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Review

Recent new examples of hidden attractors

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Abstract. Hidden attractors represent a new interesting topic in the chaos literature. These attractors have a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. Oscillations in dynamical systems can be easily localized numerically if initial conditions from its open neighborhood lead to a long-time oscillation. This paper reviews several types of new rare chaotic flows with hidden attractors. These flows are divided into to three main groups: rare flows with no equilibrium, rare flows with a line of equilibrium points, and rare flows with a stable equilibrium. In addition we describe a novel system containing hidden attractors.

1 Introduction

Most familiar examples of low-dimensional chaotic flows occur in systems having one or more saddle points. Such saddle points allow homoclinic and heteroclinic orbits and the prospect of rigorously proving the chaos when the Shilnikov condition is satisfied. Furthermore, such saddle points provide a means for locating any strange attractors by choosing an initial condition on the unstable manifold in the vicinity of the saddle point.

Recent researches by Leonov and Kuznetsov have involved categorizing periodic and chaotic attractors as either self-excited or hidden [1–11]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor (HA) has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. The classical attractors of Lorenz [12], Rssler [13], Chen [14], Sprott (cases B to S) [15], and other widely-known attractors are those excited from unstable equilibria. From a computational point of view this allows one to use a numerical method in which a trajectory started from a point on the unstable manifold in the neighborhood of an unstable equilibrium, reaches an attractor and identifies it [7]. Hidden attractors cannot be found by this method and are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

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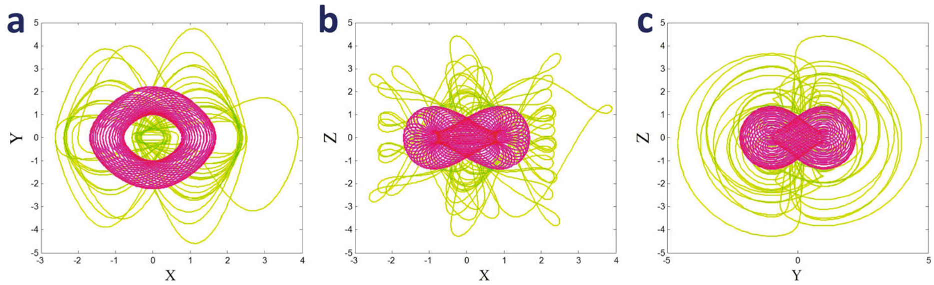

Fig. 1. The Sprott case A system with initial conditions $(0, 5, 0)$ results in a chaotic set (Green) and $(0, 1, 0)$ results a conservative torus (Red).

On the other hand in the last 3 years, there has been an interest in finding some simple rare chaotic flows with no saddle points. Almost all of these works are inspired by a single paper [15], in which a systematic examination of general three-dimensional autonomous ordinary differential equations with quadratic nonlinearities uncovered 19 distinct simple examples of chaotic flows with either five terms and two nonlinearities or six terms and one nonlinearity.

First, by modifying Sprott E, Wang and Chen presented a new chaotic system with only one stable equilibrium [16]. Then by modifying Sprott D, Wei designed a new chaotic system with no equilibria [17]. Motivated by the Weis paper, Jafari and Sprott joined this exploration and performed a systematic search to find additional three-dimensional chaotic systems with quadratic nonlinearities and no equilibria [18]. They found 17 simple systems with that property. At this stage, Leonov and Kuznetsov noticed that by their definition, these systems belong to the category of hidden attractors and encouraged continuation of that line of work. Jafari and Sprott noted that the Wang-Chen system proposed in [16] is another member of the HA category and decided to find the simplest examples of that kind [19]. Chaotic systems with a line of equilibria were the next category of rare flows which are claimed to have HAs [20]. Very recently some other new types of chaotic systems with HA have been identified. The paper is organized as follows: In Sect. 2 we investigate chaotic flows with no equilibria. Chaotic systems with stable equilibria are discussed in Sect. 3. Section 4 focuses on flows with a line of equlibria. In Sect. 5, we introduce another new chaotic system containing HAs. Finally, the paper is concluded in Sect. 6.

2 Rare flows without any equilibria

The oldest example of rare flows with no equlibria is the Sprott case A hidden attractor [15]:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 1 - y^2. \end{aligned} \tag{1}$$

Lyapunov exponents of this system are (0.0138,0,−0.0138) and the Kaplan-Yorke dimension is also 3.0. This system is a special case of the Nose-Hoover oscillator [21] and describes many natural phenomena [22]. Thus it suggests that such systems may have practical as well as theoretical importance. This is a conservative system, and thus it does not have attractors, but there is a chaotic sea coexisting with a set o nested tori which are shown in Fig. 1.

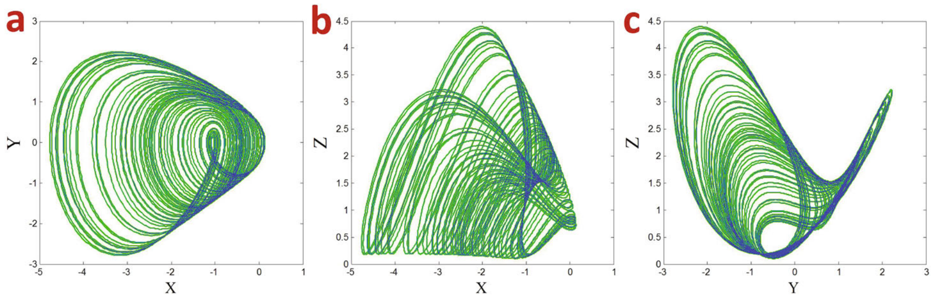

Fig. 2. Attractor of the Wei system with initial conditions $(-1.6, 0.82, 1.9)$.

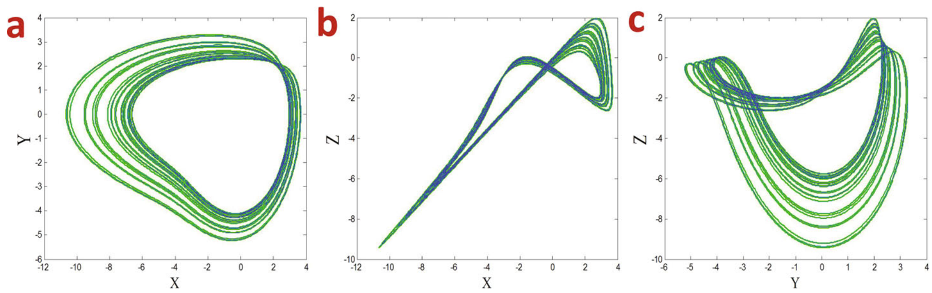

Fig. 3. Attractor of system NE_7 with initial conditions (0,2.3,0).

The first known dissipative example of this category is Wei hidden attractor [17]

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x + z\\ \dot{z} &= 2y^2 + xz - 0.35\end{aligned} \tag{2}$$

which is a modification of the Sprott case D system described in [15]. The attractor of this system is shown in Fig. 2. The Lyapunov exponents of this system are (0.0776,0,−1.5008), and the Kaplan-Yorke dimension is 2.0517. By inspiration from these systems, Jafari and Sprott performed a systematic search to find the simplest three-dimensional chaotic systems with quadratic nonlinearities and no equilibria [18]. Their search was based on the methods proposed in [15].

They found seventeen simple systems that show chaos. Except NE_1 (Sprott A), all cases were dissipative. As an example, NE_7 hidden attractor

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + z\\ \dot{z} &= -0.8x^2 + z^2 + 2\end{aligned} \tag{3}$$

is shown in Fig. 3. After that paper, many other new chaotic and hyperchaotic systems with no equilibrium were designed and used in applications [23–31].

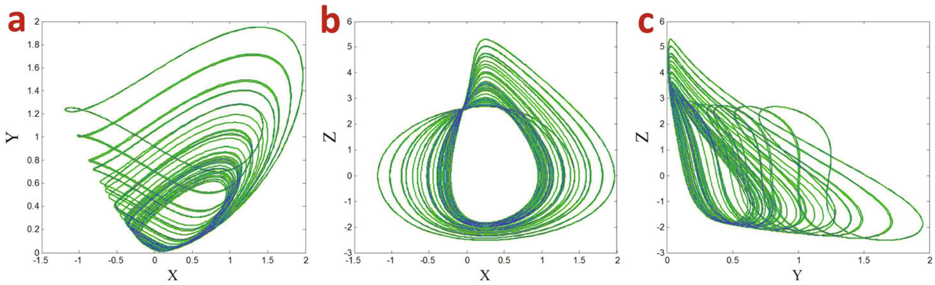

Fig. 4. Attractor of the Wang-Chen system with initial conditions (0,0,0).

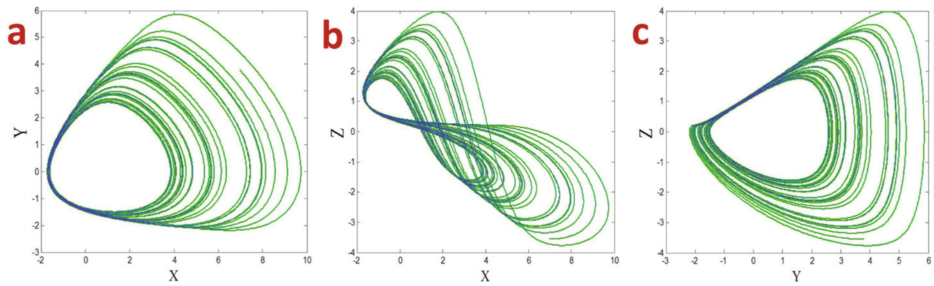

Fig. 5. Attractor of system SE_1 with initial conditions $(4, -2, 0)$.

3 Rare flows with a stable equilibrium

The first example of such rare chaotic flow (with only one stable equilibrium) was designed by Wang and Chen [16]

$$\begin{aligned} \dot{x} &= yz + 0.006 \\ \dot{y} &= x^2 - y \\ \dot{z} &= 1 - 4x. \end{aligned} \tag{4}$$

They designed it by modifying Sprott case E in [15]. Its attractor is shown in Fig. 4.

Motivated by their new finding, Jafari and Sprott expanded the list and identified 23 simple systems with that property. To do that, they performed a systematic computer search for chaos in three-dimensional autonomous systems with quadratic nonlinearities and a single equilibrium that were stable according to the Routh-Hurwitz criterion. SE_1 hidden attractor

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -x - 0.6y - 2z + z^2 - 0.4xy \end{aligned} \tag{5}$$

is one of the simplest cases found in this way (5). The equilibrium of this system is at (0,0,0) with eigenvalues given by $(-1.9548, -0.0226 \pm 0.7149i)$. Lyapunov exponents are (0.0377,0,$-$2.0377) with a Kaplan-Yorke dimension of 2.0185. Its attractor is illustrated in Fig. 5. At least one point attractor coexists with a hidden strange attractor for these types of rare chaotic flows. Many other new chaotic and hyperchaotic systems with stable equilibria have now been designed and used in applications [28,32–39].

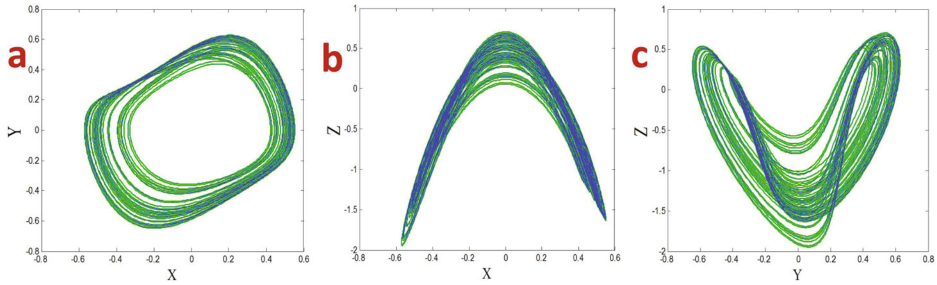

Fig. 6. Attractor of system LE_1 with initial conditions (0,0.5,0.5).

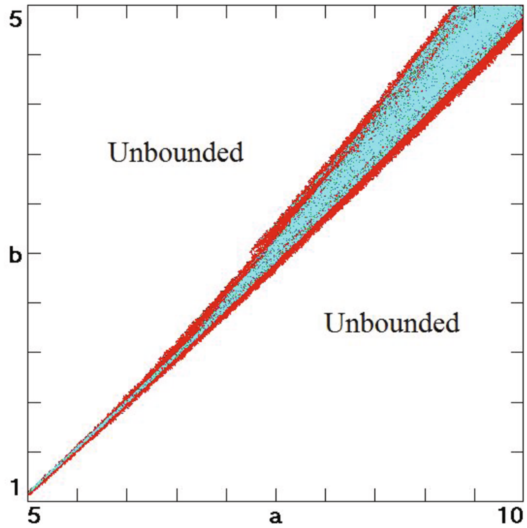

Fig. 7. Regions of various dynamical behaviors for System (7) as a function of the bifurcation parameters a and b. The chaotic regions are shown in red, the periodic (limit cycle) and quasiperiodic (torus) regions are shown in blue, and the unbounded regions are shown in white.

4 Flows with a line equilibrium

After proposing a chaotic system with any number of equilibria by Wang and Chen in [31], Jafari and Sprott in [19] introduced simple chaotic systems with a line of equilibria. They were inspired by the structure of the conservative Sprott case A system and considered a general parametric form of it with quadratic nonlinearities. With an exhaustive computer search, nine simple cases were found. LE_1 is an especially simple example with only six terms (6). This system has a line equilibrium at (0, 0, z) with no other equilibria (in other words the z-axis is the line equilibrium of this system). Its attractor is shown in Fig. 6. Eigenvalues of these systems are $((z \pm \sqrt{z^2 - 4})/2, 0)$, Lyapunov exponents are (0.0717, 0, −0.5232), and the Kaplan-Yorke dimension is 2.1927.

The strange attractor of these systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= -x - 15xy - xz \end{aligned} \tag{6}$$

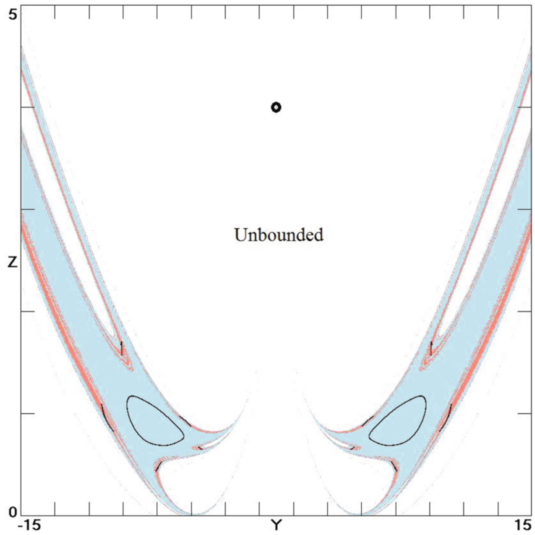

Fig. 8. Cross section in the x=0 plane of the basins of attraction for a = 8.496 and b = 4 in System (7). The blue area is the basin of the torus, the red area is the basin of the strange attractor, and unbounded regions are shown in white. Cross sections of the attractors are shown in black, and the unstable equilibrium is shown as a small circle at the center toward the top.

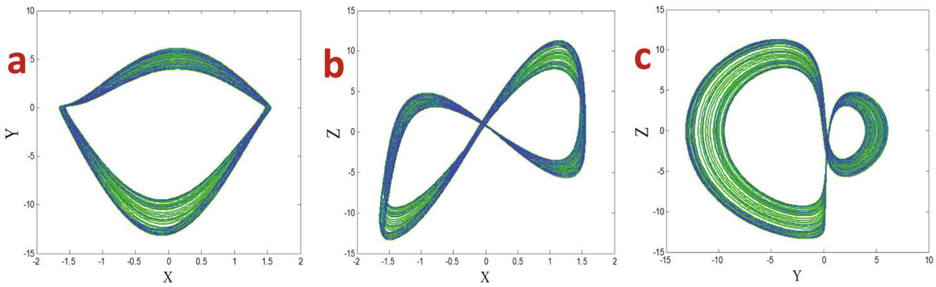

Fig. 9. Attractor of System (7) with initial conditions (0,3.9,0.7).

is hidden since there are uncountably many unstable points on the equilibrium line of which only a tiny portion intersects the basin of the chaotic attractor.

5 Other new flows with hidden attractors

Recently, Sprott and Jafari, in the search for rare chaotic flows, designed an unusual example of a three-dimensional dissipative chaotic flow with quadratic nonlinearites in which the only equilibrium is an unstable node

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= z + ax^2 - y^2 - b. \end{aligned} \tag{7}$$

For this system, the region of parameter space with bounded solutions is relatively small (Fig. 7) as is the basin of attraction (Fig. 8), which accounts for the difficulty of its discovery. Its strange attractor appears to be hidden in that it cannot be found by starting with initial conditions in the vicinity of the equilibrium, and thus it represents a new type of hidden attractor. Its attractor is shown in Fig. 9 for a = 8.888, b = 4 and (0,3.9,0.7) as the initial conditions.

6 Conclusion

Categorizing dynamical systems into systems with HAs is a new topic in dynamical systems. We described three families of third-order chaotic systems with hidden attractors (chaotic attractors in dynamical systems without any equilibrium points, with only stable equilibria, and with a line of equilibria). Furthermore, a novel interesting rare system was introduced which has a hidden attractor.

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