Nonideal Behavior of Analog Multipliers for Chaos Generation

Arturo Buscarino, *Member, IEEE*, Claudia Corradino, *Student Member, IEEE*, Luigi Fortuna, *Fellow, IEEE*, Mattia Frasca, *Senior Member, IEEE*, and Julien Clinton Sprott

***Abstract*—In this brief, nonideal behaviors of analog multipliers are explicitly taken into account in the design of nonlinear electronic circuits. The nonidealities of the analog multipliers led to further nonlinear terms that, instead of being considered as parasitic effects, are here explicitly accounted for and exploited to generate new complex dynamics, including chaos. In fact, despite the accuracy of analog devices, nonideal effects are always present, even if often neglected. An analog circuit based on a new nonlinear system is designed and tested, in which the nonideal terms in the multiplier input/output relation are essential for the onset of chaos, thus demonstrating the suitability of the approach.**

***Index Terms*—Analog multiplier, chaos, nonlinear circuits.**

I. Introduction

The study of nonlinear dynamics involves the application of methods derived from different fields. Although nonlinear mathematical models describing the behavior of real systems are studied using analytical tools and numerical simulations [1], an alternative and, in some cases, complementary approach is to use physical analogues of the mathematical models. Such analogues exploit analogies with mechanical [2], hydraulic [1], or electronic systems [3] to build an equivalent physical system that allows experimentation and includes the effects of noise and other nonideal behaviors that are always present in real systems.

Several important advantages are introduced by using electronic circuits based on the mathematical equations of a dynamical model. The investigation and characterization of the dynamical properties of the model are facilitated by collecting data with standard laboratory equipment such as oscilloscopes and data acquisition boards. Furthermore, the design of the analog circuit can benefit from detailed guidelines that can be defined [3], relying on simple low-cost components such as resistors, capacitors, and operational amplifiers. Observing complex phenomena using analog circuits thus reduces experimentation costs and increases the repeatability of sensitive experiments. Circuits can also be implemented with user-defined time scales, allowing for rapid observations in the presence of slow real dynamics. Several complex phenomena typical of nonlinear dynamics have been characterized by using electronic analogues: chaos [3], [4], hyperchaos [5], control of chaos [6], multiscroll attractors [7], [8], strange nonchaotic attractors [9], intermittency [10], synchronization, chimera states, and neuron-like dynamics [11].

In the design, electrical components are usually considered to be ideal. However, real circuit components may behave far from the ideal model. The usual design strategy is to define the working conditions in which components can be effectively considered ideal, usually avoiding the ranges in which nonideal behavior is problematic. For instance, it is known that, under specific conditions, the characteristic of diodes used to implement piecewise linear (PWL) functions can sensibly deviate from ideality [12].

In this brief, a new design strategy for circuits showing a complex behavior is presented, exploiting the nonideal properties of analog multipliers and, in particular, their dynamical effects. Apart from the nominal static model, multipliers based on the four-quadrant cell are known to have nonideal memory effects.

In this brief, a procedure to design and implement chaotic circuits considering a nonideal analog multiplier is presented. A new chaotic system with both static and dynamic nonlinearities is introduced, showing that the onset of chaos is strictly related to the presence of the normally unwanted effects. The remainder of this brief is organized as follows. In Section II, the nonideal dynamical model of two cascaded analog multipliers is presented and a procedure is adopted to evaluate its frequency-dependent parameters. In fact, most of the existing models for analog multiplier nonidealities consider constant parameters, but it turned out that, for our purposes, the model has to be refined by taking into account the broadband nature of the signals generated in the circuit. In Section III, the mathematical model of the new chaotic system is presented, while in Section IV, the design of the analog circuit and the experimental results are reported. Finally, Section V gives the conclusion.

II. Nonideal Effects in the Analog Multiplier

The most common nonlinearities in chaotic oscillators are polynomials or the products of two state variables. For this reason, nonlinear circuits designed from a mathematical chaotic system usually require analog multipliers, either voltage [14] or current multipliers [15].

Off-the-shelf analog multipliers, such as the AD633 adopted in this brief, are based on the four-quadrant multiplier [14].

Manuscript received September 2, 2015; accepted October 10, 2015. Date of publication November 25, 2015; date of current version March 24, 2016. This work has been supported in part by the EUROfusion Consortium under Association Contract ENEA-UNICT. This brief was recommended by Associate Editor E. Tlelo-Cuautle.

A. Buscarino, C. Corradino, L. Fortuna, and M. Frasca are with the Dipartimento di Ingegneria Elettrica Elettronica e Informatica, University of Catania, 95124 Catania, Italy (e-mail: arturo.buscarino@dieei.unict.it; claudia.corradino@dieei.unict.it; luigi.fortuna@dieei.unict.it; mattia.frasca@dieei.unict.it).

J. C. Sprott is with the Department of Physics, University of Wisconsin–Madison, Madison, WI 53706 USA (e-mail: sprott@physics.wisc.edu).

Color versions of one or more of the figures in this brief are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSII.2015.2503650

However, it has been proved in [13] that a static model is not sufficient to describe the input/output relationship of such multipliers. To overcome this limitation, in [13] a dynamical model considering the memory effect involving time derivatives of the inputs is introduced. According to this model, the output of the multiplier is given by

$$V_{\text{out}} = K\left(V_x(t)V_y(t) - T_A\dot{V}_x(t)V_y(t) - T_B V_x(t)\dot{V}_y(t)\right) \quad (1)$$

where $V_x(t)$ and $V_y(t)$ are the two inputs and K, T_A, and T_B are parameters characteristic of the multiplier. The values of the model parameters, that spoil the correct multiplication of the two input signals, can be estimated using a frequency-dependent analysis [13] that relies on a linear approximation assuming that the driving signals are narrowband.

This characterization works well for the case in which the signals are periodic, but it is not sufficient for the design of chaotic circuits. Chaotic signals, in fact, are characterized by a broadband spectrum. Usually, chaotic circuits produce signals with frequencies as high as 10–60 kHz or more [3]. To derive a dynamical model of the analog multiplier that can be used for chaotic circuit design, the broadband nature of the signals must be considered and the linear approximation assumed in [13] is not sufficiently detailed [16]. In this case, parameters are still constant, but their values depend on the bandwidth of the driving signals. In order to estimate the input/output relationship, a symbolic regression algorithm [17] on experimental data acquired from a cascade of two analog multipliers driven by broadband signals may be applied.

Following this approach, the two cascaded multipliers have been driven with an input noise filtered with a low-pass filter with different cutoff frequencies, and the output has been acquired using a NI-USB6255 data acquisition board with a sampling frequency $f_s = 400$ kHz. Acquired data are then used to derive the input/output relationship of the two analog multipliers through the symbolic regression algorithm that identifies the following mathematical form:

$$V_{\text{out}} = K(B)^2(x^3 - T(B)\dot{x}x^2) \quad (2)$$

where B is the bandwidth of the driving signals. It is worth noticing that higher order terms are associated with coefficients whose identified value is at least one order of magnitude smaller than T and, hence, they can be neglected. In Fig. 1, the functional dependence of the model parameters on the bandwidth B of the driving signals is shown.

The linear fitting of the model parameters evaluated from the data allows one to choose the suitable temporal rescaling for the nonlinear circuits so that the desired values of K and T are actually implemented. In particular, the further nonlinear term $T\dot{x}x^2$, whose realization does not need additional circuit components, can be suitably designed in order to generate the conditions under which chaotic dynamics are obtained. In order to show the effectiveness of the approach, in the following the design of a new chaotic circuit exploiting the presence of this further nonlinear term is presented. We remark that the focus of this brief is not the design of a circuit with a reduced number of components [18] but rather to show the occurrence of strange attractors due to the presence of nonideal terms in the cascaded analog multipliers.

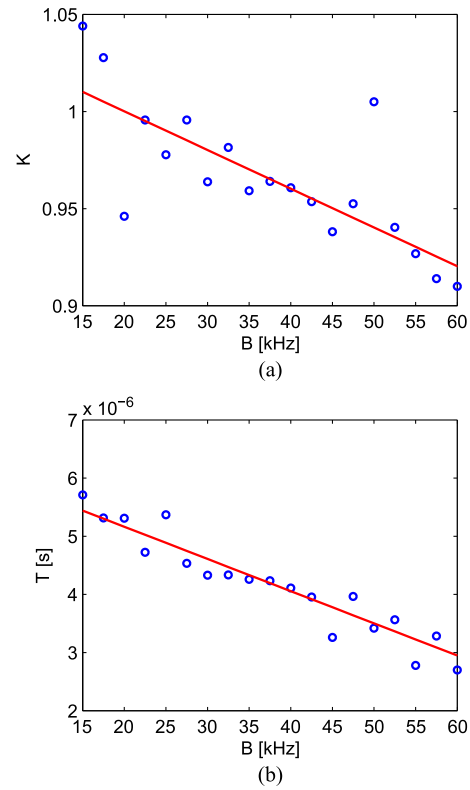

Fig. 1. Parameter estimation of the multiplier model using broadband signals. (a) K as a function of the bandwidth B of the noise input signal of the input signal and a linear fit $K = -0.002B + 1.04$. (b) T as a function of the bandwidth B of the noise input signal and a linear fit $T = -5.56 \cdot 10^{-8}B + 6.274 \cdot 10^{-6}$.

III. New Model for Chaotic Oscillations

Consider the following third-order autonomous dynamical system:

$$\begin{aligned} \dot{x} &= y - ax \\ \dot{y} &= \gamma x - Rx^3 + \Gamma z - by \\ \dot{z} &= \alpha - \beta z^2 - x^2 \end{aligned} \quad (3)$$

where a and b are positive dissipation rates and γ, R, α, β, and Γ are system parameters. This system is an extension of the conservative system proposed in [19] for gross modeling of plasma instabilities in magnetic plasma confinement devices such as the tokamak. In (3), the dissipative terms ax and by are explicitly considered, as well as a new feedback term Γz coupling the axisymmetric mode z to the second-order x–y dynamical system modeling instability amplitudes.

An important parameter for the system is γ, considered here as the bifurcation parameter. Choosing $a = b = 0.1$, $R = 1.09$, $\Gamma = 0.7$, $\alpha = 1$, and $\beta = 0.001$ gives the bifurcation diagram shown as blue in Fig. 2(a). For the range of γ considered, there is always a period-1 limit cycle with constant amplitude, similar to the behavior reported in [19] where the axisymmetric variable z has no effect on the instability amplitude.

Now reconsider (3) by including a further nonlinearity $F(x, \dot{x})$ given by

$$\begin{aligned} \dot{x} &= y - ax \\ \dot{y} &= \gamma x - \text{RF}(x, \dot{x}) + \Gamma z - by \\ \dot{z} &= \alpha - \beta z^2 - x^2 \end{aligned} \quad (4)$$

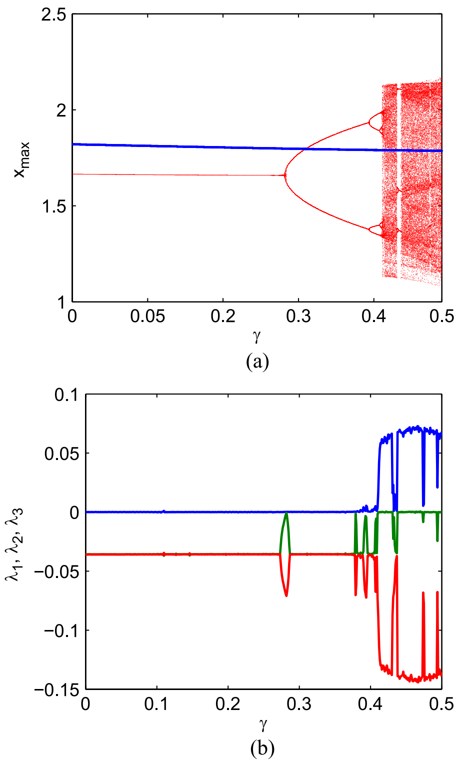

Fig. 2. (a) Bifurcation diagrams of system (3) (blue) and of system (4) (red) with respect to the bifurcation parameter γ. Local maxima of the state variable x are represented for each value of γ from 0 to 0.5. Other parameters are as indicated in the text. (b) Lyapunov spectrum for system (4) with respect to the bifurcation parameter γ, calculated according to the method in [22] for a total integration time $T = 10^7$ with step size $\Delta t = 0.01$. Other parameters are as indicated in the text.

where the form of $F(x, \dot{x})$ sensibly affects the system behavior. In particular, the dependence of F on the first-order derivative of x gives the possibility of chaotic dynamics, which is not obtained without this term. When $F(x, \dot{x}) = x^3 - \delta\dot{x}x^2$, where δ is a constant parameter, the nonlinear function consists of two terms, a static nonlinearity x^3 and a product nonlinearity $\dot{x}x^2$, involving a square and a derivative operation. This modification corresponds to the explicit introduction of nonideal terms for two cascaded analog multipliers realizing the cubic operation in the previous section. In addition, the $\dot{x}$ factor can be replaced by $y - ax$ to make (4) formally autonomous.

For $\delta = 0.16$, the system in (4) shows a bifurcation diagram with respect to γ, reported in red in Fig. 2(a), which includes a cascade of period doublings toward a chaotic window. Not shown in the figure is a region around $0.04 < \gamma < 0.08$ where a period-3 limit cycle coexists with the period-1 limit cycle. The three Lyapunov exponents shown in Fig. 2(b) confirm that the attractor is chaotic for $\gamma > 0.4$.

The attractor obtained for $\gamma = 0.45$ is shown in Fig. 3 projected onto the xy-plane. It has Lyapunov exponents given by (0.0692, 0, –0.1406) and a Kaplan–Yorke dimension of 2.4921. Its basin of attraction is relatively small, bounded, and with smooth boundaries.

In the range of the parameters considered, the occurrence of chaos is purely a consequence of the dynamical term $\dot{x}x^2$ in the nonlinear function F. In fact, setting $\delta = 0$, system (4) reduces to (3), whose bifurcation diagram [blue line in Fig. 2(a)] exhibits a period-1 limit cycle in the whole range of γ.

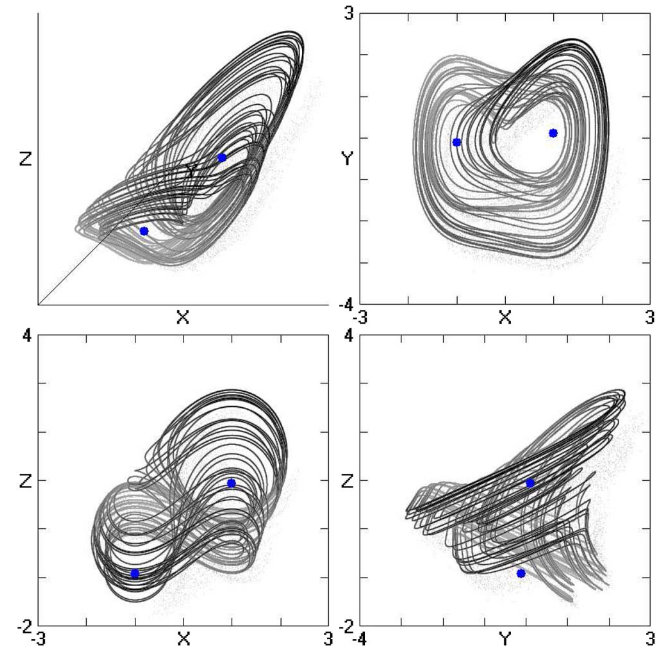

Fig. 3. Attractor for $a = b = 0.1$, $R = 1.09$, $\Gamma = 0.7$, $\alpha = 1$, $\beta = 0.001$, $\delta = 0.16$, and $\gamma = 0.45$. The two blue dots represent the saddle focus equilibrium points.

IV. Circuit Implementation

A circuit realizing the nonlinear model in (4) was designed and implemented with off-the-shelf components. In particular, the state variable approach in [3] was followed.

The designed circuit, whose electrical scheme is shown in Fig. 4, uses OP-AMPs U1A, U8B, and U2B to implement the active integrators made by a passive RC block and an algebraic adder, and the two cascaded analog multipliers U7 and U9 to realize the nonlinearity. OP-AMPs U11A and U13B allow the parameter γ to be adjusted, and OP-AMPs U10B and U12B together with diodes and resistors realize the square terms x^2 and z^2 as reported in [3].

The equations governing the circuit behavior are

$$\begin{aligned} \dot{x} &= \frac{1}{R_{C1}C_1}\left[\left(\frac{R_{F1}}{R_1}-1\right)x+\frac{R_{F2}}{R_2}y\right] \\ \dot{y} &= \frac{1}{R_{C2}C_2}\left[-\frac{R_{F2}}{R_4}F+\frac{R_3R_{F2}}{R_{14}R_{15}}x+\frac{R_{F2}}{R5}z+\left(\frac{R_{F2}}{R6}-1\right)y\right] \\ \dot{z} &= \frac{1}{R_{C3}C_3}\left[\frac{R_{F3}}{R_{10}}V-\frac{R_{F3}}{R_8}x^2-\frac{R_{F3}}{R_9}\frac{R_{33}}{R_{32}}z^2+\left(\frac{R_{F3}}{R_7}-1\right)z\right] \end{aligned} \tag{5}$$

where $V = 1$ V is a constant voltage implementing parameter α. The nonlinear function $F = x^3 - \delta\dot{x}x^2$ of (4) has parameters fixed by a temporal rescaling according to Fig. 1. Comparing function F with (2), it is possible to derive the relationships mapping the parameters of the cascaded multipliers to the circuit parameters. They are $R = K^2(R_{F2}/R_4)$ and $\delta = \kappa T$, where κ is the temporal rescaling introduced as $\tau = \kappa t$. It follows that $K = \sqrt{R(R_4/R_{F2})} = 0.93$, which, according to Fig. 1(a), is the value obtained when the spectrum of the input signal has a bandwidth of 55 kHz. Since the mathematical model has a bandwidth of 1.1 Hz, a coefficient $\kappa = 50\,000$ has been introduced, leading to a value of $T = 3.22 \cdot 10^{-6}$ as derived from Fig. 1(b). This implies that $\delta = 0.16$.

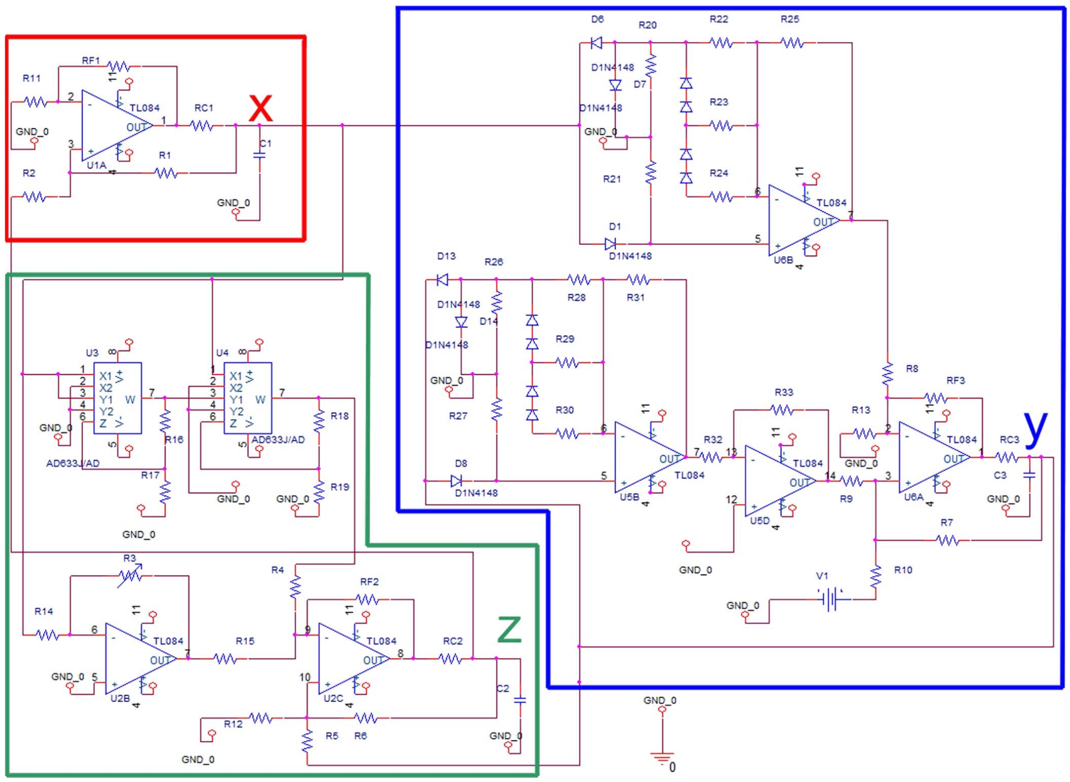

Fig. 4. Scheme of the chaotic circuit based on the nonidealities of the analog multiplier.

Since the temporal scaling factor is given by

$$\kappa = \frac{1}{R_{C1}C_1} = \frac{1}{R_{C2}C_2} = \frac{1}{R_{C3}C_3}. \quad (6)$$

$R_{C1} = R_{C2} = R_{C3} = 2\ \text{k}\Omega$ and $C_1 = C_2 = C_3 = 10$ nF have been set.

The other circuit parameters are chosen so that (5) matches (4). In particular, $R_2 = R_7 = R_8 = R_9 = R_{10} = R_{11} = R_{13} = R_{F1} = R_{F2} = R_{F3} = 100\ \text{k}\Omega$, $R_1 = R_6 = 112\ \text{k}\Omega$, $R_4 = 80\ \text{k}\Omega$, $R_5 = 142\ \text{k}\Omega$, $R_{12} = 72\ \text{k}\Omega$, $R_{14} = 200\ \text{k}\Omega$, $R_{15} = 100\ \text{k}\Omega$, $R_{16} = R_{18} = R_{33} = 1\ \text{k}\Omega$, $R_{17} = R_{19} = 9\ \text{k}\Omega$, $R_{20} = R_{21} = R_{22} = R_{23} = R_{26} = R_{27} = R_{28} = R_{29} = 10\ \text{k}\Omega$, $R_{24} = R_{30} = 4\ \text{k}\Omega$, $R_{25} = R_{31} = 20\ \text{k}\Omega$, and $R_{32} = 1\ \text{M}\Omega$, while R_3 is a 100 kΩ potentiometer.

Signals were acquired using a National Instrument (NI-USB6255) data acquisition board with a sampling frequency $f_s = 400$ kHz, and the circuit behavior with respect to γ was analyzed. The attractor shown in Fig. 5 was obtained from the circuit with $\gamma = 0.45$ and it agrees well with the simulated attractor in Fig. 3.

In Fig. 6, the experimental bifurcation diagram with respect to γ is reported. The circuit follows the same route to chaos as the model in (4) with a window of chaotic oscillation in good agreement with the predictions shown in Fig. 2(a).

Despite the correspondence of the observed behavior with that of model (4), the role of the circuitry implementing the PWL approximations has to be further discussed. As reported in [12], [20], and [21], working at high frequency, the diodes used to implement PWL functions introduce nonideal behaviors that can be captured by adopting the dynamical model proposed in [12]. In order to verify that the nonideal behavior of the PWL approximations is not responsible for the birth of chaos in the

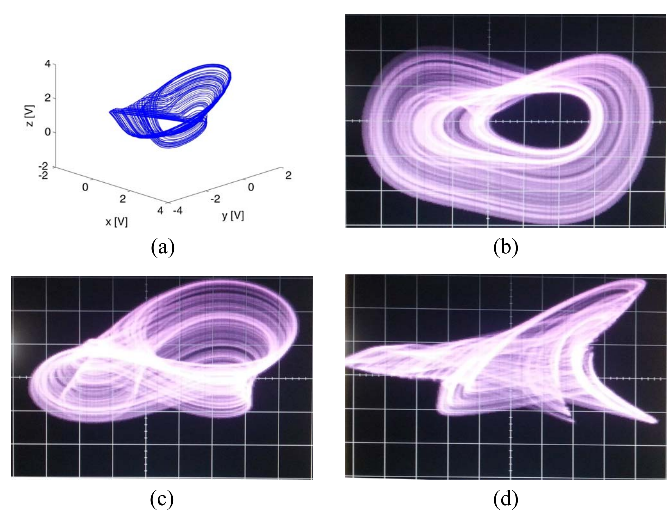

Fig. 5. (a) 3-D plot and oscilloscope traces of the attractor (b) on the xy-plane, (c) xz-plane, and (d) yz-plane from the circuit with $\gamma = 0.45$. The scale of the oscilloscope traces is 500 mV/div.

proposed circuit, the two cascaded analog multipliers implementing the function $F(x, \dot{x})$ have been replaced by the diode-based circuitry realizing a cubic nonlinearity reported in [3]. The behavior of the circuit with respect to γ is reported through the experimental bifurcation diagram traced in blue in Fig. 6, showing that a period-1 limit cycle is observed as predicted by model (4) with $\delta = 0$. No chaos has been obtained in this case.

The procedure adopted allows the design of a nonlinear circuit using only two analog multipliers to implement the whole nonlinear function $F(x, \dot{x})$. Taking into consideration an implementation of (4) not based on the introduced approach, it would

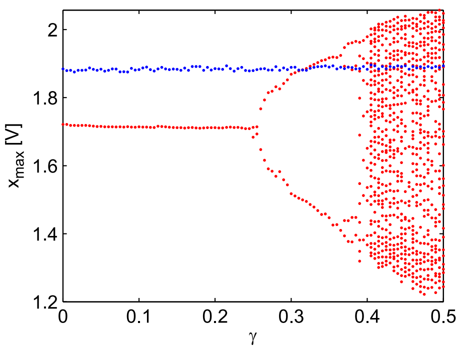

Fig. 6. Experimental bifurcation diagram with respect to the parameter γ shown in the local maxima of the state variable x for the circuit in Fig. 4 (red) and replacing analog multipliers with a diode-based device implementing the PWL approximation of $F = x^3$ (blue).

involve either a multiplier and an active differentiator block or, modifying equations to be formally autonomous, a multiplier and an integrator. In both cases, at least an analog multiplier is needed, thus introducing sensitive deviations from ideality, according to that shown in Section II. In addition, in the case of an implementation using a differentiator, it should be taken into account that it has intrinsic instabilities at high frequency. The design procedure adopted overcomes both problems.

V. Conclusion

The realization of reliable nonlinear electronic circuits for modeling complex phenomena and exploring their behavior is a fundamental topic in nonlinear science. In this brief, a new strategy to design nonlinear electronic circuits that exploits the intrinsic nonideal properties of analog multipliers has been proposed. In particular, the nonideal behavior of two cascaded analog multipliers has been characterized by deriving a dynamical model that takes into account unavoidable memory effects introduced by the four-quadrant cell. The response of analog multipliers in the presence of broadband signals, such as chaotic oscillations, has been studied, unveiling the relationship between the model parameters and the bandwidth of the input signals.

To demonstrate the effectiveness of the proposed design approach, a new chaotic circuit in which the nonideal terms of the analog multipliers play a crucial role in the onset of chaos has been introduced. The guidelines for designing reliable nonlinear electronic circuits exploiting the unwanted nonideal behaviors of analog devices introduced in this brief pave the way for the definition of a new class of nonideal chaotic circuits.

The circuit proposed here represents a new 3-D dissipative mathematical model that extends the conservative analytical model introduced in [19] by introducing a further feedback term. It represents the generalization of the so-called plasma gross behavior model introduced to mimic the occurrence of instabilities in the Joint European Torus fusion plasmas. During recent decades, significant effort has gone into deriving models that reproduce the behavior of the most relevant instabilities. The idea of incorporating nonideal behaviors can thus be adopted to tune coarse-grained parameters in order to fit the observed data.

References

[1] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. Boulder, CO, USA: Westview Press, 2001.
[2] F. C. Moon, *Chaotic Vibrations*. New York, NY, USA: Wiley, 2004.
[3] A. Buscarino, L. Fortuna, M. Frasca, and G. Sciuto, *A Concise Guide to Chaotic Electronic Circuits*. Berlin, Germany: Springer, 2014.
[4] A. Buscarino, L. Fortuna, M. Frasca, and G. Sciuto, "Design of time-delay chaotic electronic circuits," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 8, pp. 1888–1896, Aug. 2011.
[5] C. Li, J. C. Sprott, W. Thio, and Z. Huanqiang, "A new piecewise linear hyperchaotic circuit," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 61, no. 12, pp. 977–981, Dec. 2014.
[6] A. Jimenez-Triana, T. W. Kit-sang, G. Chen, and A. Gauthier, "Chaos control in Duffing system using impulsive parametric perturbations" *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 57, no. 4, pp. 305–309, Apr. 2010.
[7] M. E. Yalcin, J. A. K. Suykens, J. P. L. Vandewalle, and S. Özoguz, "Families of scroll grid attractors," *Int. J. Bifurcation Chaos*, vol. 12, no. 1, pp. 23–41, 2002.
[8] M. E. Yalcin, J. A. K. Suykens, and J. P. L. Vandewalle, "Experimental confirmation of 3- and 5-scroll attractors from a generalized Chua's circuit," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 47, no. 3, pp. 425–429, Mar. 2000.
[9] E. J. Ngamga *et al.*, "Recurrence analysis of strange nonchaotic dynamics in driven excitable systems," *Chaos, Interdiscip. J. Nonlinear Sci.*, vol. 18, no. 1, 2008, Art. ID 013128.
[10] A. Buscarino, L. Fortuna, and M. Frasca, "A new CNN-based chaotic circuit: Experimental results," *Int. J. Bifurcation Chaos*, vol. 19, no. 8, pp. 2609–2617, 2009.
[11] L. V. Gambuzza *et al.*, "Experimental investigation of chimera states with quiescent and synchronous domains in coupled electronic oscillators," *Phys. Rev. E*, vol. 90, no. 3, 2014, Art. ID 032905.
[12] E. Ortega-Torres, C. Sanchez-Lopez, and J. Mendoza-Lopez, "Frequency behavior of saturated nonlinear function series based on opamps," *Revista Mexicana de Fisica*, vol. 59, no. 6, pp. 504–510, 2013.
[13] A. Baranyi and L. O. Chua, "Dynamic model for the analog multiplier," *IEEE Trans. Circuits Syst.*, vol. CAS-29, no. 2, pp. 65–76, Feb. 1982.
[14] B. Gilbert, "A precise four-quadrant multiplier with subnanosecond response," *IEEE J. Solid-State Circuits*, vol. SSC-3, no. 4, pp. 365–373, Dec. 1968.
[15] G. Manganaro and J. Pineda de Gyvez, "A four-quadrant S^2I switched-current multiplier," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 45, no. 7, pp. 791–799, Jul. 1998.
[16] V. Rampa, "I/Q compensation of broadband direct-conversion transmitters," *IEEE Trans. Wireless Commun.*, vol. 13, no. 6, pp. 3329–3342, Jun. 2014.
[17] M. Schmidt and H. Lipson, "Distilling free-form natural laws from experimental data," *Science*, vol. 324, no. 5923, pp. 81–85, Apr. 2009.
[18] J. R. Piper and J. C. Sprott, "Simple autonomous chaotic circuits," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 57, no. 9, pp. 730–734, Sep. 2010.
[19] W. Arter, "Symmetry constraints on the dynamics of magnetically confined plasma," *Phys. Rev. Lett.*, vol. 102, no. 19, 2009, Art. ID 195004.
[20] C. Sanchez-Lopez, F. V. Fernandez, V. H. Carbajal-Gomez, E. Tlelo-Cuautle, and J. Mendoza-Lopez, "Behavioral modeling of SNFS for synthesizing multi-scroll chaotic attractors," *Int. J. Nonlinear Sci. Numerical Simul.*, vol. 14, no. 7/8, pp. 463–469, 2013.
[21] J. M. Munoz-Pacheco, E. Tlelo-Cuautle, I. Toxqui-Toxqui, C. Sanchez-Lopez, and R. Trejo-Guerra, "Frequency limitations in generating multi-scroll chaotic attractors using CFOAs," *Int. J. Electron.*, vol. 101, no. 11, pp. 1559–1569, 2014.
[22] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov exponents from a time series," *Phys. D, Nonlinear Phenom.*, vol. 16, no. 3, pp. 285–317, 1985.