Comment on "How to obtain extreme multistability in coupled dynamical systems"

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We note that extreme multistability of the type described in the referenced paper can be achieved in virtually any dynamical system by adding extraneous variables and using their initial conditions in place of the existing parameters or as additional parameters. We show several simple examples of this and show how the referenced examples are similar.

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In a recent paper, Hens *et al.* [1] described two examples of coupled oscillators in which the initial conditions appear to play the role of bifurcation parameters. We contend that the reason for this behavior is that the resulting system collapses to one of lower dimension in which the initial conditions of the extraneous equations become parameters in the remaining equations.

It is well known that certain nonautonomous dynamical systems can be made autonomous by introducing additional variables. For example, the damped, driven Ueda oscillator [2] given by

$$\dot{x} = y, \quad \dot{y} = -y - x^3 + A \cos \omega t \tag{1}$$

can be transformed to a four-dimensional autonomous system by setting $u = A \cos \omega t$ to give

$$\dot{x} = y, \quad \dot{y} = -y - x^3 + u, \quad \dot{u} = v, \quad \dot{v} = -\omega^2 u, \tag{2}$$

where the parameter A now appears in the initial conditions of $u_0 = A$ and $v_0 = 0$. The last two equations in (2) are simply a harmonic oscillator that generates a sine wave of amplitude A and frequency ω.

An even more trivial example is the diffusionless Lorenz system [3]

$$\dot{x} = y - x, \quad \dot{y} = -xz, \quad \dot{z} = xy - R, \tag{3}$$

in which the substitution $u = R$ gives

$$\dot{x} = y - x, \quad \dot{y} = -xz, \quad \dot{z} = xy - u, \quad \dot{u} = 0, \tag{4}$$

with the initial condition $u_0 = R$. Such a simple transformation can be used to remove any parameter from a dynamical system and replace it by the initial condition of an added variable whose derivative is zero.

This example can be made slightly less trivial using a substitution such as $u = R + x$ in (3) to give

$$\dot{x} = y - x, \quad \dot{y} = -xz, \quad \dot{z} = xy - u + x, \quad \dot{u} = y - x, \tag{5}$$

with the initial condition $u_0 = x_0 + R$. Here it is obvious that $\dot{u} = \dot{x}$ and thus $u(t)$ tracks $x(t)$ to within a constant given by the difference in the initial conditions u_0 and x_0.

More complicated substitutions can lead to systems where the nonuniqueness of the equations is much less apparent. For example, take $u = xy - R$ in (3) to give

$$\dot{x} = y - x, \quad \dot{y} = -xz, \quad \dot{z} = u, \quad \dot{u} = y^2 - xy - x^2 z, \tag{6}$$

with the initial condition $u_0 = x_0 y_0 - R$. The parameter R has been absorbed into the initial conditions, and it is nearly impossible to tell that the apparent four-dimensional system is really a three-dimensional system in disguise with an extraneous fourth equation nonlinearly related to the other three.

As a final example, consider a variant of Eq. (3) given by

$$\dot{x} = y - x, \quad \dot{y} = -xz, \quad \dot{z} = xy - u, \quad \dot{u} = v, \quad \dot{v} = -v. \tag{7}$$

This system is four dimensional as one can verify by calculating the rank of its Jacobian matrix or by counting the number of nonidentical Lyapunov exponents, but it collapses to a three-dimensional system after the initial transient has decayed because v asymptotically approaches zero and u approaches a constant given by $R = u_0 + v_0$.

In the referenced paper [1], the authors consider two coupled Rössler systems [4] that are synchronized such that all three of their variables track one another to within a constant that depends on the initial conditions. In simplified form, their equations can be written as

$$\begin{aligned} &\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1 + a(x_2 - e_2), \\ &\dot{x}_3 = b - cx_3 + x_1 x_3, \quad \dot{e}_1 = -e_1, \\ &\dot{e}_2 = e_1, \quad \dot{e}_3 = -ce_3, \end{aligned} \tag{8}$$

where $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, and $e_3 = x_3 - y_3$. The y_i variables refer to the second Rössler system that becomes synchronized to the corresponding x_i variables after the initial transient has decayed and thus no longer needs to be considered. From the $\dot{e}_1$ equation of (8), it is clear that $e_1 = 0$ after the initial transient has decayed, and thus $\dot{e}_2 = 0$ and e_2 is a constant. Therefore, what the authors have effectively done is to introduce into the Rössler system an additional constant term ae_2 in the $\dot{x}_2$ equation and used the initial conditions of the extraneous $\dot{e}_2$ equation to determine its value. Note that system (8) is a master-slave system similar to system (2) in which the first half of the equations depends on the second half, but the second half is independent of the first half.

The authors' second autocatalator [5] example can be written as

$$\dot{x}_1 = \mu(\kappa + x_3) - x_1\left(1 + x_2^2\right),$$
$$\sigma\dot{x}_2 = (x_1 - e_1)\left(1 + x_2^2\right) - x_2, \tag{9}$$
$$\delta\dot{x}_3 = x_2 - x_3, \quad \dot{e}_1 = \mu e_3, \quad \dot{e}_2 = -e_2, \quad \delta\dot{e}_3 = -e_3,$$

where $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, and $e_3 = x_3 - y_3$. The y_i variables refer to the second autocatalator model that becomes synchronized to the corresponding x_i variables after the initial transient has decayed and thus no longer needs to be considered. From the $\dot{e}_2$ and $\dot{e}_3$ equations of (8), it is clear that $e_2 = 0$ and $e_3 = 0$ after the initial transient has decayed. Thus $\dot{e}_1$ is zero, which means that e_1 is a constant that depends on the initial conditions. Therefore, the extraneous $\dot{e}_1$ equation is being used simply to determine the value of the parameter e_1 in the $\dot{x}_2$ equation. System (9) is also a master-slave system similar to system (2) in which the first half of the equations depends on the second half, but the second half is independent of the first half.

Thus we contend that any dynamical system with an attractor can be made to exhibit extreme multistability by replacing one of its bifurcation parameters by an additional variable whose value is constant or that approaches a constant in the asymptotic limit $t \to \infty$ and whose value is determined by the initial conditions. However, the converse is not true. Most high-dimensional systems cannot be reduced to a lower-dimensional system with parameters determined by the initial conditions except in special situations such as two identical systems with carefully chosen coupling for which several examples have now been provided [1,6–8]. This leaves the false impression that extreme multistability is unusual and difficult to achieve, and that it is somehow special or surprising.

In studying dynamical systems, it is important to verify that all the equations are independent and contribute to the dynamics. The presence of extraneous equations and their apparent additional dimensions can lead to false conclusions such as the claim that the initial conditions are bifurcation parameters or that a system has infinitely many equilibria [9]. However, what might be viewed as a flaw in Ref. [1] may actually be a virtue since it suggests that plotting bifurcation diagrams versus initial conditions is a useful means for identifying extraneous equations in models of dynamical systems.

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