International Journal of Bifurcation and Chaos, Vol. 25, No. 10 (2015) 1530025 (14 pages)

DOI: 10.1142/S0218127415300256

Constructing Chaotic Systems with Total Amplitude Control

Chunbiao Li*
School of Electronic and Information Engineering, Nanjing University of Information Science and Technology, Nanjing 210044, P. R. China
Engineering Technology Research and Development Center of Jiangsu, Circulation Modernization Sensor Network, Jiangsu Institute of Commerce, Nanjing 211168, P. R. China
goontry@126.com

Julien Clinton Sprott
Department of Physics, University of Wisconsin – Madison, Madison, WI 53706, USA
sprott@physics.wisc.edu

Zeshi Yuan[†] and Hongtao Li[‡]
School of Electronic and Optical Engineering, Nanjing University of Science and Technology, Nanjing 210094, P. R. China
[†]yzs644613931@gmail.com
[‡]floodlee@126.com

Received February 27, 2015

A general method is introduced for controlling the amplitude of the variables in chaotic systems by modifying the degree of one or more of the terms in the governing equations. The method is applied to the Sprott B system as an example to show its flexibility and generality. The method may introduce infinite lines of equilibrium points, which influence the dynamics in the neighborhood of the equilibria and reorganize the basins of attraction, altering the multistability. However, the isolated equilibrium points of the original system and their stability are retained with their basic properties. Electrical circuit implementation shows the convenience of amplitude control, and the resulting oscillations agree well with results from simulation.

Keywords: Amplitude control; Sprott B system; line equilibrium points; multistability.

1. Introduction

Amplitude control of chaotic signals is important for engineering applications since appropriate amplitude must be obtained for generation and transmission of the signals [Li *et al.*, 2005; Wang *et al.*, 2010; Yu *et al.*, 2008; Li & Sprott, 2013; Li *et al.*, 2014b, 2015]. Once a chaotic system is designed, an additional linear transformation is usually necessary to obtain the desired size of the attractor so that the amplitude does not exceed the limitation of the circuit elements. The broadband nature of chaotic signals makes it difficult to design a linear amplifier. Moreover, if the amplitude of the variables requires further adjusting, several

*Author for correspondence

parameters or coefficients in the system may need to be controlled to avoid altering the chaotic nature of the signal. An independent amplitude control parameter is thus desired that preserves the Lyapunov exponent spectrum [Li & Wang, 2009; Li *et al.*, 2012; Li *et al.*, 2015] or that proportionally rescales the exponents [Li & Sprott, 2014a] without introducing new bifurcations except perhaps caused by using inappropriate initial conditions when the system is multistable [Li & Sprott, 2014a].

Some chaotic systems have an amplitude control parameter when all the terms in the governing equations are monomials of the same degree except for one whose coefficient then determines the amplitude of all the variables [Li & Sprott, 2014a, 2014b; Li & Wang, 2009; Li *et al.*, 2012]. However, most chaotic systems are not of that form, but they can usually be modified to achieve the goal. As pointed out [Li *et al.*, 2015], the chaos in dynamic systems usually survives when the amplitude information in some of the variables is removed, or when additional amplitude information is added. Therefore, the signum function and absolute-value function can be applied to decrease or increase the degree of the terms in the equations so as to isolate a single term of different degree whose coefficient then provides the desired amplitude control. However, the degree modification may yield infinite lines of equilibrium points, which in turn will reorganize the basins of attraction for multistability, which can cause difficulty unless the initial conditions are appropriately rescaled.

In this paper, we illustrate the approach of providing a total amplitude control parameter in a chaotic system by degree modification using the signum and absolute-value functions. In Sec. 2, the degree modification is applied to the Sprott B system, showing both degree-decreasing with the signum function and degree-increasing with the absolute-value function. The amplitude control with different scales is analyzed. In Sec. 3, the properties of the modified Sprott B systems are explored, including equilibria and multistability. Electronic circuit implementation is presented in Sec. 4. Conclusions and discussions are given in the last section.

2. Degree Modification for Amplitude Control

Here we select the Sprott B system as an example because this system is simple and has two quadratic terms, two linear terms, and one constant term, which provides relatively comprehensive cases for demonstrating the degree modification for amplitude control. The Sprott B system [Sprott, 1994, 2010] is given as

$$\dot{x} = yz, \quad \dot{y} = x - y, \quad \dot{z} = a - xy. \tag{1}$$

This system has rotational symmetry with respect to the z-axis as evidenced by its invariance under the coordinate transformation $(x, y, z) \to (-x, -y, z)$, and it has a partial amplitude control parameter hidden in the coefficient of the xy term, which controls the amplitude of x and y, but not z. The parameter a is a constant, and when $a = 1$, the corresponding strange attractor is as shown in Fig. 1.

There are at least two methods to make all but one of the terms the same degree and thereby achieve total amplitude control. One is to unify them to be first degree (linear), and the other is to unify them to be second degree (quadratic). The signum operation retains the polarity information while removing the amplitude information. Applying it to one of the factors in the quadratic terms reduces the degree of those terms from 2 to 1. Consequently, the remaining constant term gives total amplitude control because it is the only term with degree different from unity.

There are four methods of linearization as follows:

$$\begin{aligned} \dot{x} &= z\,\mathrm{sgn}(y), \\ \dot{y} &= x - y, \\ \dot{z} &= m - ay\,\mathrm{sgn}(x), \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{x} &= z\,\mathrm{sgn}(y), \\ \dot{y} &= x - y, \\ \dot{z} &= m - ax\,\mathrm{sgn}(y), \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{x} &= y\,\mathrm{sgn}(z), \\ \dot{y} &= x - y \\ \dot{z} &= m - ay\,\mathrm{sgn}(x), \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{x} &= y\,\mathrm{sgn}(z), \\ \dot{y} &= x - y, \\ \dot{z} &= m - ax\,\mathrm{sgn}(y). \end{aligned} \tag{5}$$

Two parameters a and m represent the bifurcation parameter and the amplitude parameter, either of which leads the system to different dynamics

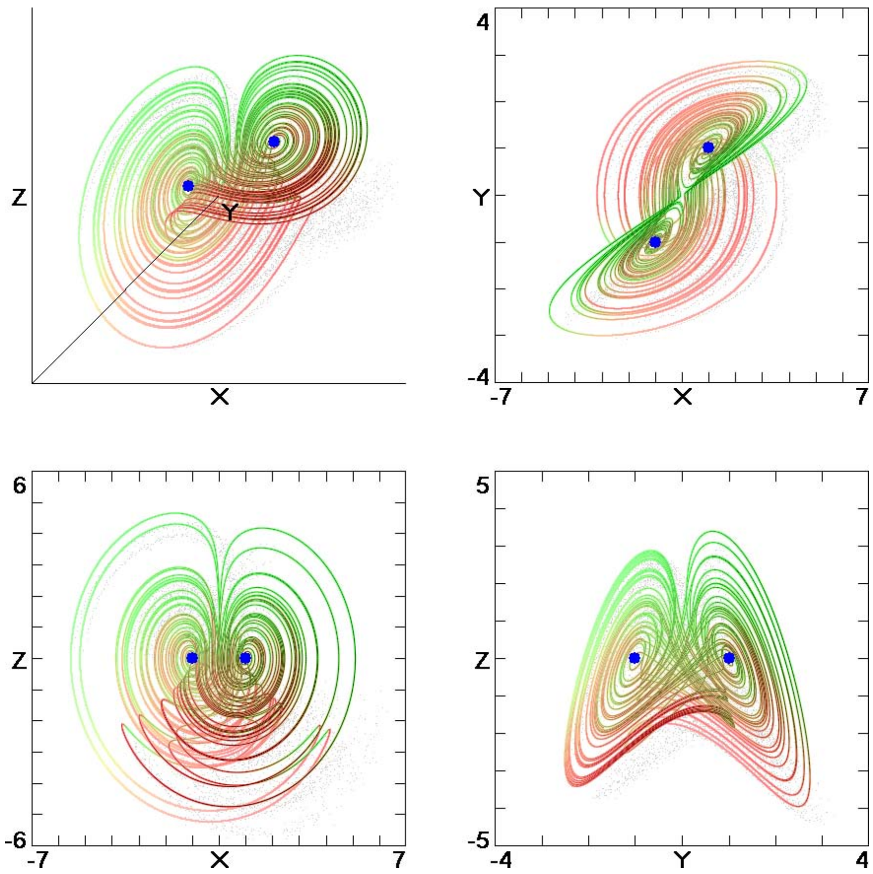


Fig. 1. Four views of the strange attractor in Eq. (1) for $a = 1$ with initial conditions $(1, 0, -1)$ and LEs $= (0.2101, 0, -1.2101)$. The colors indicate the value of the local largest Lyapunov exponent with positive values in green and negative values in red. Two equilibrium points are shown as blue dots.

or only determines the size of the attractor correspondingly. To understand the connections in the above systems, we consider that there are two kinds of coupling for each variable, one associated with its amplitude information and the other with its polarity information. Using a solid line or a dotted line with arrow to represent respectively the amplitude and the polarity of each variable that influences the derivative of another variable, leads to the structures shown in Fig. 2 describing the above four systems. In the original Sprott B system, there are twelve connections, six of which are amplitude coupling, and the other six are polarity coupling. After the linearization based on two signum operations, there are ten connections, four of which are amplitude coupling, and the remaining six are polarity coupling, as shown in Fig. 2. We see that there are pure polarity connections between two variables, marked as the dotted line in Figs. 2(b)–2(d), which indicates that the systems (3)–(5) risk

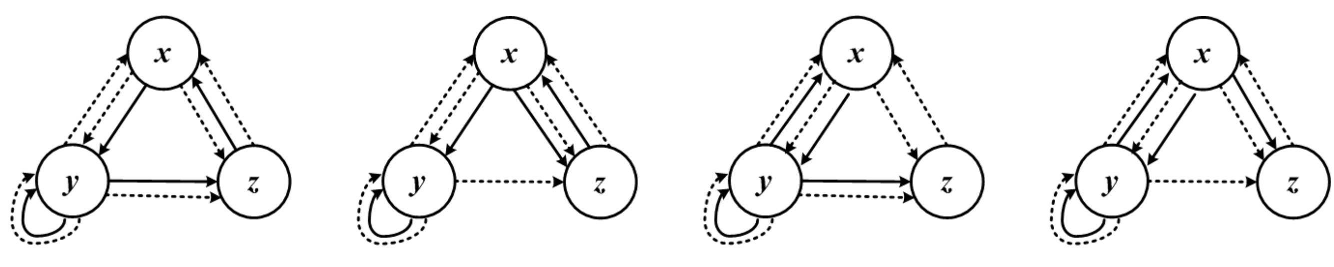


Fig. 2. The amplitude and polarity connections in the network structures: (a) system (2), (b) system (3), (c) system (4) and (d) system (5).

losing chaos. System (3) fails to give chaos because its amplitude link with the variable y is destroyed. As shown in Fig. 2(c) and 2(d), the amplitude connection between the variables y and z is preserved by an indirect coupling, which plays an important role in the topology of the Sprott B system. The rest of the polarity feedback from the variable z into the variable x shows that the first two dimensions more likely oscillate as an independent two-variable driving system according to the polarity of the variable z and correspondingly the systems (4) and (5) can hardly survive chaos but gives a solution with a very different look from the regular Sprott B. The structure (a) of system (2) most

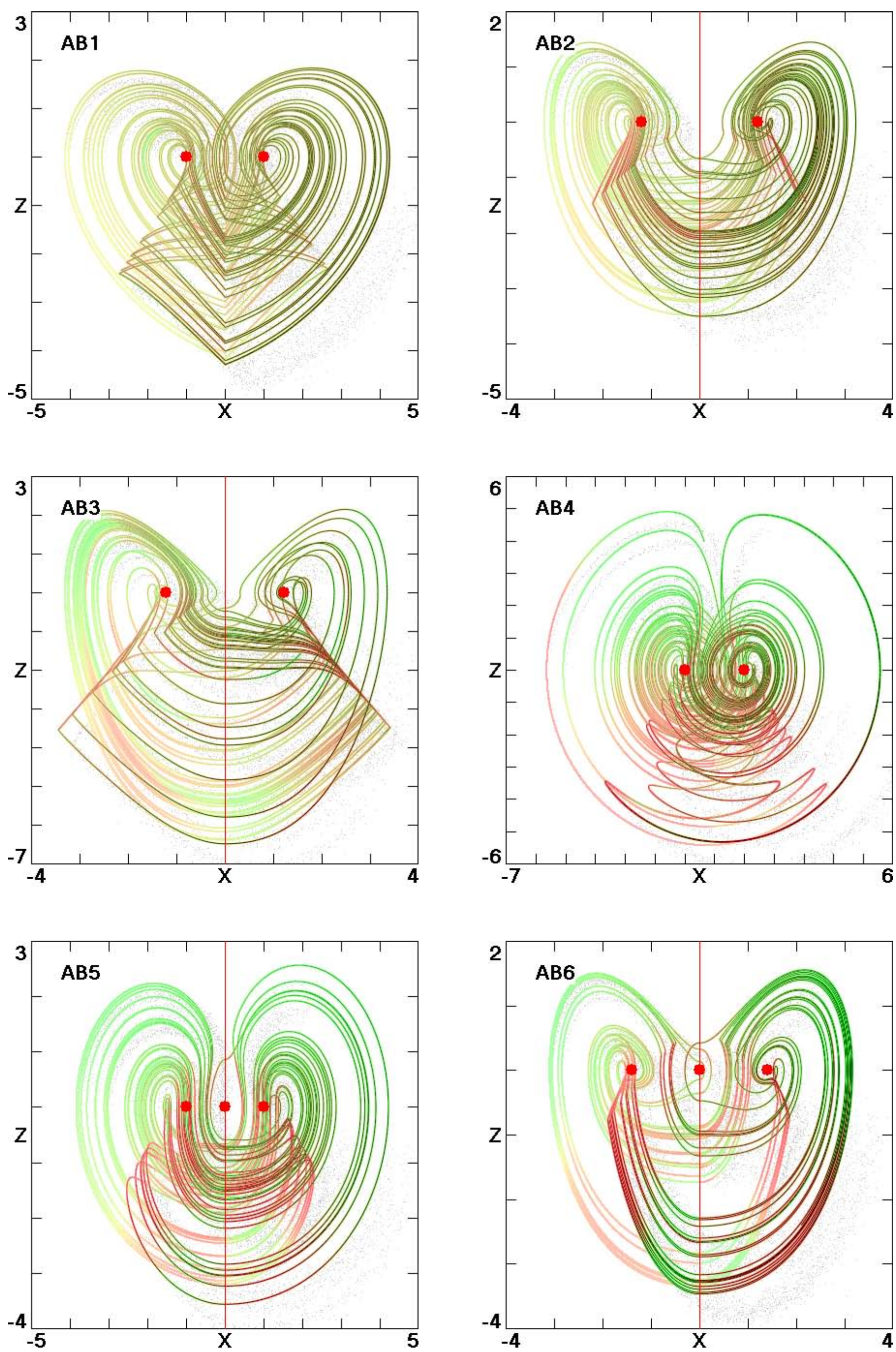


Fig. 3. Strange attractors observed from amplitude-controllable Sprott B systems in Table 1 for $m = 1$ projected onto the x–z phase plane. The colors indicate the value of the local largest Lyapunov exponent with positive values in green and negative values in red, the equilibrium points are shown in red.

Table 1. Amplitude-controllable Sprott B systems.

Models	Equations	Parameters	Equilibria	Eigenvalues	LEs	D_{KY}
AB0	$\begin{cases}\dot{x} = yz,\\ \dot{y} = x - y,\\ \dot{z} = a - mxy,\end{cases}$	$a = 1$ $m = 1$	$\sqrt{\frac{a}{m}}, \sqrt{\frac{a}{m}}, 0$ $-\sqrt{\frac{a}{m}}, -\sqrt{\frac{a}{m}}, 0$	$(-1.3532, 0.1766 \pm 1.2028i)$ $(-1.3532, 0.1766 \pm 1.2028i)$	0.2101 0 -1.2101	2.1736
AB1	$\begin{cases}\dot{x} = z\,\mathrm{sgn}(y)\\ \dot{y} = x - y\\ \dot{z} = m - ay\,\mathrm{sgn}(x)\end{cases}$	$a = 1$ $m = 1$	$\frac{m}{a}, \frac{m}{a}, 0$ $-\frac{m}{a}, -\frac{m}{a}, 0$	$(-1.4656, 0.2328 \pm 0.7926i)$ $(-1.4656, 0.2328 \pm 0.7926i)$	0.1191 0 -1.1191	2.1065
AB2	$\begin{cases}\dot{x} = z\,\mathrm{sgn}(y)\\ \dot{y} = x - y\\ \dot{z} = a\lvert x\rvert - mxy\end{cases}$	$a = 1.3$ $m = 1$	$\frac{a}{m}, \frac{a}{m}, 0$ $0, 0, z$ $-\frac{a}{m}, -\frac{a}{m}, 0$	$(-1.5448, 0.2724 \pm 0.8760i)$ $0, 0, -1$ $(-1.5448, 0.2724 \pm 0.8760i)$	0.0906 0 -1.0906	2.0831
AB3	$\begin{cases}\dot{x} = z\,\mathrm{sgn}(y)\\ \dot{y} = x - y\\ \dot{z} = a\lvert x\rvert - mxy\lvert y\rvert\end{cases}$	$a = 1.5$ $m = 1$	$\sqrt{\frac{a}{m}}, \sqrt{\frac{a}{m}}, 0$ $0, 0, z$ $-\sqrt{\frac{a}{m}}, -\sqrt{\frac{a}{m}}, 0$	$(-1.8637, 0.4319 \pm 1.1930i)$ $0, 0, -1$ $(-1.8637, 0.4319 \pm 1.1930i)$	0.1486 0 -1.1486	2.1294
AB4	$\begin{cases}\dot{x} = yz\\ \dot{y} = x\lvert x\rvert - y\lvert x\rvert\\ \dot{z} = m - axy\end{cases}$	$a = 1$ $m = 1$	$\sqrt{\frac{m}{a}}, \sqrt{\frac{m}{a}}, 0$ $-\sqrt{\frac{m}{a}}, -\sqrt{\frac{m}{a}}, 0$	$(-1.3532, 0.1766 \pm 1.2028i)$ $(-1.3532, 0.1766 \pm 1.2028i)$	0.2355 0 -1.2677	2.1858
AB5	$\begin{cases}\dot{x} = yz\\ \dot{y} = x\lvert x\rvert - y\lvert x\rvert\\ \dot{z} = m\lvert x\rvert - axy\end{cases}$	$a = 1$ $m = 1$	$\frac{m}{a}, \frac{m}{a}, 0$ $0, y, 0$ $0, 0, z$ $-\frac{m}{a}, -\frac{m}{a}, 0$	$(-1.4656, 0.2328 \pm 0.7926i)$ $(\sqrt{a}\lvert y\rvert, 0, -\sqrt{a}\lvert y\rvert)$ $0, 0, 0$ $(-1.4656, 0.2328 \pm 0.7926i)$	0.0993 0 -1.1783	2.0843
AB6	$\begin{cases}\dot{x} = yz\\ \dot{y} = x\lvert x\rvert - y\lvert x\rvert\\ \dot{z} = a\lvert xy\rvert - mxy\lvert y\rvert\end{cases}$	$a = 1.4$ $m = 1$	$\frac{a}{m}, \frac{a}{m}, 0$ $0, y, 0$ $0, 0, z$ $-\frac{a}{m}, -\frac{a}{m}, 0$	$(-2.1964, 0.3982 \pm 1.2612i)$ $(\sqrt{m\lvert y\rvert}\lvert y\rvert i, 0, -\sqrt{m\lvert y\rvert}\lvert y\rvert i$ $0, 0, 0$ $(-2.1964, 0.3982 \pm 1.2612i)$	0.1006 0 -1.2861	2.0782

closely matches the original topological structure with well-balanced coupling of amplitude and polarity and therefore gives a chaotic attractor as shown in Fig. 3 (Case AB1) even without any revision of the parameters. It resembles the usual Sprott B attractor in Fig. 1 except for discontinuities in the direction of the flow.

Alternately, we can modify system (1) so that the constant term and one of the quadratic terms has degree-1 so that the coefficient of the remaining quadratic term becomes an amplitude parameter. Since the chaotic solution is bounded, it is reasonable to increase the degree of the constant by multiplying it by the absolute-value of one of the variables leading to system AB2. For the same reason as in the complete linearization, some modifications of the Sprott B system do not give chaos since the network dimension collapses from 3 to 2 because one of the variables is not part of any feedback loop. The sole nonlinear term need not be quadratic, and system AB3 shows the case where it is cubic. The properties of these modified Sprott B systems are listed in Table 1, the Lyapunov Exponents (LEs) and the Kaplan–Yorke Dimension (D_{KY}) are also included.

Furthermore, we can modify all but one of the terms to be quadratic. Since the normal Sprott B system has two quadratic terms, two linear terms, and one constant term, it is necessary to increase the degree of two of the three nonquadratic terms. By introducing absolute-value functions, a new amplitude-controllable Sprott B system (AB4) with all quadratic terms except one is derived, where the linear terms in the second dimension are multiplied by an absolute-value of the variable x, and the remaining constant term is an amplitude parameter. From the rotational symmetry of the system, it is reasonable to apply an absolute-value to the variable y to get a degree increase. The only nonquadratic term can be degree-1 or any degree higher than 2. Table 1 gives some other cases with the term of first degree (AB5) and third degree (AB6), respectively. Comparing with the first three cases in Table 1, instead of the signum function ignoring the amplitude of the variables, here absolute-value functions are used to modify the amplitude of the terms giving a higher degree. From Table 1 and the corresponding attractors in Fig. 3, we see that the isolated equilibrium points and their stability and the main structure of the phase trajectory are preserved in these modified systems. The similarity of all the attractors for the modified systems is evidence that the essential dynamics are retained despite the appearance of additional equilibria. Since all of these systems have five terms, there is a single bifurcation parameter taken as a, and by design a single amplitude parameter taken as m, both put arbitrarily into the $\dot{z}$ equation.

3. Analysis of the Amplitude Scaling

Generally, autonomous chaotic flows include some positive feedback to compensate for the dissipation. Each variable has three essential factors: amplitude, phase, and frequency. Therefore, any change in the variables will result in some possible alteration of the flow dynamics. Since the symmetric structure of the system depends more on the polarity information of the variables, modification of the amplitude information will not usually destroy the fundamental structure. Namely, the structure of rotational symmetry with respect to the z-axis will be preserved by its invariance under the coordinate transformation $(x, y, z) \rightarrow (-x, -y, z)$, $(\frac{x}{|x|}, \frac{y}{|y|}, z) \rightarrow (-\frac{x}{|x|}, -\frac{y}{|y|}, z)$ and $(x|x|, y|y|, z) \rightarrow (-x|x|, -y|y|, z)$. That is to say, the signum operation removes the amplitude information while retaining the phase information, and the absolute-value function adds only amplitude information without changing the phase information. Therefore, these operations can preserve the chaos after degree modification when the parameters are reassembled for rescaling the size of the attractor or controlling its bifurcations.

Because of the boundedness of the variables, a chaotic system will reach a new balance of amplitude after the signum or absolute-value operation. Consequently, its coefficient will control the amplitude without changing the sign of the Lyapunov exponents provided one remains in the basin of attraction [Li *et al.*, 2015]. If the coefficient controls both the amplitude and frequency, the Lyapunov exponents will change in magnitude as a result of the time rescaling [Li & Sprott, 2014a].

Specifically, the unified degree in the systems with all but one term of degree-1 gives only amplitude control of the variables. The constant m in AB1 controls the amplitude of all three variables in proportion to m, while the coefficient m of the only quadratic term in AB2 controls the amplitude of all three variables in proportion to $\frac{1}{m}$. The coefficient

m of the only cubic term in AB3 controls the amplitude of all three variables in proportion to $\frac{1}{\sqrt{m}}$. The corresponding coefficient also rescales the coordinates of the equilibrium points as shown in Table 1.

However, the unified degree in the systems with all terms quadratic except for one will introduce both amplitude and frequency control of the variables [Li & Sprott, 2014a]. The constant term m in AB4 scales the amplitude and frequency according to $\sqrt{m}$, while the coefficient m of the only linear term in AB5 controls the amplitude and frequency according to m. The coefficient m of the only cubic term in AB6 controls the amplitude and frequency in proportion to $\frac{1}{m}$. As an illustration, consider a simultaneous amplitude and frequency control of AB6 by the transformation $x \to mx, y \to my, z \to mz, t \to \frac{t}{m}$. Then the equations of AB6 transform back to: $\dot{x} = yz, \dot{y} = x|x| - y|x|, \dot{z} = a|xy| - xy|y|$, which means that the coefficient m of the cubic term $xy|y|$ in the $\dot{z}$ equation can rescale the amplitude and frequency according to $\frac{1}{m}$.

4. Equilibria and Multistability

The equilibrium points play an important role in the degree modification. As shown in Table 1, the amplitude control can be identified also from those retained isolated equilibrium points whose coordinates are revised proportionally with the amplitude parameters, but the stability of each isolated equilibrium point is preserved. In other words, if the degree modification introduces additional isolated equilibrium points or changes the stability of the equilibria, amplitude modification may not occur because the revised system will not retain its chaos or will give a strange attractor with a different manifold.

On the other hand, the additional absolute-value terms will usually yield additional lines of equilibrium points [Li *et al.*, 2014c; Jafari & Sprott, 2013] when the constant term disappears and the rank of the Jacobian matrix at some of these lines is not full. New introduced redundant absolute-value terms may have a common factor with the linear or

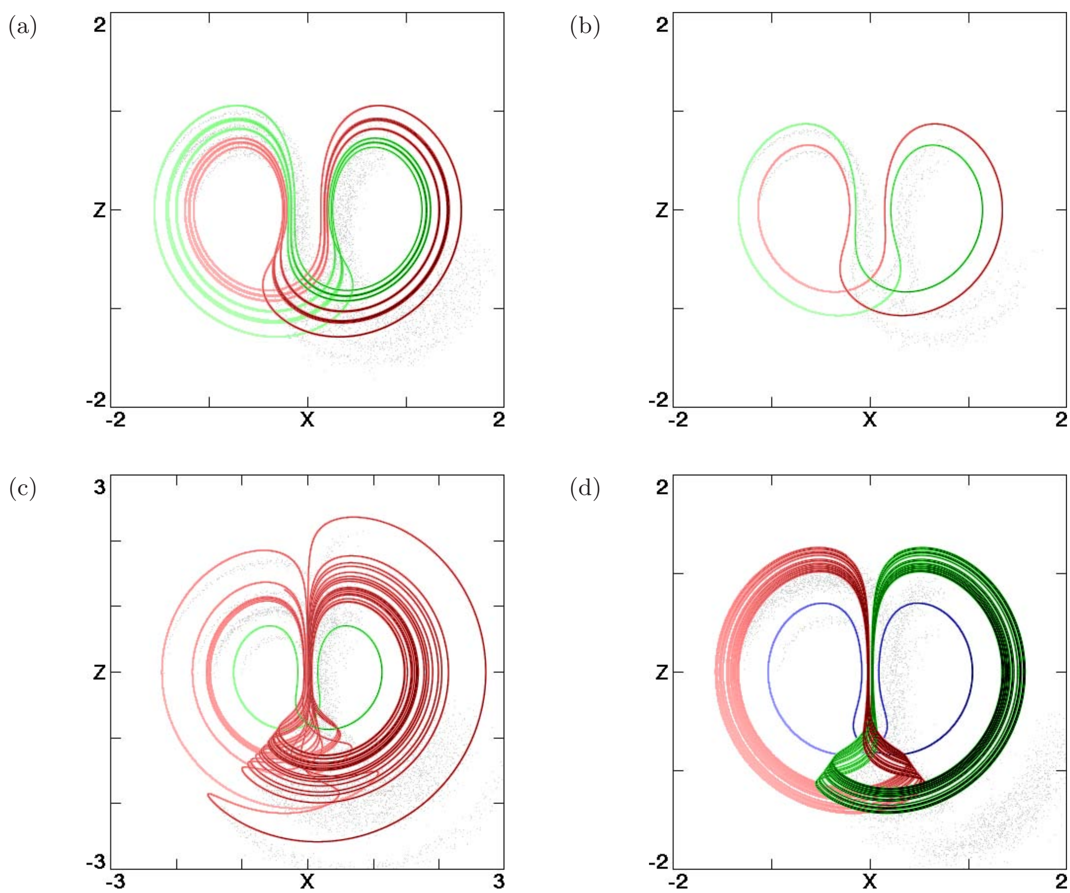


Fig. 4. Phase portrait of coexisting attractors in the x–z plane: (a) $a = 0.32$ with initial conditions $(\pm 0.58, \pm 1, -1)$, (b) $a = 0.28$ with initial conditions $(\pm 0.15, \pm 0.11, 0.10)$, (c) $a = 0.23$ with initial conditions $(0.15, 0.14, 0)$ for green and $(0.3, 0.3, -1)$ for red and (d) $a = 0.16$, initial conditions $(0.98, 0.72, 0.3)$ for blue and symmetric initial conditions $(\pm 0.45, 0, -1)$ for green and red.

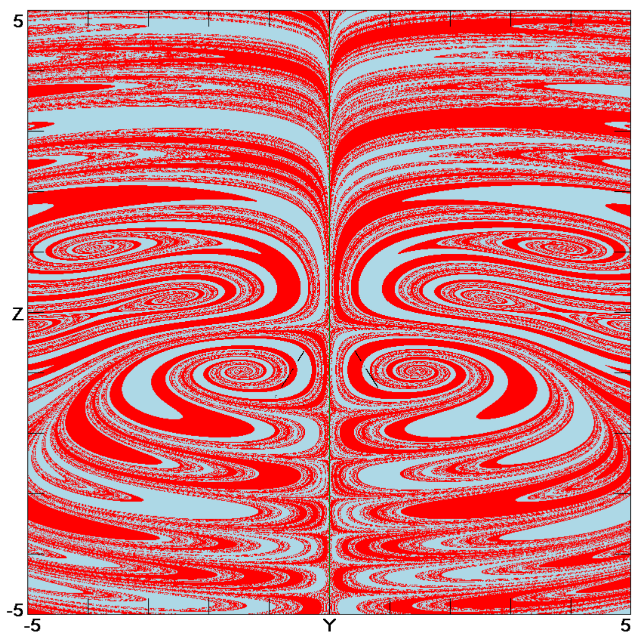


Fig. 5. Cross-section for $x = 0$ of the basins of attraction for the symmetric pair of strange attractors (light blue and red) for AB0 with $a = 0.32$.

nonlinear terms, and therefore give birth to a line or to two perpendicular lines of equilibrium points. Generally, the new introduced lines of equilibria are "safe" for the systems and are usually unstable or neutrally stable. As shown in Table 1, there is one line of equilibria in systems AB2 and AB3, and there are two perpendicular lines of equilibria in AB5 and AB6. The eigenvalues of the single line of equilibrium points in systems AB2 and AB3 are $(0, 0, -1)$ showing that the line is stable in one direction, while the eigenvalues of the two perpendicular line equilibrium points in system AB5 are $(0, 0, 0)$ for the line $(0, 0, z)$ and $(\sqrt{a}|y|, 0, -\sqrt{a}|y|)$ for the line $(0, y, 0)$, indicating that one of the lines is neutrally stable, while the other line contains unstable saddle nodes. The new lines of equilibrium points will influence the dynamics along with the alteration of the form of the nonlinearities, the effect of which can be partially observed by the rearrangement of the basins of attraction in multistable systems. The Sprott B system and its diverse modified

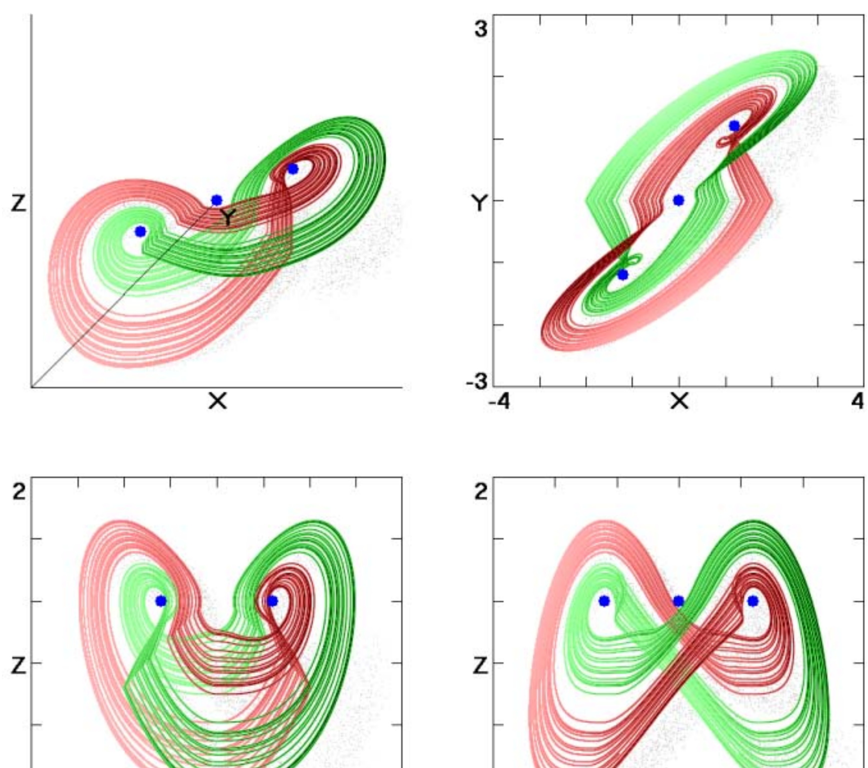


Fig. 6. Symmetric pair of strange attractors for $(a, m) = (1.2, 1)$ with initial conditions $(x_0, y_0, z_0) = (0, \pm 1, 1)$ in AB2. The green and red attractors correspond to two symmetric initial conditions, and the equilibrium points are shown as blue dots.

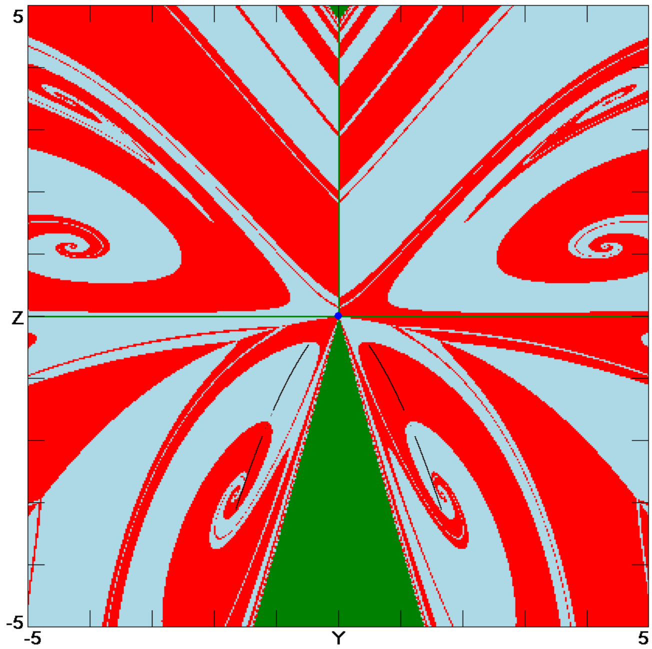


Fig. 7. Cross-section for $x = 0$ of the basins of attraction for the symmetric pair of strange attractors (light blue and red) for AB2 with $a = 1.2, m = 1$.

versions provide a good candidate for observing disturbed multistability.

The Sprott B system was selected as an example for amplitude control precisely because of its symmetric structure and resulting multistability. The Sprott B system has four regimes of multistability for appropriate choice of the parameters, including a symmetric pair of limit cycles, a symmetric pair of strange attractors, and limit cycles coexisting with strange attractors, as shown in Fig. 4. The coexisting symmetric pair of strange attractors at $a = 0.32$ has Lyapunov exponents $(0.0058, 0, -1.0058)$, the coexisting symmetric pair of limit cycles at $a = 0.28$ has Lyapunov exponents $(0, -0.0587, -0.9413)$, while the coexisting strange attractor and limit cycle at $a = 0.23$ have Lyapunov exponents $(0.0662, 0, -1.0662)$ and $(0, -0.0194, -0.9806)$, respectively, and the coexisting symmetric pair of strange attractors and a symmetric limit cycle at $a = 0.16$ have Lyapunov exponents $(0.0160, 0, -1.0160)$ and $(0, -0.0446, -0.9554)$, respectively. The basins of attraction for

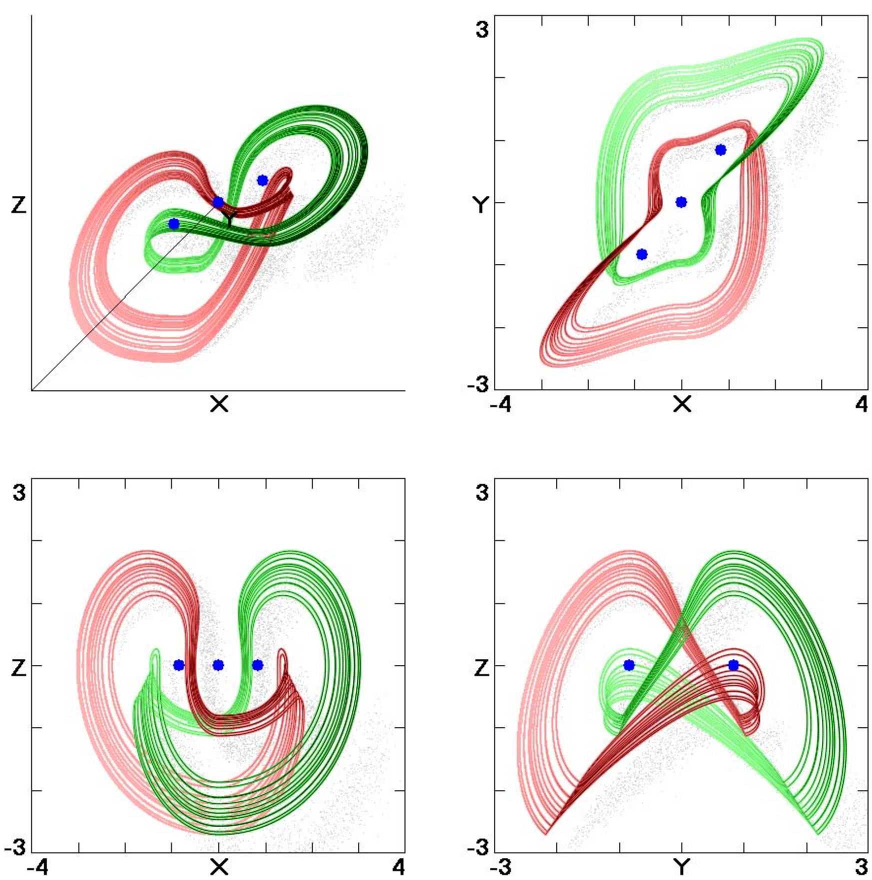


Fig. 8. Symmetric pair of strange attractors for $(a, m) = (1.2, 1)$ with initial conditions $(x_0, y_0, z_0) = (0, \pm 1, 1)$ in AB5. The green and red attractors correspond to two symmetric initial conditions, and the equilibrium points are shown as blue dots.

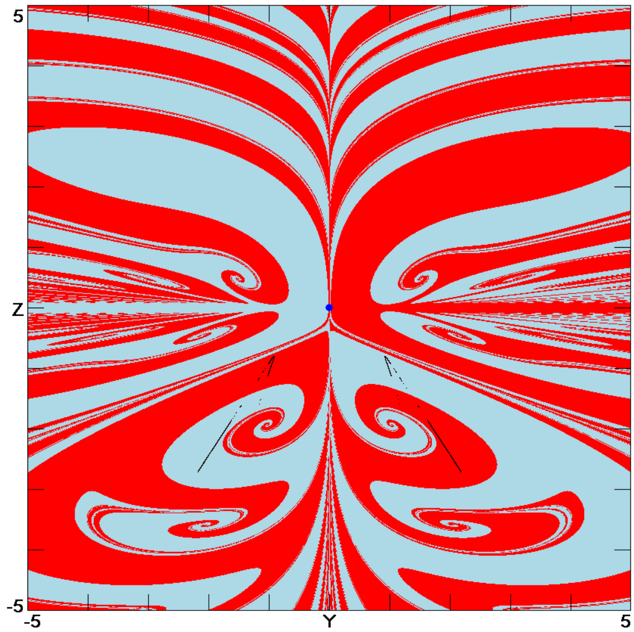
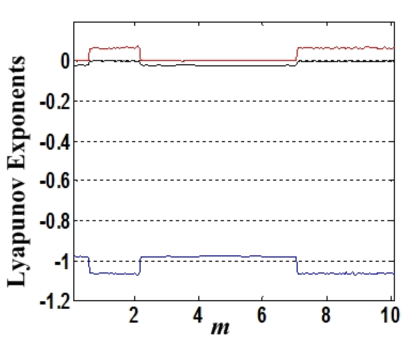


Fig. 9. Cross-section for $x = 0$ of the basins of attraction for the symmetric pair of strange attractors (light blue and red) for AB5 with $a = 1.2, m = 1$.

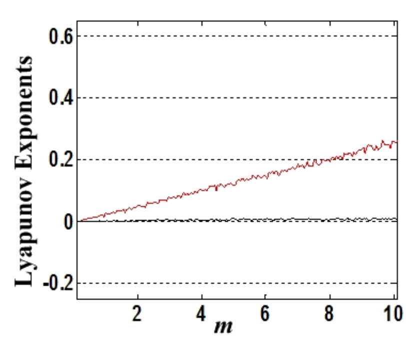
the coexisting strange attractors when $a = 0.32$ are shown in Fig. 5, indicating its fractal structure.

There are similar coexisting attractors in systems AB2, AB3, AB5, and AB6. Even the systems without any lines of equilibrium points, such as AB1 and AB4, still have coexisting attractors. The system AB1 has coexisting symmetric and asymmetric attractors, while the system AB4 has symmetric pairs of limit cycles, which merge into a single symmetric one before the chaos onsets. When $a = 0.31$, the system AB1 has a symmetric pair of strange attractors with Lyapunov exponents $(0.0137, 0, -1.0137)$ at the initial condition $(\pm 1.00, \pm 0.79, 0.92)$. When $a = 0.52$, it has a symmetric strange attractor coexisting with a symmetric limit cycle having Lyapunov exponents $(0.0701, 0, -1.0701)$ at the initial condition $(-0.18, -2.16, 0.32)$ and Lyapunov exponents $(0, -0.0195, -0.9805)$ at the initial condition $(1.02, 2.15, 0.72)$, respectively. Both of the revisions of AB1 and AB4 have robust symmetric solutions over a relatively large range of the bifurcation parameter a.

Meanwhile, the intrusion of lines of equilibria will alter the multistability and correspondingly rearrange the basins of attraction. Except for possibly introducing new multistability along the line, the modifications more often suppress the multistability than enhance it. The system AB2 with a neutrally stable line of equilibrium points has coexisting strange attractors at $a = 1.2$ as shown in Fig. 6. The corresponding Lyapunov exponents of the two coexisting attractors are $(0.0387, 0, -1.0387)$, and the Kaplan–Yorke dimension is $D_{\mathrm{KY}} = 2.0373$. The system AB5 with two lines of equilibrium points shows a similar symmetric pair of interlinked strange attractors at $a = 1.2$ as shown in Fig. 8. The corresponding Lyapunov exponents of the two coexisting attractors are $(0.0251, 0, -1.0346)$, and the Kaplan–Yorke dimension is $D_{\mathrm{KY}} = 2.0243$. The corresponding basins of attraction are shown in Figs. 7 and 9. The basins of attraction for the two chaotic attractors are indicated by light blue and red, respectively. The basins have the expected symmetry about the z-axis and a fractal structure. The full spread of red and light blue in the whole plane in Fig. 9 indicates the two lines of equilibrium points are both unstable, while the basins of attraction for AB2, shown in Fig. 7, are damaged since the line of equilibria is neutrally stable. Since these basin plots are just one slice of a 3-D basin taken in the plane of the equilibria, it is



(a)



(b)

Fig. 10. Lyapunov exponents when the amplitude parameter m varies in $[0, 10]$: (a) $a = 0.23$ with initial conditions $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$ for AB0 and (b) $a = 1.2$ with initial conditions $(x_0, y_0, z_0) = (0, 4, -4)$ for AB5.

not surprising that the plot has a discontinuity there especially if two of the eigenvalues are zero. Further exploration shows that the green basin for the system AB2 indicates new extra multistability, where the variable z stretches (evolves) to negative infinity while the variables x and y oscillate with an increasing amplitude or with a small constant amplitude. AB2 and AB5 both have isolated equilibria that border the basins of attraction of the strange attractors, and thus all these attractors are self-excited. All the reorganized basins are different from others [Jafari & Sprott, 2013] where the line of equilibrium points threads the attractor with different stability for separate segments of the line, indicating that the corresponding attractor is hidden [Leonov *et al.*, 2011, 2012; Leonov & Kuznetsov, 2013].

However, the existence of multistability can degrade the amplitude control because fixed initial conditions may switch basins of attraction and therefore trigger a state-shift among the coexisting attractors [Li & Sprott, 2014a]. The initial conditions need to be rescaled when the amplitude parameter varies to adjust the variables. Figure 10(a) shows that in the normal Sprott B system (AB0) when the partial amplitude parameter m, namely, the coefficient of the quadratic term in the third dimension, varies from 0 to 10, there are states of limit cycle and chaos. Figure 10(b) shows that the Lyapunov exponents increase with the amplitude parameter since it also controls the frequency of the variables. There are no notches in Fig. 10(b) because the coexisting attractors are a symmetric pair with the same Lyapunov exponents, which still indicates that the same initial conditions may not safely result in a desired state. One can check that the same initial conditions make the system AB5 locate on the left and right attractors alternatively at $m = 1, 2, 3$.

5. Electrical Circuit Implementation

The electrical circuits for the amplitude-controllable Sprott B systems are designed with the main structure for different unified degrees, as shown in Fig. 11. The structure has the same terms in the first and second dimensions for both triplets of cases. For the first three cases, i.e. AB1, AB2, AB3, $S1 = -z\,\mathrm{sgn}(y)$, $S2 = -x$, $S3 = y$; for the second three cases, i.e. AB4, AB5, AB6, $S1 = -yz$, $S2 = -x|x|$, $S3 = y|x|$. The third dimension provides the corresponding feedback of $G_m(x, y)$ and $G_a(x, y)$, by which the amplitude of the variables and the bifurcation dynamics of the systems may be controlled or observed. In accordance with the structure in Table 1, the feedback for the first three systems is $G_{m1}(x, y) = -1$, $G_{m2}(x, y) = xy$, $G_{m3}(x, y) = xy|y|$, $G_{a1}(x, y) = y\,\mathrm{sgn}(x)$, $G_{a2}(x, y) = G_{a3}(x, y) = -|x|$ when $S1 = -z\,\mathrm{sgn}(y)$, $S2 = -x$, $S3 = y$. The feedback for the second three systems is $G_{m4}(x, y) = -1$, $G_{m5}(x, y) = -|x|$, $G_{m6}(x, y) = xy|y|$, $G_{a4}(x, y) = G_{a5}(x, y) = xy$, $G_{a6}(x, y) = -|xy|$ when $S1 = -yz$, $S2 = -x|x|$, $S3 = y|x|$. The circuit elements for $G_m(x, y)$ and $G_a(x, y)$ are shown in Fig. 12. The special switch elements [Li *et al.*, 2014b, 2015] can be used to decrease the required number of multipliers.

To realize the systems in Table 1, we select smaller capacitors, $C_1 = C_2 = C_3 = 1\,\mathrm{nF}$, for higher

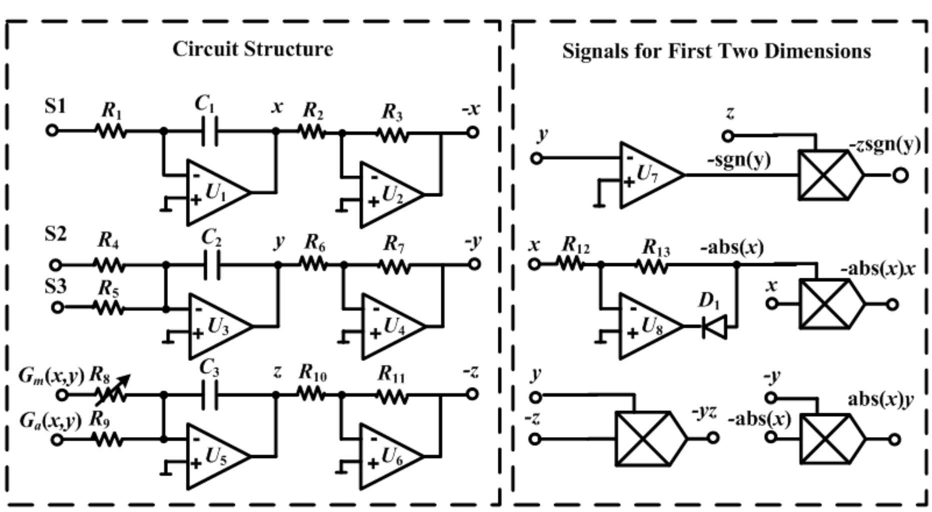


Fig. 11. Circuit structure and signal provider for the first two dimensions of the AB systems.

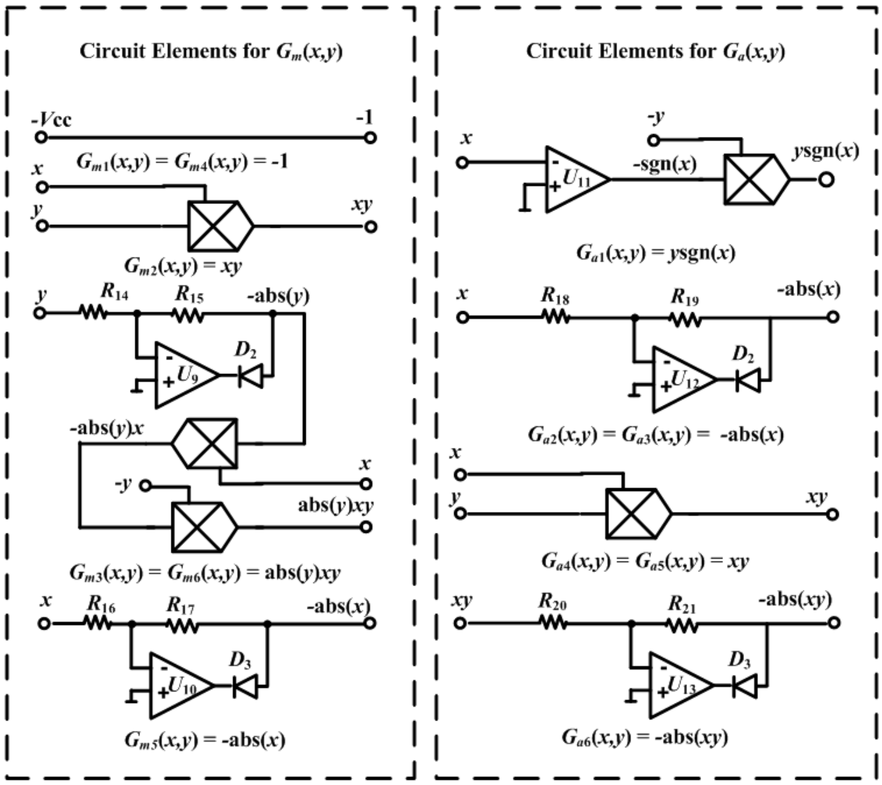


Fig. 12. Circuit elements providing the inputs for the third dimensions of the AB systems.

frequency. The resistors for the absolute-value are $R_{12} = R_{13} = R_{14} = R_{15} = R_{16} = R_{17} = R_{18} = R_{19} = R_{20} = R_{21} = 470\,\Omega$. The operational amplifiers are TL084 ICs powered by 10 volts. The multipliers are all AD633, and the diodes are 1N3659. The resistors for the phase inverters are $R_2 = R_3 = R_6 = R_7 = R_{10} = R_{11} = 100\,\text{k}\Omega$, and $V_{cc} = 1\,\text{V}$. The resistors R_1, R_4, R_5, R_8, R_9 in the adder line are critically determined by the system parameters. The resistors R_1, R_4, R_5 are set to $100\,\text{k}\Omega$ for the unit coefficients in the first two dimensions. R_8 is set to $100\,\text{k}\Omega$ for unit value in the amplitude terms. R_9 is also set to $100\,\text{k}\Omega$ except $76.9\,\text{k}\Omega$ for AB2, $66.7\,\text{k}\Omega$ for AB3, and $71.4\,\text{k}\Omega$ for AB6. The corresponding phase trajectories from the oscilloscope shown in Fig. 13 agree well with the predictions of Fig. 3.

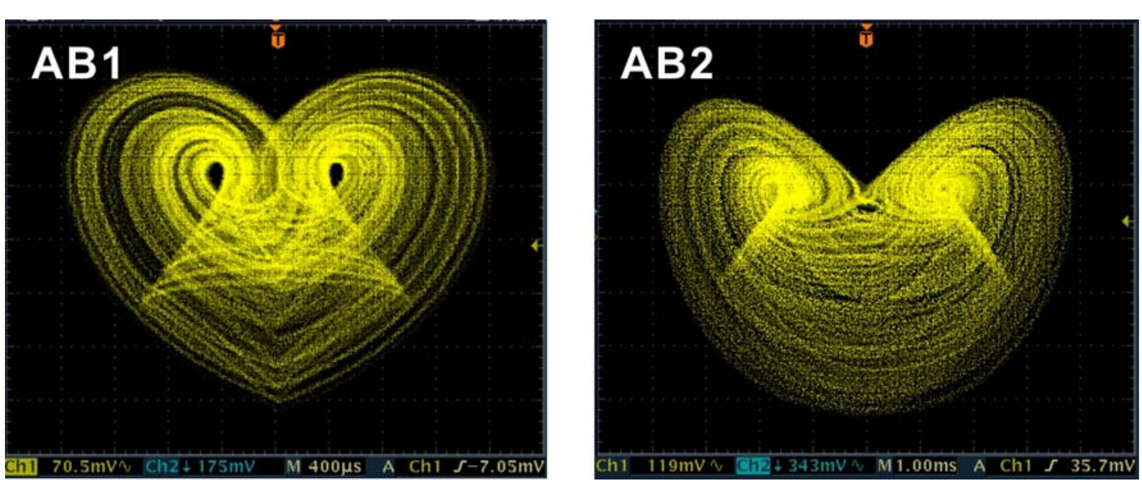


Fig. 13. Experimental phase portraits of the AB systems in the x–z plane observed from the oscilloscope for comparison with Fig. 3.

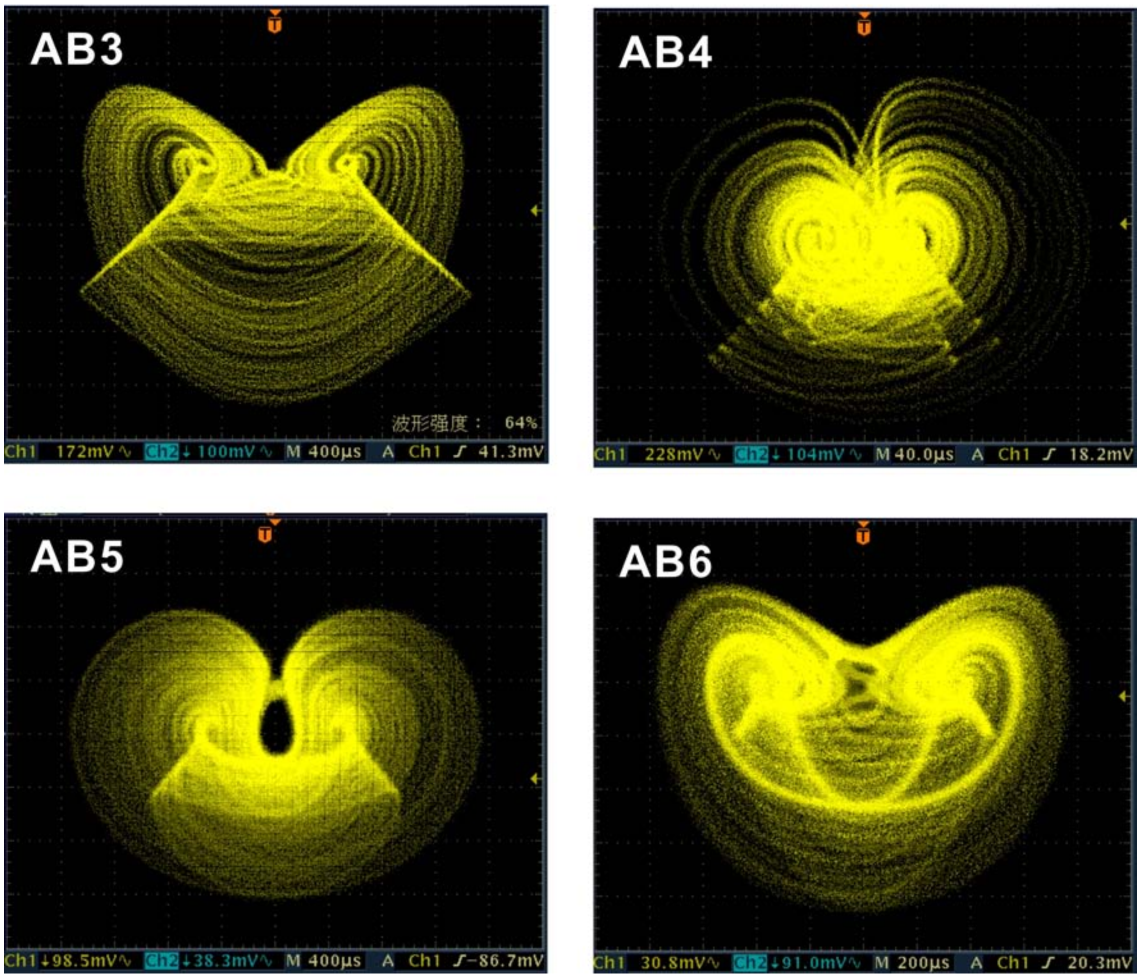


Fig. 13. (*Continued*)

6. Discussion and Conclusions

It is useful to rescale a chaotic signal by an independent amplitude parameter in order to simplify the broadband amplifier design in electronic engineering applications. Most chaotic systems fail to give an amplitude-controllable signal due to the linear and nonlinear terms of different degrees. If all the terms have the same degree except one, the coefficient of the lone term with a different degree will be a functional independent amplitude controller.

Because each of the variables in chaotic systems includes polarity and amplitude information, the degree of the linear or nonlinear terms can be increased by using an absolute-value function or decreased by using a signum function. The basic equilibria (except for proportional revision in amplitude) and their stability must be retained when choosing among the many options to guarantee the degree modification will work. This degree-modification procedure often leads to a chaotic system but generally requires that the parameters be readjusted to recover the chaos. Besides the modification of the chaotic system providing an amplitude parameter for rescaling the variables, lines of equilibrium points are often introduced, which in turn influence the dynamics especially when the system is multistable.

Acknowledgments

This work was supported financially by the Jiangsu Overseas Research and Training Program for University Prominent Young and Middle-aged Teachers and Presidents, the 4th 333 High-level Personnel Training Project (Grant No. BRA2013209) and the National Science Foundation for Postdoctoral General Program and Special Founding Program of the People's Republic of China (Grant Nos. 2011M500838 and 2012T50456) and Postdoctoral Research Foundation of Jiangsu Province (Grant No. 1002004C).

References

Jafari, S. & Sprott, J. C. [2013] "Simple chaotic flows with a line equilibrium," *Chaos Solit. Fract.* **57**, 79–84.

Leonov, G. A., Vagaitsev, V. I. & Kuznetsov, N. V. [2011] "Localization of hidden Chua's attractors," *Phys. Lett. A* **375**, 2230–2233.

Leonov, G. A., Vagaitsev, V. I. & Kuznetsov, N. V. [2012] "Hidden attractor in smooth Chua systems," *Physica D* **241**, 1482–1486.

Leonov, G. A. & Kuznetsov, N. V. [2013] "Hidden attractors in dynamical systems from hidden oscillations in Hilbert–Kolmogov, Aizerman, and Kalman problems to hidden chaotic attractors in Chua circuits," *Int. J. Bifurcation and Chaos* **23**, 1330002-1–69.

Li, Y., Chen, G. & Tang, W. K. S. [2005] "Controlling a unified chaotic system to hyperchaotic," *IEEE Trans. Circuits Syst.-II*: *Exp. Briefs* **52**, 204–207.

Li, C. & Wang, D. [2009] "An attractor with invariable Lyapunov exponent spectrum and its Jerk circuit implementation," *Acta Phys. Sin.* **58**, 764–770 (in Chinese).

Li, C., Wang, J. & Hu, W. [2012] "Absolute term introduced to rebuild the chaotic attractor with constant Lyapunov exponent spectrum," *Nonlin. Dyn.* **68**, 575–587.

Li, C. & Sprott, J. C. [2013] "Amplitude control approach for chaotic signals," *Nonlin. Dyn.* **73**, 1335–1341.

Li, C. & Sprott, J. C. [2014a] "Chaotic flows with a single nonquadratic term," *Phys. Lett. A* **378**, 178–183.

Li, C. & Sprott, J. C. [2014b] "Finding coexisting attractors using amplitude control," *Nonlin. Dyn.* **78**, 2059–2064.

Li, C., Sprott, J. C. & Thio, W. [2014a] "Bistability in a hyperchaotic system with a line equilibrium," *J. Exp. Theor. Phys.* **118**, 494–500.

Li, C., Sprott, J. C. & Thio, W. [2014b] "A new piecewise-linear hyperchaotic circuit," *IEEE Trans. Circuits Syst.-II*: *Exp. Briefs* **61**, 977–981.

Li, Q., Hu, S., Tang, S. & Zeng, G. [2014c] "Hyperchaos and horseshoe in a 4D memristive system with a line of equilibria and its implementation," *Int. J. Circ. Theor. Appl.* **42**, 1172–1188.

Li, C., Sprott, J. C. & Thio, W. [2015] "Linearization of the Lorenz system," *Phys. Lett. A* **379**, 888–893.

Sprott, J. C. [1994] "Some simple chaotic flows," *Phys. Rev. E* **50**, R647–R650.

Sprott, J. C. [2010] *Elegant Chaos*: *Algebraically Simple Chaotic Flows* (World Scientific, Singapore).

Wang, H., Cai, G., Miao, S. & Tian, L. [2010] "Nonlinear feedback control of a novel hyperchaotic system and its circuit implementation," *Chin. Phys. B* **19**, 030509.

Yu, S. M., Tang, W. K. S., Lu, J. H. & Chen, G. [2008] "Generation of $n \times m$-wing Lorenz-like attractors from a modified Shimizu–Morioka model," *IEEE Trans. Circuits Syst.-II*: *Exp. Briefs* **55**, 1168–1172.