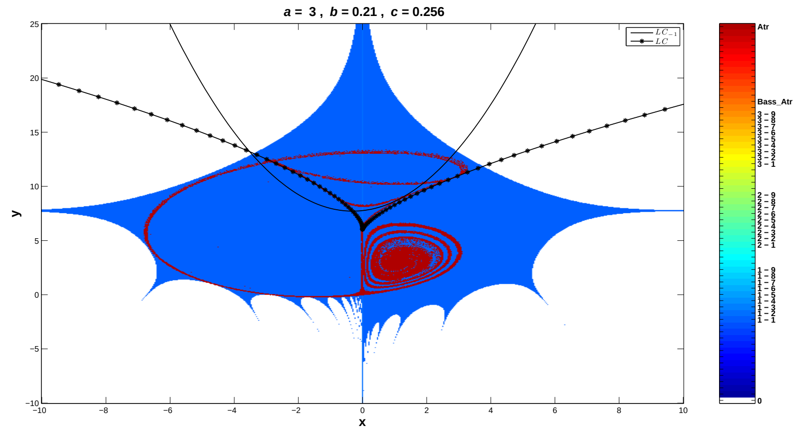


Figure 5: The blue and orange points denote the basins of P_1 and P_2 for the map $T_{a,b,c}$

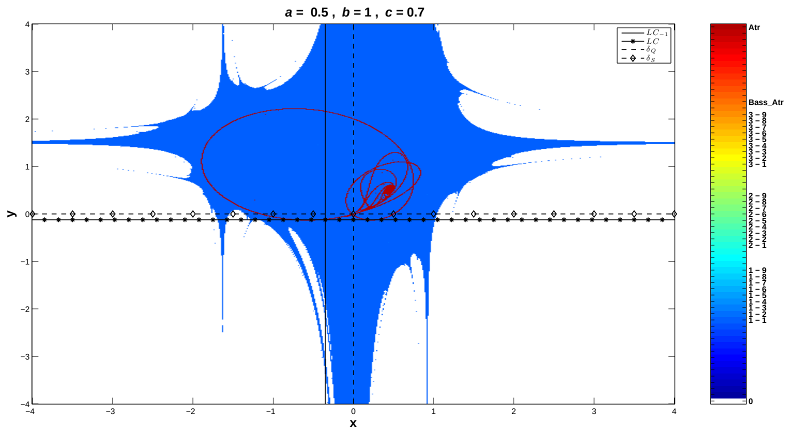


Figure 6: The homoclinic bifurcation and closed invariant curve around the focus P_2

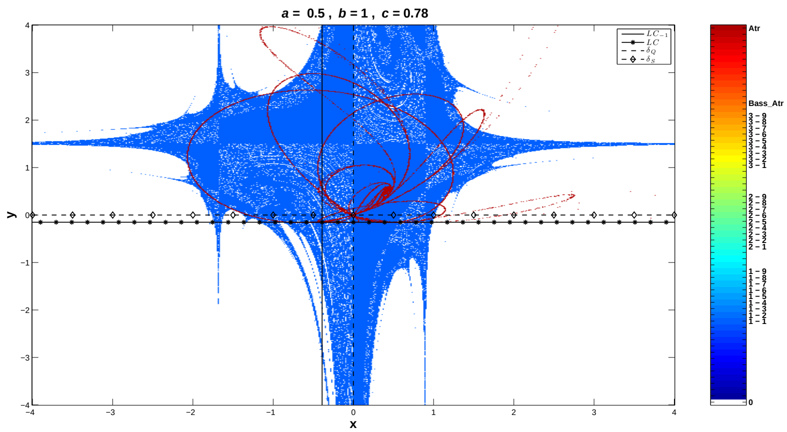


Figure 7: The homoclinic closed curve and loops issuing from the origin