International Journal of Bifurcation and Chaos, Vol. 26, No. 8 (2016) 1650137 (8 pages)

DOI: 10.1142/S0218127416501376

Simple Chaotic Flow with Circle and Square Equilibrium

Tomas Gotthans
Department of Radio Electronics,
Brno University of Technology,
Technicka 3082/12, Brno 616 00, Czech Republic
gotthans@feec.vutbr.cz

Julien Clinton Sprott
Department of Physics,
University of Wisconsin – Madison,
Madison, WI 53706, USA
sprott@physics.wisc.edu

Jiri Petrzela
Department of Radio Electronics,
Brno University of Technology, Technicka 3082/12,
Brno 616 00, Czech Republic

Received February 18, 2016

Simple systems of third-order autonomous nonlinear differential equations can exhibit chaotic behavior. In this paper, we present a new class of chaotic flow with a square-shaped equilibrium. This unique property has apparently not yet been described. Such a system belongs to a newly introduced category of chaotic systems with hidden attractors that are interesting and important in engineering applications. The mathematical model is accompanied by an electrical circuit implementation, demonstrating structural stability of the strange attractor. The circuit is simulated with PSpice, constructed, and analyzed (measured).

Keywords: Chaos; circle equilibrium; square equilibrium; attractor; bifurcation.

1. Introduction

Over the past three decades, finding new chaotic systems has attracted the attention of many researchers. Generating chaotic attractors may help one to understand the dynamics of real world systems. After many years of intensive research, numerous chaotic systems have been described. Recent developments include chaotic systems without any equilibrium points [Jafari & Sprott, 2013; Pham *et al.*, 2014; Tahir *et al.*, 2015; Pham *et al.*, 2016], with a single stable equilibrium [Molaie *et al.*, 2013], with a line of equilibrium points [Jafari & Sprott, 2013], and with a circular equilibrium [Gotthans & Petrzela, 2015]. In fact, many chaotic systems with unique equilibrium points have recently been presented [Wei *et al.*, 2015; Sprott, 2010, 2015; Sprott *et al.*, 2015]. Yet many undiscovered systems surely exist. The goal of this work is not only to present a new system with unique properties, but to extend the general knowledge about such systems. In this paper, we introduce a new category of systems with infinitely many equilibrium points arranged in a square. Because of its PWL (piecewise-linear) vector field, this category poses numerical challenges [Li *et al.*, 2015; Petrzela, 2012]. The system satisfies two of the three conditions for novelty [Sprott, 2011].

2. Mathematical Model

An extension of [Gotthans & Petrzela, 2015] shows that even simpler examples of systems with a circle of equilibrium points exist, one example of which is given by

$$\begin{aligned}\dot{x} &= z,\\ \dot{y} &= -z(ay + by^2 + xz),\\ \dot{z} &= x^2 + y^2 - 1,\end{aligned} \tag{1}$$

where a and b are constants. For $a = 5$, $b = 3$ and initial conditions $i_c = (0, 0, 0)^T$, this system has chaotic solutions with Lyapunov exponents $(0.0291, 0, -0.0697)$ and a Kaplan–Yorke dimension of 2.4172.

Linearizing the system by pieces yields a piecewise-linear (PWL) system given by

$$\begin{aligned}\dot{x} &= z,\\ \dot{y} &= -z(ay + b|y|) - x|z|,\\ \dot{z} &= |x| + |y| - 1,\end{aligned} \tag{2}$$

with a square equilibrium and solutions as shown in Fig. 1. Note that the attractors surround and link the equilibrium square. For $a = 5$, $b = 3$ and initial conditions $i_c = (0, 0, 0)^T$, this system has chaotic solutions with Lyapunov exponents $(0.0507, 0, -0.2272)$ and a Kaplan–Yorke dimension of 2.2230. It has a small Class 4 (finite) basin of attraction [Sprott & Xiong, 2015] with a smooth basin boundary.

Since system (2) is a set of piecewise-linear equations, an analytical solution in each linearized region can be obtained. When the solution reaches a boundary, its value can be used as the initial condition for the next analytical solution. Unfortunately, such an approach is impractical due to its algebraic complexity. System (2) can be further modified to have a rectangular equilibrium by changing the last state equation to $\dot{z} = |\frac{x}{\alpha^2}| + |\frac{y}{\beta^2}| - 1$, but that generalization leads to nothing new since it merely amounts to a rescaling of x and y, and thus it will not be considered.

The equilibrium points of system (2) can be obtained by solving $\dot{x} = 0$, $\dot{y} = 0$, and $\dot{z} = 0$,

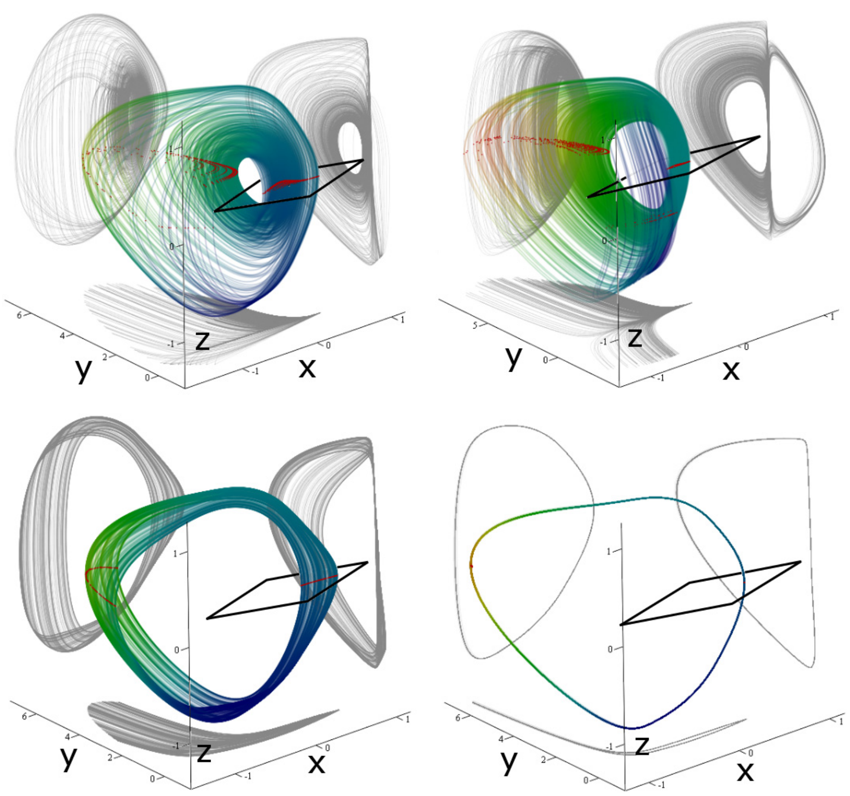


Fig. 1. Numerical integration. The individual state projections (corresponding to top left: $a = 5$ and $b = 3$, top right: $a = 10$ and $b = 3$, bottom left: $a = 3.5$ and $b = 3$ and bottom right: $a = 5$ and $b = 4$). The black quadrangle represents the square equilibrium. In the figures, Poincaré sections at $z = 0$ are shown in red. The gray plots are projections of the attractor onto the different axes.

giving

$$x = \mathbb{R},$$
$$y = \begin{cases} -1 - x, & \text{if } -1 < x \leq 0 \\ 1 + x, & \text{if } -1 < x \leq 0 \\ 1 - x, & \text{if } 0 < x < 1 \\ -1 + x, & \text{if } 0 < x < 1, \end{cases} \quad (3)$$
$$z = 0.$$

Dynamical motion in the close neighborhood of the equilibrium square is determined by the eigenvalues and associated eigenspaces established along this structure [Gotthans & Petrzela, 2015]. In order to estimate the Jacobian matrix, the partial derivatives of the state variables are required. The derivation of $|\cdot|$ can be obtained in several ways, for example, as condition statements (step functions), or as $\frac{x}{|x|}$. Alternatively, we use $\mathrm{sgn}(\cdot)$ for simplicity, where the function is not differentiable at 0.

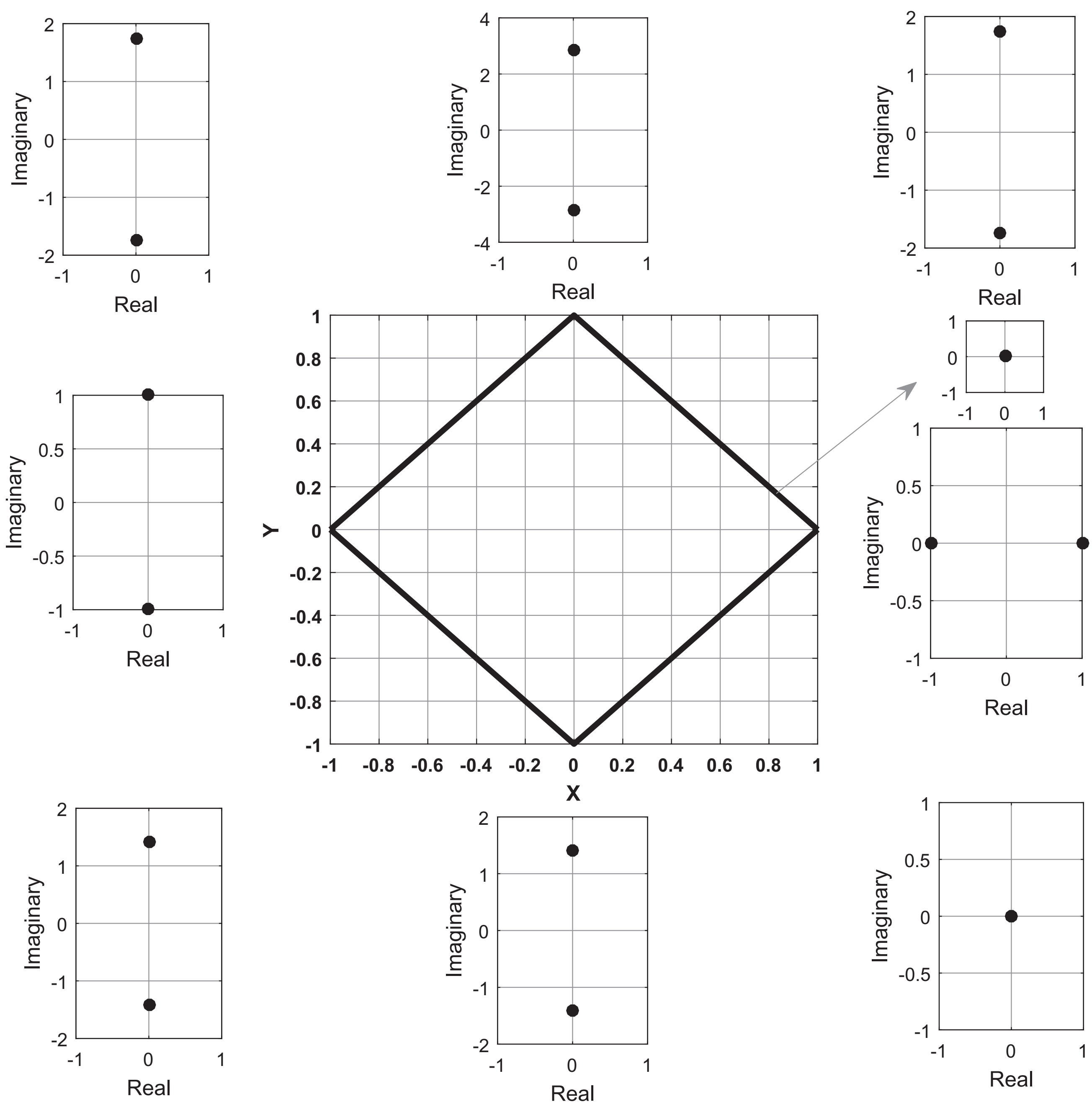


Fig. 2. For values $a = 5$ and $b = 3$ with $i_c = (0, 0, 0)^T$, the behavior of the eigenvalues $\lambda_{2,3}$ along the square equilibrium.

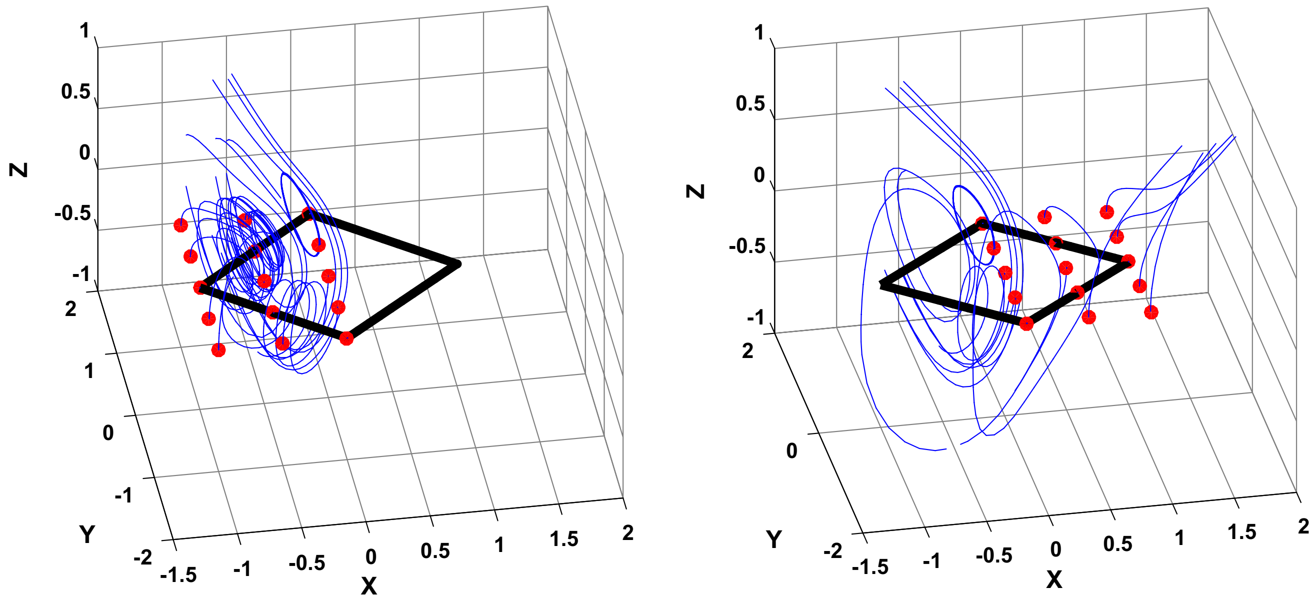


Fig. 3. Dynamical motion (blue curves) for initial conditions (red dots) near the equilibrium square (black).

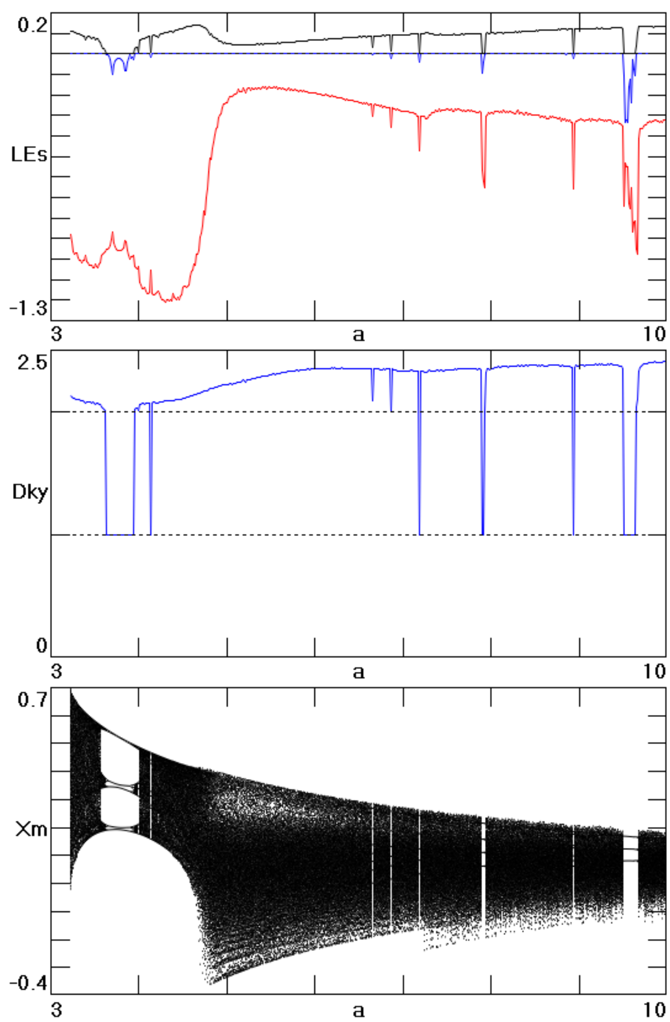


Fig. 4. Lyapunov exponents, Kaplan–Yorke dimension, and local maxima of x as a function of a for $b = 3$.

Note that the statements can always be substituted into the equations. In the case of Eq. (2), a state-dependent linearization matrix can be established as

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 \\ -|z| & -a \cdot z - b \cdot z \cdot \mathrm{sgn}(y) & -a \cdot y - b \cdot |y| - x \cdot \mathrm{sgn}(z) \\ \mathrm{sgn}(x) & \mathrm{sgn}(y) & 0 \end{pmatrix}. \tag{4}$$

The local behavior along the equilibrium square is determined by the eigenvalues, i.e. the roots of the parameterized characteristic equation

$$\begin{aligned} \det(\mathbf{J} - \lambda) = {} & -\lambda^2(a \cdot z + \lambda) + \mathrm{sgn}(x) \\ & \cdot [a \cdot z + \lambda + b \cdot z \cdot \mathrm{sgn}(y)] \\ & - \mathrm{sgn}(y) \cdot \{b \cdot \lambda |y| + |z| \\ & + \lambda[a \cdot y + b \cdot z \cdot \lambda + x \cdot \mathrm{sgn}(z)]\} \\ = {} & 0. \end{aligned} \tag{5}$$

The equilibrium points lie in the plane $z = 0$ and have eigenvalues given by

$$\lambda_1 = 0$$

$$\lambda_{2,3} = \pm\sqrt{\mathrm{sgn}(x) - a \cdot y \cdot \mathrm{sgn}(y) - b \cdot |y| \cdot \mathrm{sgn}(y)}. \tag{6}$$

The behavior along the square equilibrium located as noted in Eq. (3) can be seen in Fig. 2. A pair of purely imaginary eigenvalues represent an unstable center equilibrium. That means there

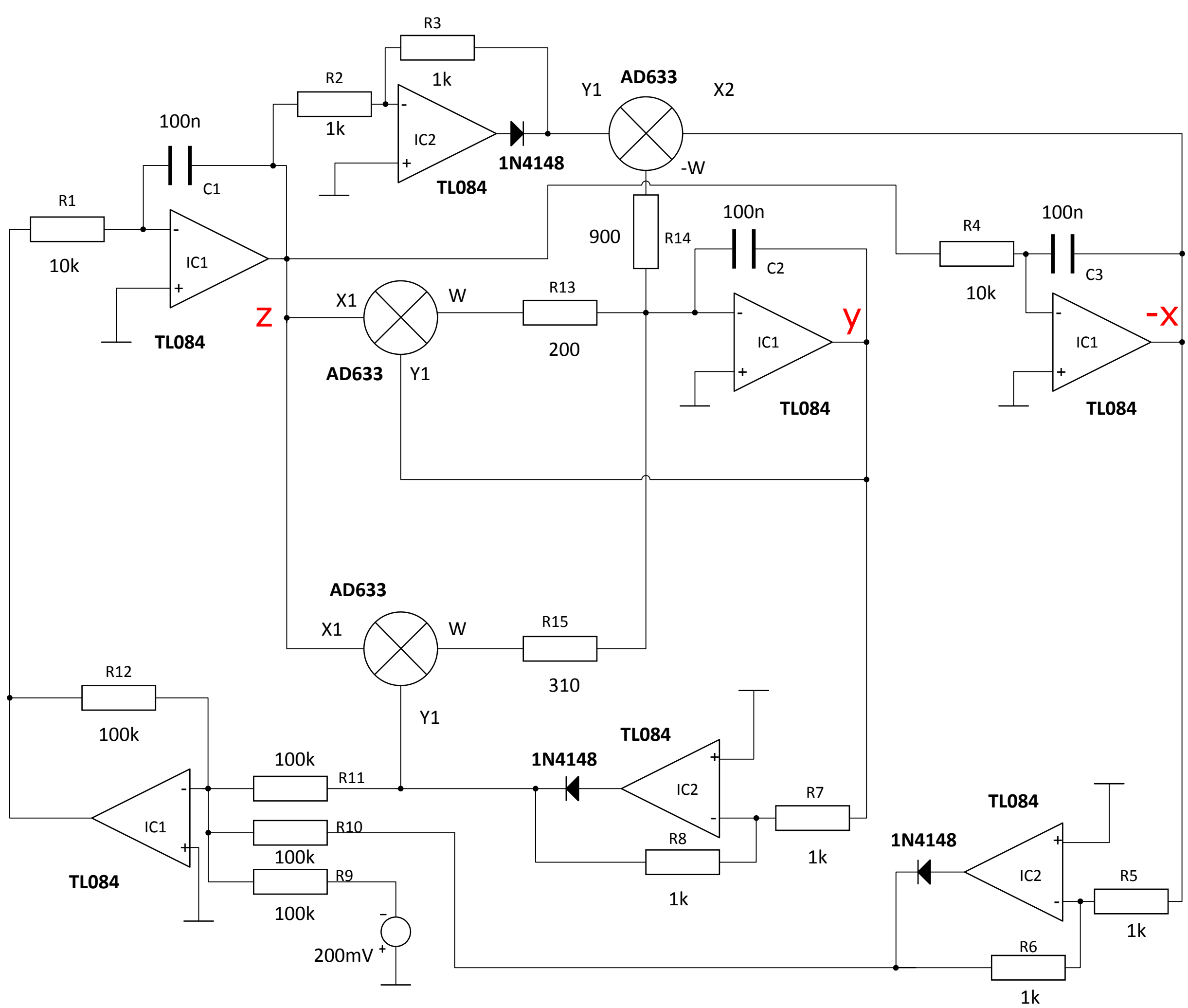


Fig. 5. Circuitry realization of proposed system.

are concentric periodic orbits around the equilibrium lines. Such phenomena can also be observed in Fig. 3. The pair of purely real eigenvalues imply an unstable saddle. Three-dimensional Bogdanov–Takens equilibria ($\lambda_{1,2,3} = 0$) are also present and are located at $(x, y, z) = (0.5, -0.5, 0)$ and $(x, y, z) = (0.825, 0.175, 0)$.

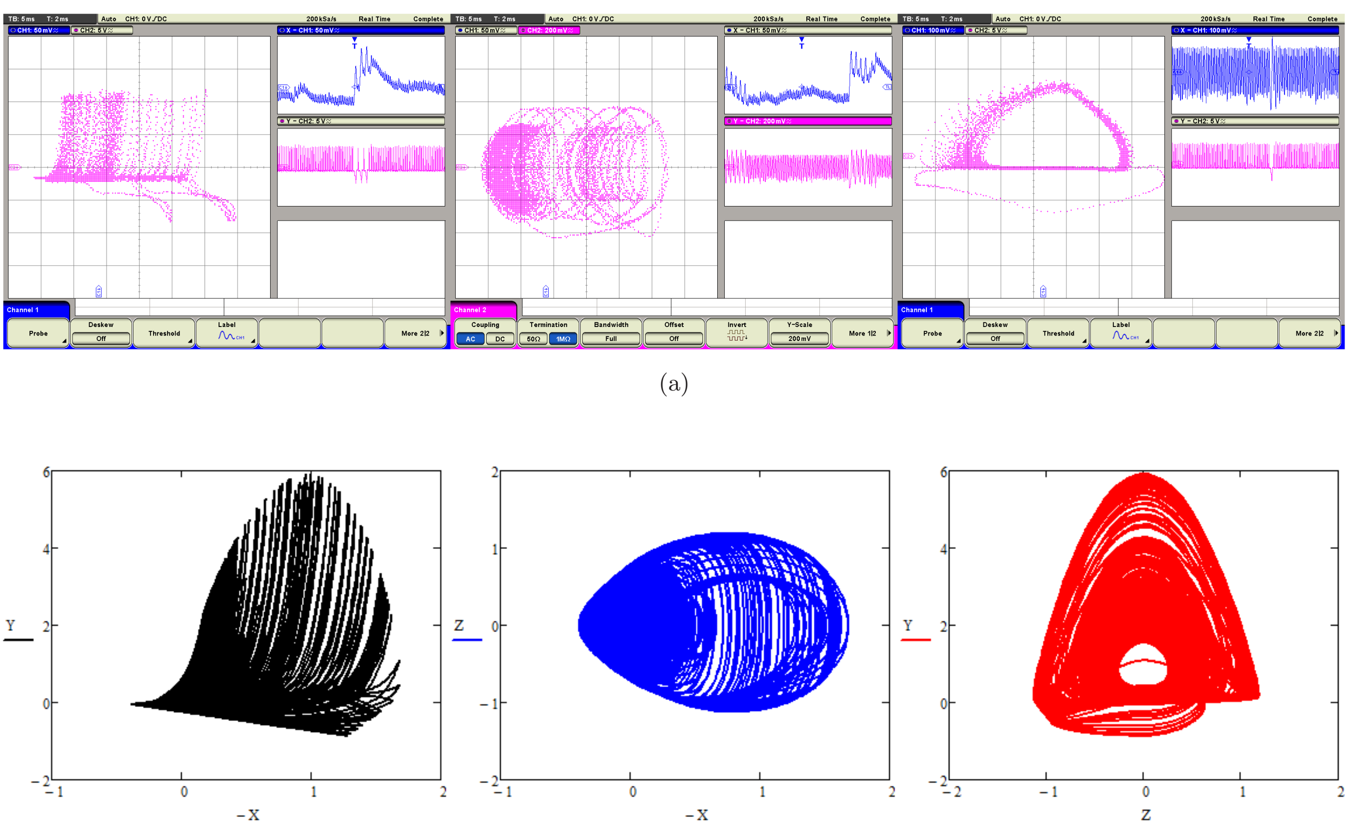
The dynamical motion near the equilibrium square is shown in Fig. 3. The negative value of the state variable x creates a periodic motion along the equilibrium. After reaching some point, chaotic motion can be observed (it can be considered as a very long transient). Therefore, initial conditions must be chosen carefully.

The dynamics of system (2) is examined by varying the parameter a with $b = 3$. Figure 4 shows the Lyapunov exponents, the Kaplan–Yorke dimension [Leonov & Kuznetsov, 2015], and the local maxima of x for $3 < a < 10$ with $b = 3$. They indicate that system (2) is chaotic for a large range of a except for a few small windows of periodicity.

3. Experimental Verification

Synthesis of electronic circuits is not only a way to accurately model a nonlinear dynamical system, but it is also a way to test the structural stability of the system. There are several ways to practically realize chaotic oscillators [Petrzela *et al.*, 2011]. To synthesize a circuit from the differential equations in system (2), integrator synthesis was chosen. Only a few basic building blocks are necessary: inverting integrators (TL084), summing amplifiers (TL084), three multipliers (AD633), and diodes (1N4148) (for absolute value modeling). The analog multiplier has all the nodes not displayed in Fig. 5 connected to ground.

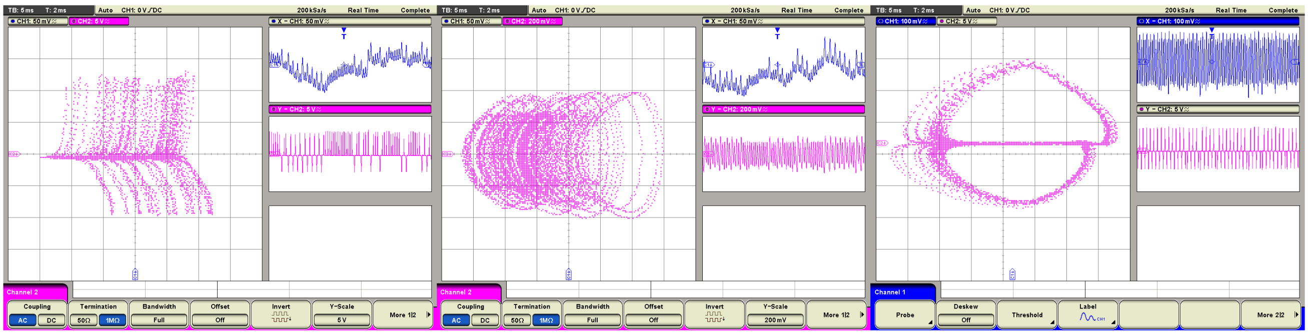
First, the proposed topology is verified with the PSpice 16.0 circuit simulator. The individual state variables are easily measured at the output nodes of the lossless integrators. Then the circuit was constructed, and the state variables were measured by a Rohde&Schwarz RTM 1052 oscilloscope.



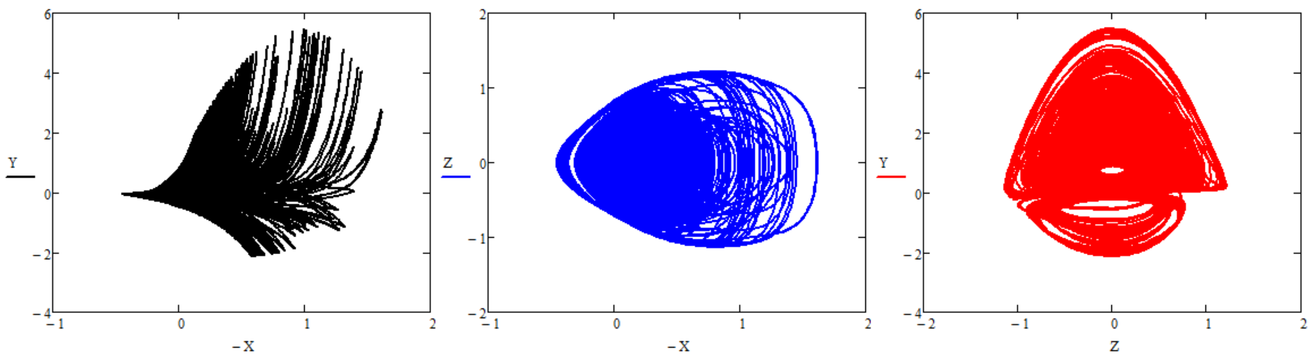
(a)

(b)

Fig. 6. (a) Measurements and (b) simulations on individual state variables (corresponding to $a = 5$ and $b = 3$).

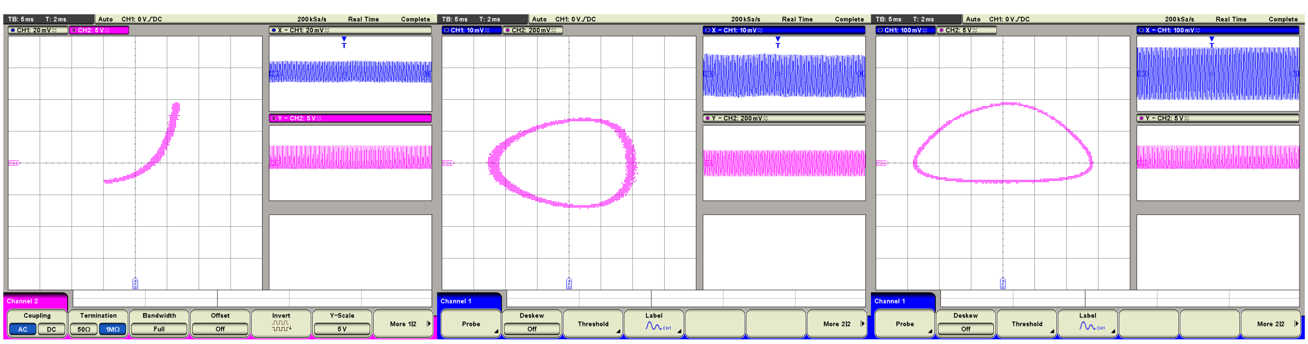


(a)

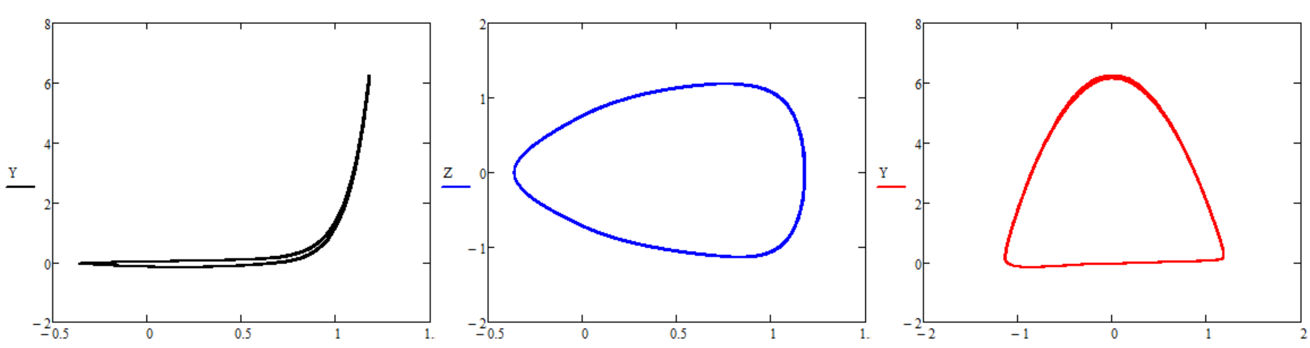


(b)

Fig. 7. (a) Measurements and (b) simulations on individual state variables (corresponding to $a = 5$ and $b = 2.2$).



(a)



(b)

Fig. 8. (a) Measurements and (b) simulations on individual state variables (corresponding to $a = 5$ and $b = 4$).

The screenshots as shown in Figs. 6–8 agree well with numerical calculations.

4. Conclusion

In this paper a three-dimensional system with a circular equilibrium was presented as an initial system. A PWL modification of the system with a square equilibrium was presented, and a variant with a rectangular equilibrium was mentioned. After providing a numerical analysis, the behavior around regions of the square equilibrium was described. Then the Lyapunov exponents, Kaplan–Yorke dimension and bifurcations were shown as a function of the bifurcation parameter a.

Finally we used an analog circuit to confirm the structural stability of the proposed system. The circuit was simulated, constructed, and tested. The measurements are in good agreement with numerical simulations. The proposed system is apparently new and may be the simplest such system with a square equilibrium and chaotic solutions, thus making it yet another example of a system with a hidden chaotic attractor [Leonov & Kuznetsov, 2013; Zelinka, 2016].

Acknowledgments

Research described in this paper was financed by Czech Ministry of Education in the frame of National Sustainability Program under grants LO1401 and GA15-22712S. For research, infrastructure of the SIX Center was used.

References

Gotthans, T. & Petrzela, J. [2015] "New class of chaotic systems with circular equilibrium," *Nonlin. Dyn.* **81**, 1143–1149.

Jafari, S. & Sprott, J. C. [2013] "Simple chaotic flows with a line equilibrium," *Chaos Solit. Fract.* **57**, 79–84.

Leonov, G. A. & Kuznetsov, N. V. [2013] "Hidden attractors in dynamical systems. From hidden oscillations in Hilbert–Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits," *Int. J. Bifurcation and Chaos* **23**, 1330002-1–69.

Leonov, G. A. & Kuznetsov, N. V. [2015] "A short survey on Lyapunov dimension for finite dimensional dynamical systems in Euclidean space," arXiv:1510.03835 (42 pages).

Li, C., Sprott, J. C. & Thio, W. [2015] "Linearization of the Lorenz system," *Phys. Lett. Sec. A: Gen. Atom. Solid State Phys.* **379**, 888–893.

Molaie, M., Jafari, S., Sprott, J. C. & Golpayegani, S. [2013] "Simple chaotic flows with one stable equilibrium," *Int. J. Bifurcation and Chaos* **23**, 1350188-1–7.

Petrzela, J., Gotthans, T. & Hrubos, Z. [2011] "Modeling deterministic chaos using electronic circuits," *Radioengineer.* **20**, 438–444.

Petrzela, J. [2012] "Optimal piecewise-linear approximation of the quadratic chaotic dynamics," *Radioengineer.* **21**, 20–28.

Pham, V., Volos, C., Jafari, S., Wei, Z. & Wang, X. [2014] "Constructing a novel no-equilibrium chaotic system," *Int. J. Bifurcation and Chaos* **24**, 1450073-1–6.

Pham, V., Vaidyanathan, S., Volos, C., Jafari, S. & Kingni, S. T. [2016] "A no-equilibrium hyperchaotic system with a cubic nonlinear term," *Optik-Int. J. Light and Electron Opt.* **127**, 3259–3265.

Sprott, J. C. [2010] *Elegant Chaos: Algebraically Simple Chaotic Flows* (World Scientific, Singapore).

Sprott, J. C. [2011] "A proposed standard for the publication of new chaotic systems," *Int. J. Bifurcation and Chaos* **21**, 2391–2394.

Sprott, J. C. [2015] "Strange attractors with various equilibrium types," *Eur. Phys. J. Special Topics* **224**, 1409–1419.

Sprott, J. C. & Xiong, A. [2015] "Classifying and quantifying basins of attraction," *Chaos* **25**, 083101-1–7.

Sprott, J. C., Jafari, S., Pham, V. & Hosseni, Z. S. [2015] "A chaotic system with a single unstable node," *Phys. Lett. A* **379**, 2030–2036.

Tahir, F. R., Jafari, S., Pham, V. & Wang, X. [2015] "A novel no-equilibrium chaotic system with multiwing butterfly attractors," *Int. J. Bifurcation and Chaos* **25**, 1550056-1–11.

Wei, Z., Sprott, J. C. & Chen, H. [2015] "Elementary quadratic chaotic flows with a single non-hyperbolic equilibrium," *Phys. Lett. Sect. A: Gen. Atom. Solid State Phys.* **379**, 2184–2187.

Zelinka, I. [2016] "Evolutionary identification of hidden chaotic attractors," *Engin. Appl. Artif. Intell.* **50**, 159–167.