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Categorizing Chaotic Flows from the Viewpoint of Fixed Points and Perpetual Points

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Perpetual points represent a new interesting topic in the literature of nonlinear dynamics. This paper introduces some chaotic flows with four different structural features from the viewpoint of fixed points and perpetual points.

Keywords: Perpetual points; fixed points; chaotic flows.

1. Introduction

Recently, many new chaotic flows have been discovered that are not associated with a saddle point, including ones without any equilibrium points, with only stable equilibria, or with a line containing infinitely many equilibrium points [Jafari & Sprott, 2013, 2015; Jafari *et al.*, 2013; Jafari *et al.*, 2015b; Kingni *et al.*, 2014; Lao *et al.*, 2014; Molaie *et al.*, 2013; Pham *et al.*, 2014a; Pham *et al.*, 2014b; Pham *et al.*, 2014c; Pham *et al.*, 2014d; Pham *et al.*, 2015; Shahzad *et al.*, 2015; Tahir *et al.*, 2015; Pham *et al.*, 2016; Goudarzi *et al.*, 2016]. The attractors of these categories have been called hidden attractors [Leonov & Kuznetsov, 2014; Leonov *et al.*, 2014; Leonov & Kuznetsov, 2011; Leonov *et al.*, 2011, 2012; Leonov *et al.*, 2015; Leonov & Kuznetsov, 2013a, 2013b, 2013c; Bragin *et al.*, 2011; Kuznetsov *et al.*, 2010; Kuznetsov *et al.*, 2011; Leonov, 2010; Kiseleva *et al.*, 2017; Andrievsky *et al.*, 2016; Kiseleva *et al.*, 2016; Bianchi *et al.*, 2016; Kuznetsov *et al.*, 2016b]. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or aircraft wing. The classical attractors of Lorenz [Lorenz, 1963], Rössler [Rössler, 1976], Chen [Chen & Ueta, 1999], Sprott (cases B to S) [Sprott, 1994], and other well-known attractors are excited from unstable equilibria. Thus, one can find these attractors by starting a trajectory from a point on the unstable manifold in the neighborhood of an unstable equilibrium [Leonov *et al.*, 2011].

One of the interesting topics in nonlinear dynamics that was recently proposed is perpetual points [Prasad, 2015a; Dudkowski *et al.*, 2015; Prasad, 2015b; Jafari *et al.*, 2015a]. It has been shown that these points can be used to locate hidden attractors and to find coexisting attractors in multistable systems [Prasad, 2015a].

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In this note, we introduce some chaotic flows with four different structural features from the viewpoint of fixed points (equilibria) and perpetual points. We believe that this categorization can help researchers to investigate the common and uncommon features between fixed points (equilibria) and perpetual points and the possible roles of both of them in forming strange attractors.

In the next section we describe perpetual points in a simple way. Section 3 introduces (a) a chaotic system that has simultaneous fixed points and a perpetual point, (b) a new chaotic flow with a fixed point but without any perpetual points, (c) a chaotic system with perpetual points but without any fixed points, and (d) a chaotic system with neither fixed points nor perpetual points, thus showing that neither fixed points nor perpetual points are needed for a system to exhibit chaos. Finally, Sec. 4 gives conclusions.

2. Perpetual Points

In a general dynamical system, we have

$$\begin{aligned} v_1 &= \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n)\\ v_2 &= \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n)\\ &\vdots\\ v_n &= \dot{x}_n = f_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{1}$$

$x_1, x_2, \ldots, x_n$ are dynamical variables (states), $v_1, v_2, \ldots, v_n$ are the time derivatives of the states (velocities) and $f_1(X), f_2(X), \ldots, f_n(X)$ are the evolution equations (velocity vectors). The fixed points of the above system are points $(x_1^*, x_2^*, \ldots, x_n^*)$ at which the velocities of all states are zero. The fixed points are an important structural feature of systems [Prasad, 2015a; Ott, 2002; Strogatz, 2014].

As shown in the following equations, acceleration is the time derivative of velocity

$$\begin{aligned} a_1 &= \ddot{x}_1 = \dot{x}_1\frac{\partial f_1}{\partial x_1} + \dot{x}_2\frac{\partial f_1}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_1}{\partial x_n}\\ &= v_1\frac{\partial f_1}{\partial x_1} + v_2\frac{\partial f_1}{\partial x_2} + \cdots + v_n\frac{\partial f_1}{\partial x_n}\\ a_2 &= \ddot{x}_2 = \dot{x}_1\frac{\partial f_2}{\partial x_1} + \dot{x}_2\frac{\partial f_2}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_2}{\partial x_n}\\ &= v_1\frac{\partial f_2}{\partial x_1} + v_2\frac{\partial f_2}{\partial x_2} + \cdots + v_n\frac{\partial f_2}{\partial x_n}\\ &\vdots\\ a_n &= \ddot{x}_n = \dot{x}_1\frac{\partial f_n}{\partial x_1} + \dot{x}_2\frac{\partial f_n}{\partial x_2} + \cdots + \dot{x}_n\frac{\partial f_n}{\partial x_n}\\ &= v_1\frac{\partial f_n}{\partial x_1} + v_2\frac{\partial f_n}{\partial x_2} + \cdots + v_n\frac{\partial f_n}{\partial x_n} \end{aligned} \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are the second derivatives of the states (accelerations). Perpetual points are points like $(x_1^*, x_2^*, \ldots, x_n^*)$ at which all the accelerations are zero but the velocities are not [Prasad, 2015a].

3. Four Categories of Chaotic Flows Depending on the Existence of Fixed Points and Perpetual Points

3.1. *A chaotic flow with fixed points and perpetual points*

Jafari and Sprott in [Molaie *et al.*, 2013] introduced simple chaotic systems with a line of equilibria. They were inspired by the structure of the conservative Sprott case A system. Equation (3) is a mathematical form of the first one of those simple systems with only six terms.

$$\begin{aligned} \dot{x} &= y\\ \dot{y} &= -x + yz\\ \dot{z} &= -x - 15xy - xz. \end{aligned} \tag{3}$$

By setting the right-hand side of these equations to zero, the fixed points are given by

$$\begin{aligned} \dot{x} &= y = 0 \to y = 0\\ \dot{y} &= -x + yz = 0 \to x = 0\\ \dot{z} &= -x - 15xy - xz = 0. \end{aligned} \tag{4}$$

Having $x = 0$ and $y = 0$, $\dot{z}$ will be zero independent of the value of z, and thus there is a line of equilibria at $(0, 0, z)$ with eigenvalues $(\frac{z+\sqrt{z^2-4}}{2}, \frac{z-\sqrt{z^2-4}}{2}, 0)$ and no other equilibria. Calculation of the perpetual points gives the following equations

$$\begin{aligned} \ddot{x} &= \dot{y} = -x + yz = 0\\ \ddot{y} &= -\dot{x} + \dot{y}z + y\dot{z} = -y - xy - 15xy^2 - xyz = 0\\ \ddot{z} &= -\dot{x} - 15\dot{x}y - 15x\dot{y} - \dot{x}z - x\dot{z}\\ &= -y - 15y^2 - yz + x^2 + 15x^2y + x^2z = 0. \end{aligned} \tag{5}$$

The solution of these equations has two real values, one of which is the line of equilibria, and the

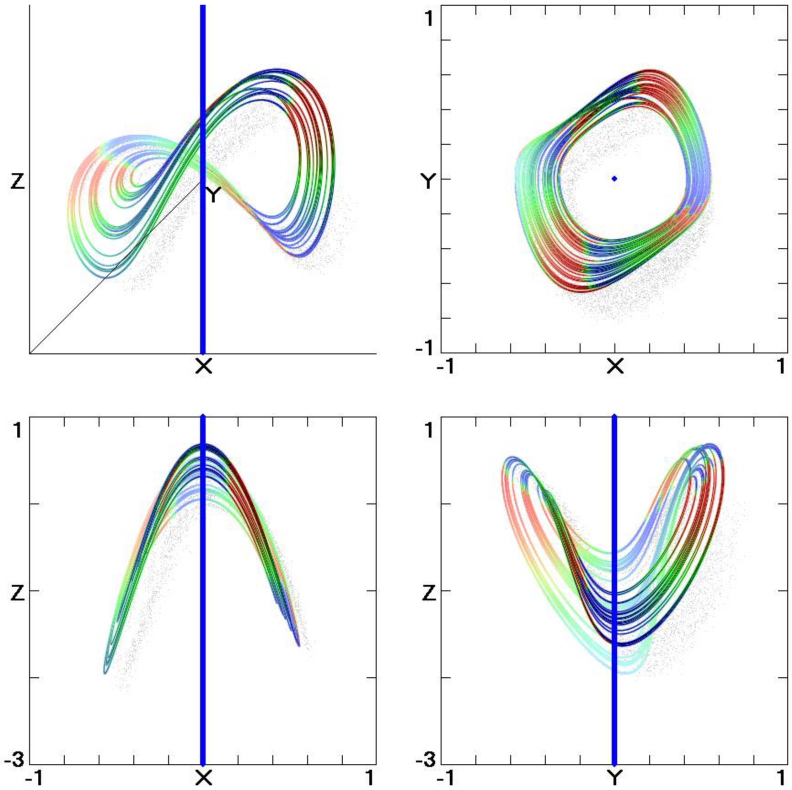

Fig. 1. The strange attractor and its projections for system (3) with initial conditions $(0, 0.5, 0.5)$.

other is the only perpetual point of the system, given approximately by $(-0.4675, 0.2185, -2.1391)$. Projections of the strange attractor are shown in Fig. 1. Lyapunov exponents are a measure of divergence of nearby trajectories. The positive largest Lyapunov exponent is often considered as an indication of chaotic behavior [Hilborn, 2000; Sprott, 2003]. In other words, the sensitive dependence on initial conditions is one of the basic features of chaos and Lyapunov exponents provide quantitative measures of response sensitivity of a dynamical system to small changes in initial conditions [Skokos, 2010]. There are different methods for calculating Lyapunov exponents which sometimes result in different values [Kuznetsov *et al.*, 2014a; Kuznetsov & Leonov, 2005; Kuznetsov *et al.*, 2014b; Leonov & Kuznetsov, 2007; Leonov *et al.*, 2016; Leonov, 2016; Leonov & Mokaev, 2016; Kuznetsov *et al.*, 2016a; Leonov & Kuznetsov, 2015]. Lyapunov exponents of the system calculated using Wolf's algorithm [Wolf *et al.*, 1985] (with initial conditions $(0, 0.5, 0.5)$ and computation time 60 000) are $(0.0717, 0, -0.5232)$, and the Kaplan–Yorke dimension is 2.1371. We believe that the Wolf algorithm has acceptable results with enough calculation time for Lyapunov exponents. In this and the following figures, the local largest Lyapunov exponent is mapped into a color palette, with red indicating the most positive value and blue the most negative value, and the blue line or dots are the positions of the fixed points.

3.2. *A chaotic flow with a fixed point but without any perpetual points*

In the search for a chaotic flow with a fixed point but without any perpetual points, we designed a flow with cubic nonlinearities [Eq. (6)] in which there is only one fixed point and no perpetual points:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -x - 4x^2y - x^2z. \end{aligned} \tag{6}$$

It has one fixed point in the origin:

$$\begin{aligned} \dot{x} &= y = 0 \\ \dot{y} &= z = 0 \\ \dot{z} &= -x - 4x^2y - x^2z = 0 \to x = 0 \end{aligned} \tag{7}$$

with eigenvalues $(\frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i, -1)$. Applying Eq. (2) for calculation of perpetual points gives

$$\begin{aligned}
&\ddot{x} = \dot{y} = z = 0\\
&\ddot{y} = \dot{z} = -x - 4x^2y = 0\\
&\ddot{z} = -\dot{x} - 8xy\dot{x} - 4x^2\dot{y} - 2xz\dot{x} - x^2\dot{z}\\
&\quad = -y - 8xy^2 = 0.
\end{aligned} \tag{8}$$

Setting the right-hand side of the above equations to zero gives the solution

$$x = y = z = 0. \tag{9}$$

Since the only solution is a fixed point, there are no perpetual points. Projections of the strange attractor are shown in Fig. 2. Lyapunov exponents of this system are $(0.1642, 0, -0.6399)$ (with initial conditions $(0, 1, 0.7)$ and computation time 60 000), and the Kaplan–Yorke dimension is 2.2566.

3.3. *A chaotic flow without any fixed points but with perpetual points*

One of the simplest dissipative examples of chaotic flows with no equilibria is the Wei system [Wei, 2011] which is a modification of the Sprott case D system in [Sprott, 1994] [Eq. (10)].

$$\begin{aligned}
\dot{x} &= -y\\
\dot{y} &= x + z\\
\dot{z} &= 2y^2 + xz - 0.35.
\end{aligned} \tag{10}$$

Applying Eq. (2) to calculate the perpetual points gives

$$\begin{aligned}
&\ddot{x} = -\dot{y} = -x - z = 0\\
&\ddot{y} = \dot{x} + \dot{z} = -y + 2y^2 + xz - 0.35 = 0\\
&\ddot{z} = 4y\dot{y} + \dot{x}z + x\dot{z}\\
&\quad = 4y(x+z) - yz + 2xy^2 + x^2z - 0.35x = 0.
\end{aligned} \tag{11}$$

Thus this system has the perpetual points

$$\left(0, \frac{5+\sqrt{95}}{20}, 0\right), \quad \left(0, \frac{5-\sqrt{95}}{20}, 0\right). \tag{12}$$

Projections of the strange attractor are shown in Fig. 3. The Lyapunov exponents of this system are $(0.0776, 0, -1.5008)$ (with initial conditions

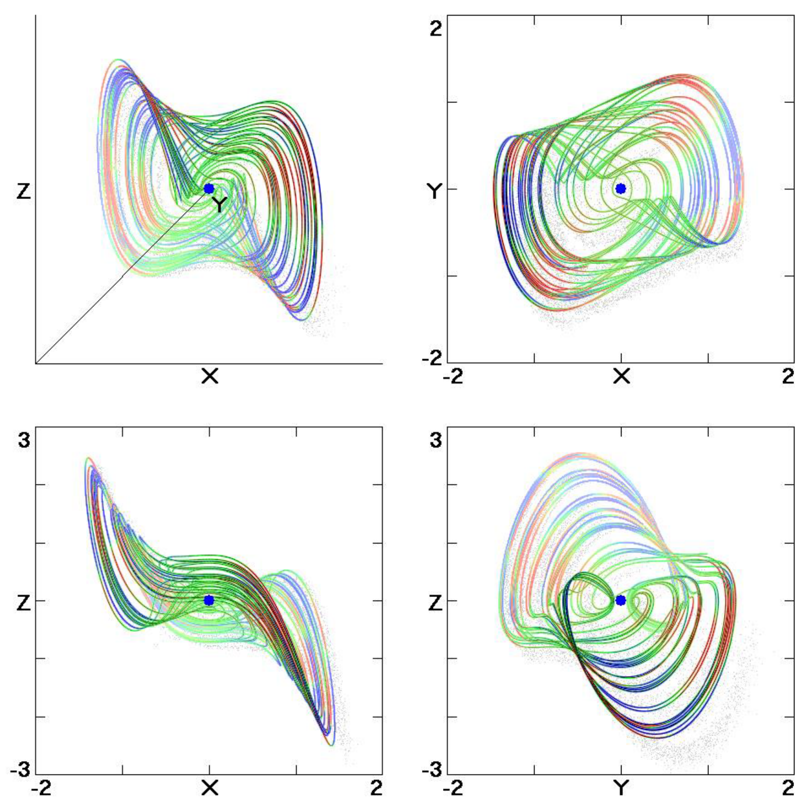

Fig. 2. The strange attractor and its projections for system (6) with initial conditions $(0, 1, 0.7)$.

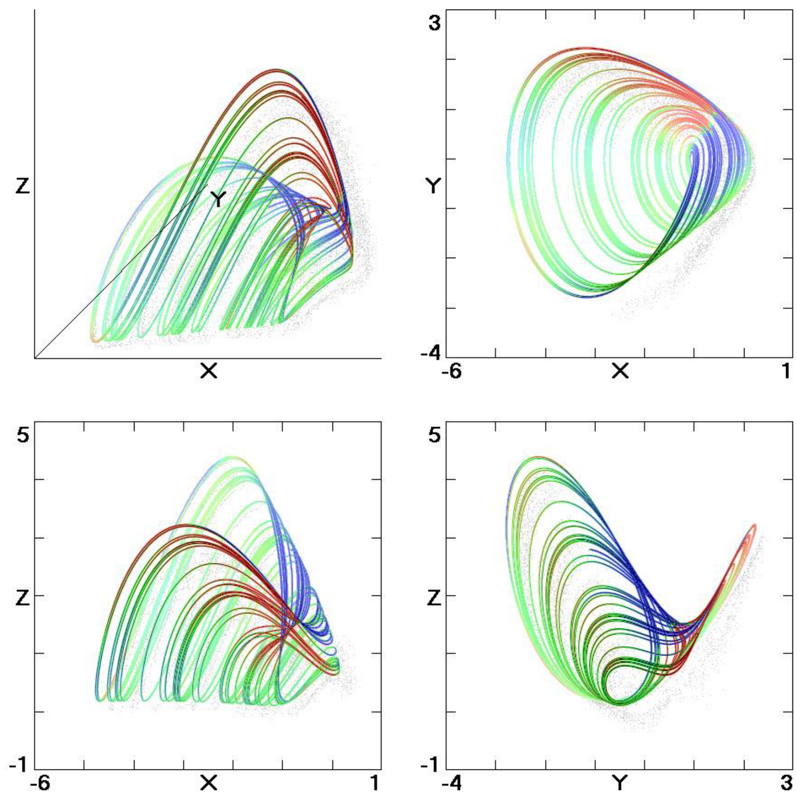

Fig. 3. The strange attractor and its projections for system (10) with initial conditions $(0, 0.37, 1)$.

$(0, 0.37, 1)$ and computation time 60 000), and the Kaplan–Yorke dimension is 2.0517.

3.4. *A chaotic flow without any fixed points and without any perpetual points*

Consider the Nosé system [Hoover *et al.*, 2016] [Eq. (13)] which has conservative tori surrounded by a chaotic sea:

$$\dot{x} = \frac{y}{z^2}, \quad \dot{y} = -x, \qquad \dot{z} = w, \quad \dot{w} = \frac{y^2}{z^3} - \frac{2}{z}. \tag{13}$$

This system does not have any fixed points or perpetual points. Setting the right-hand side of these equations to zero shows that there is a conflict between the first and last equations as follows:

$$\dot{x} = \frac{y}{z^2} = 0 \to y = 0 \qquad \dot{w} = \frac{y^2}{z^3} - \frac{2}{z} = 0 \to -\frac{2}{z} = 0. \tag{14}$$

Hence, this system does not have any fixed points. For calculating perpetual points, Eq. (2) applied to the system gives

$$\begin{aligned}
\ddot{x} &= \frac{\dot{y}z^2 - 2yz\dot{z}}{z^4} = \frac{-xz^2 - 2yzw}{z^4} \\
&= 0 \to xz = -2yw \\
\ddot{y} &= -\dot{x} = -\frac{y}{z^2} = 0 \to y = 0 \\
\ddot{z} &= \dot{w} = \frac{y^2}{z^3} - \frac{2}{z} = 0 \to -\frac{2}{z} = 0 \\
\ddot{w} &= \frac{2y\dot{y}z^3 - 3z^2\dot{z}y^2}{z^6} + \frac{2\dot{z}}{z^2} \\
&= \frac{-2yxz^3 - 3z^2wy^2}{z^6} + \frac{2w}{z^2} = 0.
\end{aligned} \tag{15}$$

This system does not have any perpetual points, but it has a chaotic solution for initial conditions $(3, 3, 1, 0)$. Some projections of the chaotic sea are shown in Fig. 4. The system is conservative with Lyapunov exponents $(0.0019, 0, 0, -0.0019)$ (for initial conditions $(3, 3, 1, 0)$ and computation time 60 000).

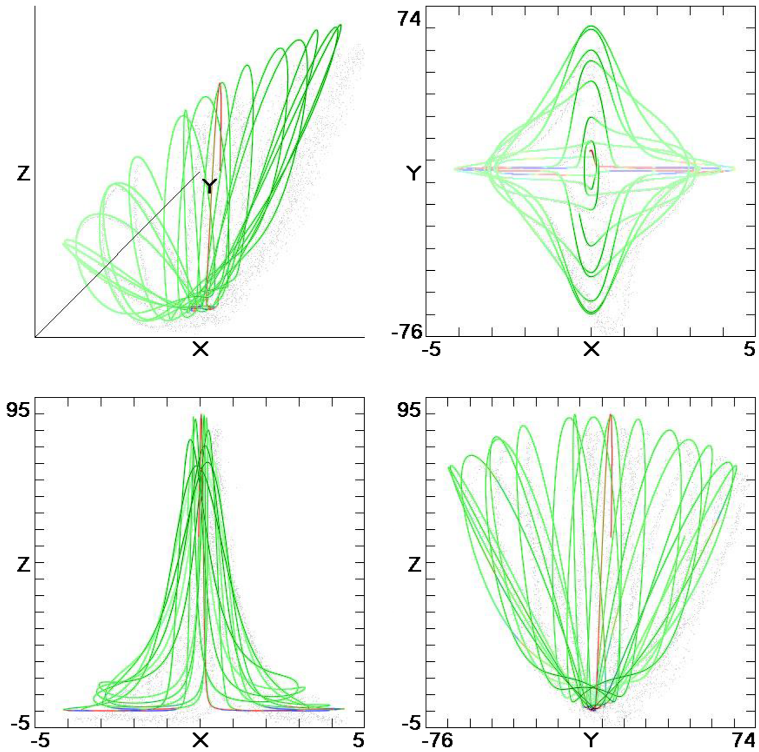

Fig. 4. Some projections of the chaotic sea of the 4D system (13) with initial conditions $(3, 3, 1, 0)$.

4. Conclusion

Categorizing dynamical systems into systems with or without perpetual points is a new topic in nonlinear dynamics. This paper studies the application of perpetual points in the existence of strange attractors. It reviews four types of chaotic flows: flows having fixed points and a perpetual point, flows with a fixed point but without perpetual points, flows with no fixed points but with perpetual points, and finally flows with neither fixed points nor perpetual points. In other words, systems in all four combinations of having or not having fixed points and perpetual points can show chaotic behavior. Thus, the existence of strange attractors cannot be demonstrated by the existence of fixed points and perpetual points. These categories may be useful for further research on perpetual points.

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