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A Simple Chaotic Flow with a Plane of Equilibria

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Using a systematic computer search, a simple four-dimensional chaotic flow was found that has the unusual feature of having a plane of equilibria. Such a system belongs to a newly introduced category of chaotic systems with hidden attractors that are important and potentially problematic in engineering applications.

Keywords: Chaotic flows; surface equilibrium; hidden attractors.

1. Introduction

It is widely recognized that mathematically simple systems of nonlinear differential equations can exhibit chaos. With the advent of fast computers, it is now possible to explore the entire parameter space of these systems with the goal of finding parameters that result in some desired characteristics of the system.

Recently, many new chaotic flows have been discovered that are not associated with a saddle point, including ones without any equilibrium points, with only stable equilibria, or with a line containing infinitely many equilibrium points [Jafari *et al.*, 2013; Molaie *et al.*, 2013; Jafari & Sprott, 2013, 2015; Jafari *et al.*, 2015; Jafari *et al.*, 2014; Pham *et al.*, 2014a; Pham *et al.*, 2015; Pham *et al.*, 2014b; Kingni *et al.*, 2014; Shahzad *et al.*, 2015; Sprott *et al.*, 2015; Pham *et al.*, 2014c; Tahir *et al.*, 2015; Gotthans & Petržela, 2015; Wang & Chen, 2012, 2013; Wei, 2011; Pham *et al.*, 2014d]. The attractors for such systems have been called hidden attractors [Kuznetsov & Leonov, 2011; Leonov *et al.*, 2014; Leonov & Kuznetsov, 2013; Leonov *et al.*, 2015; Leonov *et al.*, 2011; Leonov *et al.*, 2012; Leonov & Kuznetsov, 2013; Bragin *et al.*, 2011], and that accounts for the difficulty of discovering them since there is no systematic way to choose initial conditions except by extensive numerical search. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or aircraft wing.

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In this paper, we introduce a new rare category of chaotic systems (note that there is a specific definition for rare attractors [Chudzik *et al.*, 2011; Klokov & Zakrzhevsky, 2011] which we do not mean to imply) with hidden attractors: systems with surfaces of equilibria. Although in such systems the basin of attraction may intersect the equilibrium surface in some sections, there are usually uncountably many points on the surface that lie outside the basin of attraction of the chaotic attractor, and thus it is impossible to identify the chaotic attractor for sure by choosing an arbitrary initial condition in the vicinity of the unstable equilibria. In other words, from a computational point of view these attractors are hidden, and knowledge about the equilibria does not help in their localization. On the other hand, to the best of our knowledge, there are no chaotic systems with surfaces of equilibria in the literature. The goal of this paper is to describe a new category of hidden attractor and expand the list of known mathematically simple hidden chaotic attractors. We will show that such systems should be constructed in a dimension greater than three. Thus, we perform a systematic computer search for chaos in four-dimensional autonomous systems with quadratic nonlinearities that have been designed so that there will be equilibrium surfaces.

It should be noted that based on [Sprott, 2011] any new proposed chaotic system should satisfy at least one of the following three conditions:

(1) The system should credibly model some important unsolved problem in nature and shed insight on that problem.
(2) The system should exhibit some behavior previously unobserved.
(3) The system should be simpler than all other known examples exhibiting the observed behavior.

Our proposed system satisfies both the second and third conditions.

2. Simple Chaotic Flows with Surfaces of Equilibria

In the search for chaotic flows with surfaces of equilibria, we followed a simple procedure. Consider the general parametric form of quadratic three-dimensional flows:

$$\dot{x} = Q_1(x,y,z), \quad \dot{y} = Q_2(x,y,z), \quad \dot{z} = Q_3(x,y,z), \tag{1}$$

in which

$$\begin{aligned} Q_1 &= k_1x + k_2y + k_3z + k_4x^2 + k_5y^2 + k_6z^2 \\ &\quad + k_7xy + k_8xz + k_9yz + k_{10}, \\ Q_2 &= k_{11}x + k_{12}y + k_{13}z + k_{14}x^2 + k_{15}y^2 + k_{16}z^2 \\ &\quad + k_{17}xy + k_{18}xz + k_{19}yz + k_{20}, \\ Q_3 &= k_{21}x + k_{22}y + k_{23}z + k_{24}x^2 + k_{25}y^2 + k_{26}z^2 \\ &\quad + k_{27}xy + k_{28}xz + k_{29}yz + k_{30}. \end{aligned} \tag{2}$$

In order to have a surface on which all the points are an equilibrium, there should be a multiplying factor like $f(x,y,z)$ in all the equations, so that an equilibrium surface occurs whenever $f(x, y,z) = 0$. Thus the equations to be examined are

$$\dot{x} = f(x,y,z)Q_1, \quad \dot{y} = f(x,y,z)Q_2, \quad \dot{z} = f(x,y,z)Q_3. \tag{3}$$

For almost all of the common chaotic flows, such multiplying factors are easily found but are observed to have little effect on the strange attractors other than to translate the attractor so that it does not intersect the surface. Thus we consider four-dimensional systems where nontrivial surfaces of equilibria exist.

Consider the following structure:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a_1y + a_2z + a_3xy + a_4xz + a_5yz \\ &\quad + a_6yw + a_7zw + a_8y^2 + a_9z^2, \\ \dot{w} &= a_{10}y + a_{11}z + a_{12}xy + a_{13}xz + a_{14}yz \\ &\quad + a_{15}yw + a_{16}zw + a_{17}y^2 + a_{18}z^2. \end{aligned} \tag{4}$$

With the following conditions, this system will have a plane of equilibrium in $(x, 0, 0, w)$:

$$\begin{aligned} &|a_6| + |a_7| \neq 0, \\ &|a_3| + |a_4| + |a_{12}| + |a_{13}| \neq 0. \end{aligned} \tag{5}$$

An exhaustive computer search was done considering thousands of combinations of coefficients a_1 through a_{18} and initial conditions, seeking dissipative cases for which the largest Lyapunov exponent is greater than 0.001. For each case that was found, the space of coefficients was searched for values that are deemed elegant [Sprott, 2010], by which we mean that as many coefficients as possible are set to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits. The simplest case we found in this way is:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= yw - azw,\\ \dot{w} &= bxz - y^2 + z^2\end{aligned} \tag{6}$$

for which $a = 4$ and $b = 0.1$ give chaotic solutions. This system is symmetric under the transformation $(x, y, z, w) \to (-x, -y, -z, w)$ and dissipative with a symmetric pair of attractors shown with various projections in Fig. 1 for initial conditions $(\mp 30, \pm 0.5, \pm 0.1, 0)$.

The Jacobian matrix, eigenvalues of the equilibria, Lyapunov exponents, and Kaplan–Yorke dimension are shown in Table 1. Two of the eigenvalues are zero everywhere, corresponding to directions tangent to the plane. In the perpendicular directions, we can distinguish three regions of the equilibrium plane, all unstable: For $w < -1/4$, the equilibria are unstable nodes with two positive eigenvalues. For $-1/4 < w < 0$, the equilibria are unstable spirals with a complex conjugate pair of eigenvalues and a positive real part. For $w > 0$, the equilibria are saddle points with a positive and a negative real eigenvalue. Thus the plane is nowhere attracting.

In order to describe how the plane of equilibria at $y = z = 0$ and the attractors manage to coexist, we study the trajectory in the y–z plane and how it approaches the plane at $(y, z) = (0, 0)$.

As seen in Fig. 2, the trajectory goes near to (0, 0) but it does not reach it. In Fig. 3, the distance of the trajectory from the equilibria plane (which is $\sqrt{(y^2 + z^2)}$) is plotted versus time. It can be seen that this distance can be very small but not zero. In this time interval, the minimum of the distance is 4.9×10^{-4}. So we can say that the trajectory does not intersect with the plane of equilibria, but goes near it and maybe around it (note that it can go around a plane in a four-dimensional space since

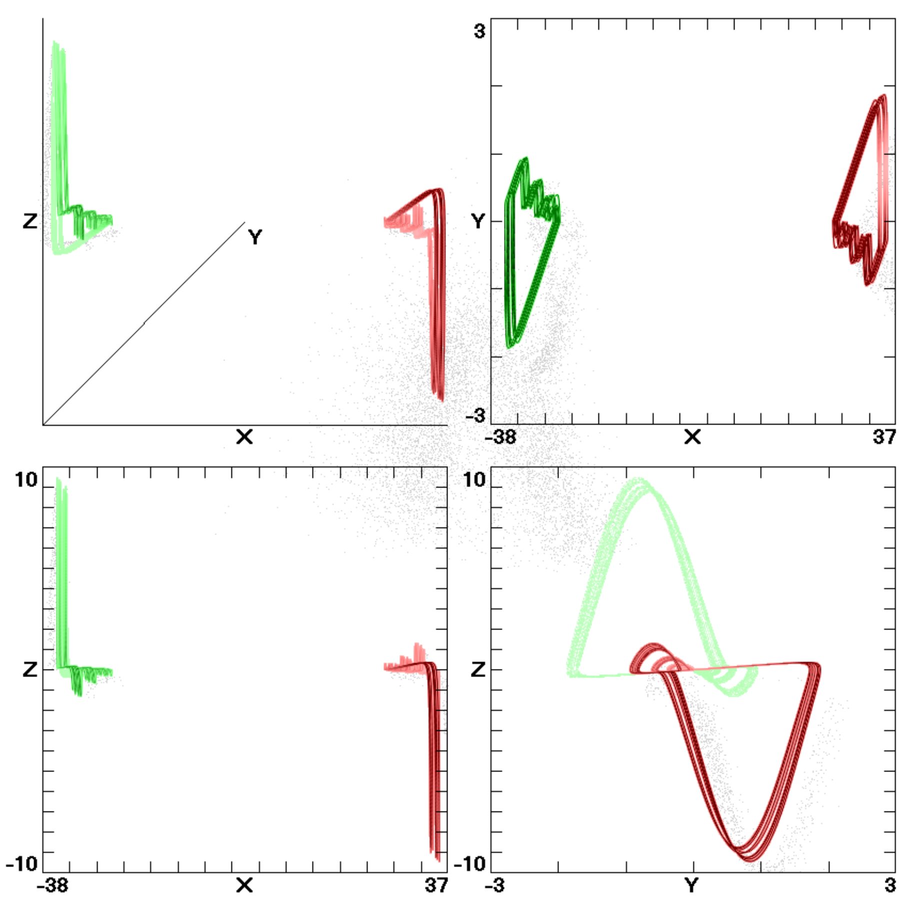
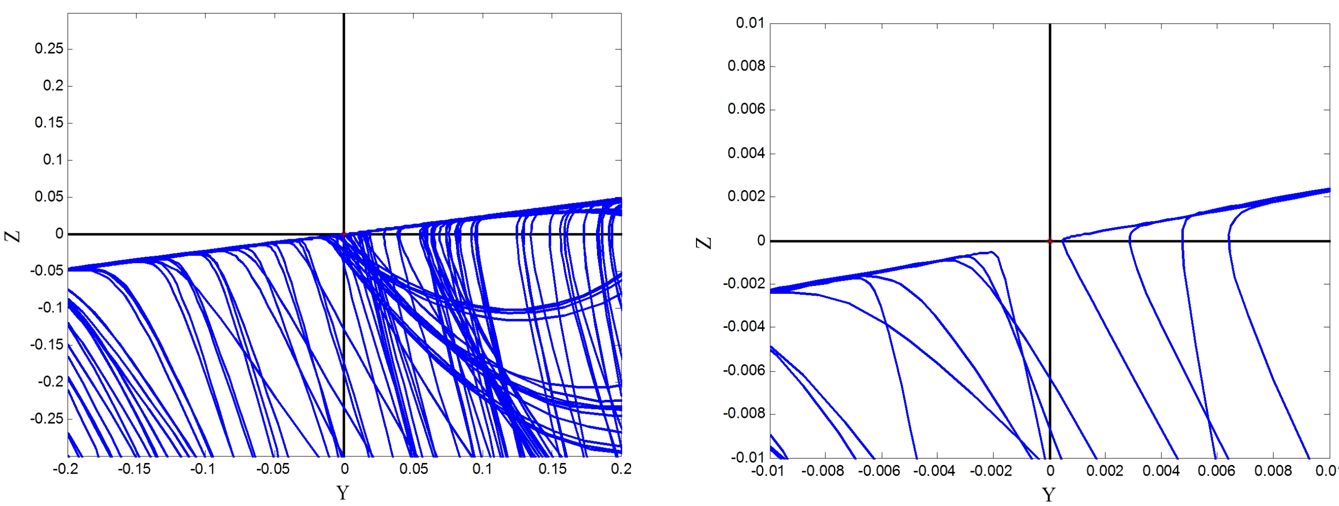


Fig. 1. Projections of the strange attractors onto various planes.

Table 1. The Jacobian matrix, eigenvalues of the equilibria, Lyapunov exponents, and Kaplan–Yorke dimension of the system (6) with $a = 4$ and $b = 0.1$.

Jacobian Matrix	Eigenvalue	LEs	D_{KY}
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & w & -4w & 0 \\ 0 & 0 & 0.1x & 0 \end{bmatrix}$	0 0 $-2w \pm \sqrt{4w^2 + w}$	0.0066 0 −0.0321 −2.0991	2.2049



(a) (b)

Fig. 2. Two projections of the strange attractor onto the y–z plane which have been zoomed in around $(y, z) = (0, 0)$.

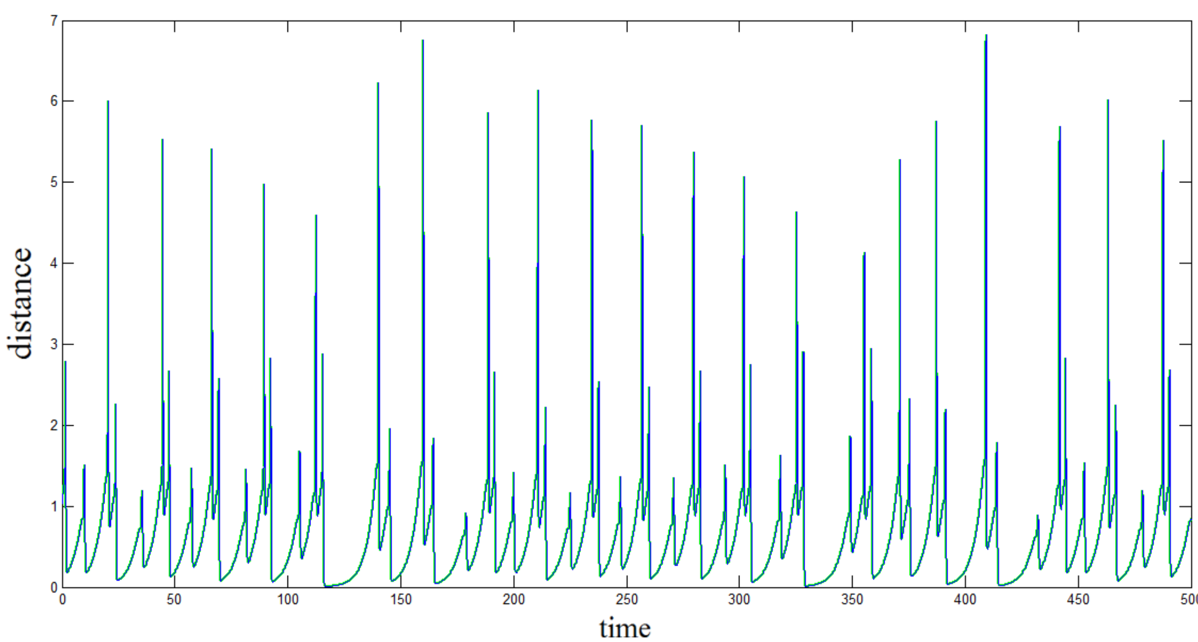


Fig. 3. The distance of the trajectory from the equilibria plane (which is $\sqrt{(y^2 + z^2)}$) versus time.

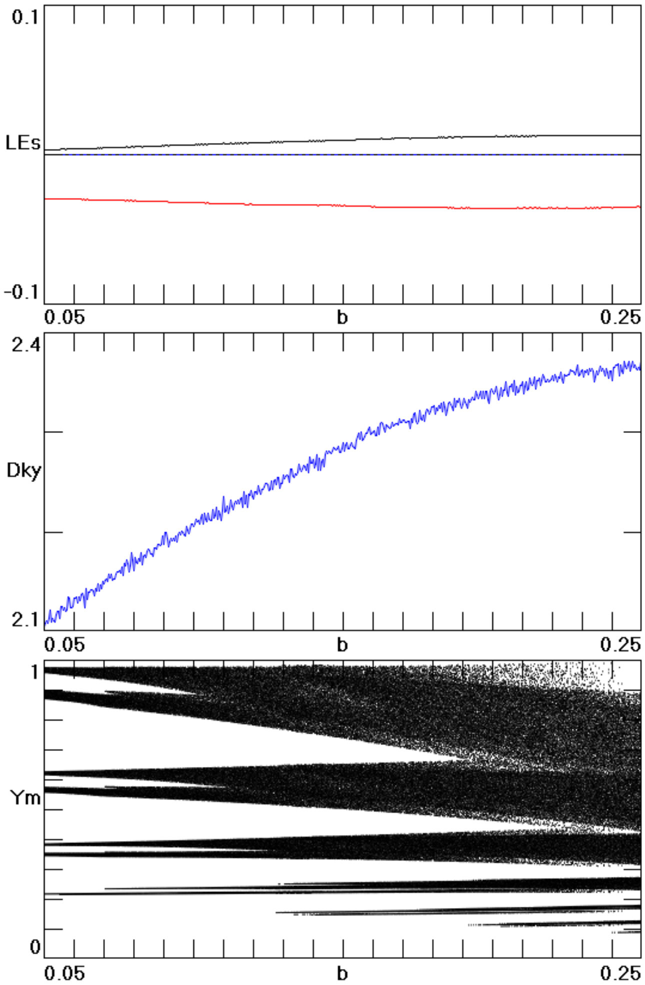


Fig. 4. Largest three Lyapunov exponents, Kaplan–Yorke dimension, and local maximum of y versus b.

even an infinite plane does not divide the space into two disjoint regions).

To illustrate the robustness of the chaos and convergence of the Lyapunov exponents, Fig. 4 shows the three largest Lyapunov exponents, the Kaplan–Yorke dimension, and the local maxima of y for one of the attractors of this system for $0.05 < b < 0.3$. Outside of that range, most of the solutions are unbounded, and thus the attractor is destroyed in a global bifurcation.

3. Conclusion

We introduced a simple four-dimensional chaotic flow with the unusual feature of having a surface of equilibria. From a computational point of view, this system has a hidden attractor. It is interesting that the trajectory in this system goes very close to the surface of equilibria, but does not intersect with it. We are now studying systems in which the surface of equilibria is curved, including closed surfaces such as spheres and ellipsoids.

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