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Constructing chaotic systems with conditional symmetry

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Abstract Asymmetric dynamical systems sometimes admit a symmetric pair of coexisting attractors for reasons that are not readily apparent. This phenomenon is called conditional symmetry and deserves further explanation and exploration. In this paper, a general method for constructing such systems is proposed in which the asymmetric system restores its original equation when some of the variables are subjected to a symmetric coordinate transformation combined with a special offset boosting. Two regimes of this conditional symmetry are illustrated in chaotic flows where a symmetric pair of attractors resides in asymmetric basins of attraction.

Keywords Conditional symmetry · Offset boosting · Multistability

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1 Introduction

Multistability has attracted considerable interest, and it exists in physics [1–5], chemistry [6], biological systems [7,8], neural networks [9,10] and even in ice sheet [11]. Sometimes multistability is even intertwined with chaos in various physical systems [12,13], coupled oscillators [14–16] and other single dynamical systems [17–34]. Symmetric dynamical systems often admit a symmetric pair of coexisting attractors [17–29], and these coexisting attractors sometimes persist when the symmetry is broken, which provides a guide for predicting multistability, but the resulting attractors generally lose their symmetry. However, there is another regime of multistability in which coexisting attractors come from an asymmetric structure [30,32–34]. The intrinsic mechanisms leading to such multistability are complicated and are associated with the specific structure or other unique properties. Different local and global bifurcations [30], weak attraction from a stable equilibrium point [31], the initial-condition-dependent trace of the Jacobian matrix [32], or even a symmetry return [33,34] may all permit coexisting attractors, which gives clues for finding multistability. Asymmetric structures that become locally symmetric when some of the variables evolve in their definition domain is called conditional symmetry [30]. Such systems appear asymmetric in their structure, but they have a symmetric pair of attractors just as in a symmetric system. To illustrate the mechanism, we propose a general method for producing such systems that has not been previously described. Two regimes of conditional symmetry (conditional reflection symmetry and conditional rotational symmetry) are constructed in which a symmetric pair of strange attractors reside in asymmetric basins of attraction.

2 Route for constructing conditional symmetry

Definition 2.1 Suppose there is a differential dynamical system, $\dot{X} = F(X)$ $(X = (x_1, x_2, \ldots, x_N))$. If $u_{i_0} = x_{i_0} + c$ ($i_0 \in \{1, 2, \ldots, N\}$, c is an arbitrary constant) is subject to the same governing equations after introducing a single constant associated with c into one and only one of the other equations, i.e., $\dot{Y} = F(Y)(Y = (x_1, x_2, \ldots, u_{i_0}, \ldots, x_N)$ $(i_0 \in \{1, 2, \ldots, N\}))$, then the system is **variable-boostable** [34] since it has the freedom to offset boost the variable x_{i_0} by varying the value of c. Setting x_{i_0} to $x_{i_0} + c$ will introduce in the x_{i_0} variable a new constant c which will change the average value of the variable x_{i_0}.

Lemma 2.1 *For a variable-boostable dynamical system* $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$ $(X = (x_1, x_2, \ldots, x_{i_0}, \ldots, x_N))$, *there exists one and only one* $f_{j_0}(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N)$ $(i_0 \neq j_0)$ *satisfying* $f_{j_0}(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N) = h_{j_0}$ $(x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N) + kx_{i_0}$, *where k is a nonzero constant.*

Proof For a variable-boostable dynamical system in the variable x_{i_0}, $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots,$ $f_N(X))$ $(X = (x_1, x_2, \ldots, x_{i_0}, \ldots, x_N))$, according to Def. 2.1, there exists only a dimension $f_{j_0}(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N)$, $(i_0 \neq j_0)$, that makes $f_{j_0}(x_1, x_2, \ldots, x_{i_0} + c, \ldots, x_N) - f_{j_0}(x_1, x_2, \ldots,$ $x_{i_0}, \ldots, x_N) = g(c)$. Here c is an arbitrary constant, and $g(c)$ is not identically zero. Obviously, $g(c) = f_{j_0}(0, 0, \ldots, c, \ldots, 0) - f_{j_0}(0, 0,$ $\ldots, 0, \ldots, 0)$, specifically, $g(0) = 0$. If $\left.\frac{\partial f_{j_0}}{\partial x_{i_0}}\right|_{(0,0,\ldots,0)} = k$, $\left.\frac{dg}{dc}\right|_{c=0} = \lim\limits_{c\to 0} \frac{g(c)-g(0)}{c} = \lim\limits_{c\to 0} \frac{f_{j_0}(0,0,\ldots,c,\ldots,0)-f_{j_0}(0,0,\ldots,0,\ldots,0)}{c} = \left.\frac{\partial f_{j_0}}{\partial x_{i_0}}\right|_{(0,0,\ldots,0)} = k$, and therefore $\left.\frac{\partial f_{j_0}}{\partial x_{i_0}}\right|_{(x_1,x_2,\ldots,x_N)} = \lim\limits_{c\to 0} \frac{f_{j_0}(x_1,x_2,\ldots,x_{i_0}+c,\ldots,x_N)-f_{j_0}(x_1,x_2,\ldots,x_{i_0},\ldots,x_N)}{c} = \lim\limits_{c\to 0} \frac{g(c)-g(0)}{c} = \left.\frac{dg}{dc}\right|_{c=0} = k$. Integration with respect to the variable x_{i_0} on both sides for $\left.\frac{\partial f_{j_0}}{\partial x_{i_0}}\right|_{(x_1,x_2,\ldots,x_N)} = k$, gives $f_{j_0}(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N) = h_{j_0}(x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N) + kx_{i_0}$. When $j \neq j_0$ $(j \in \{1, 2, \ldots, N\})$, $f_j(x_1, x_2, \ldots, x_{i_0} + c, \ldots, x_N) - f_j(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N) = 0$, so a similar derivation gives $f_j(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N) = h_j$ $(x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N)$. □

Definition 2.2 For a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$ $(X = (x_1, x_2, \ldots, x_N))$, if there exists a variable substitution: $u_{i_1} = -x_{i_1}, u_{i_2} = -x_{i_2}, \ldots, u_{i_k} = -x_{i_k}, u_i = x_i$, (here $1 \le i_1, \ldots, i_k \le N$, , $i_1, \ldots, i_k$ are not identical, $i \in \{1, 2, \ldots, N\} \backslash \{i_1, \ldots, i_k\}$) satisfying $\dot{U} = F(U)$ $(U = (u_1, u_2, \ldots,$ $u_N))$, then the system $\dot{X} = F(x)$ $(X = (x_1, x_2, \ldots,$ $x_N))$ is symmetric. Specifically, for a three-dimensional dynamical system, $\dot{X} = F(X)$ $(X = (x_1, x_2, x_3))$: If $x_{i_0} = -u_{i_0} (i_0 \in \{1, 2, 3\})$ is subject to the same governing equation, the system is **reflection symmetric.** If $x_{i_0} = -u_{i_0}, x_{j_0} = -u_{j_0} (i_0, j_0 \in \{1, 2, 3\}, i_0 \neq j_0)$ is subject to the same governing equation, the system is **rotational symmetric.** If $x_1 = -u_1, x_2 = -u_2, x_3 = -u_3$ is subject to the same governing equation, the system is **inversion symmetric**.

Definition 2.3 For a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$ $(X = (x_1, x_2, \ldots, x_N))$, if there exists a variable substitution: $u_{i_0} = x_{i_0} + c_0$, $u_i = x_i$, (here c_0 is a nonzero constant, $i_0 \in \{1, 2, \ldots, N\}$, and $i \in \{1, 2, \ldots, N\} \backslash \{i_0\}$), which makes the deduced system $\dot{U} = F^*(U) = (f_1^*(U),$ $f_2^*(U), \ldots, f_N^*(U))$ $(U = (u_1, u_2, \ldots, u_N))$ asymmetric, but when $f_{j_0}^*(U)$ $(1 \le j_0 \le N, j_0 \neq i_0)$is revised, the system becomes symmetric, then system $\dot{X} = F(X)$ $(X = (x_1, x_2, \ldots, x_N))$ is defined as **conditionally symmetric**. Specifically, a three-dimensional system can have conditional reflection symmetry or conditional rotational symmetry, but conditional inversion symmetry is not possible since one dimension must be left to satisfy the shifting inversion condition.

From **Lemma** 2.1, if there exists a variable substitution, $u_{j_0} = -x_{j_0}, u_{j_1} = -x_{j_1}, u_{j_2} = -x_{j_2}, \ldots, u_{j_l} = -x_{j_l}, u_j = x_j$, $(1 \le j_0, j_1, j_2, \ldots, j_l \le N, i_0, j_0, j_1,$ $j_2, \ldots, j_l$ are not equal each other, and $j \in \{1, 2,$ $\ldots, N\} \backslash \{j_0, j_1, j_2, \ldots, j_l\})$ subject to $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_{j_0-1}(X),$ $h_{j_0}(x_1, x_2,$

$\ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N\big), \quad f_{j_0+1}(X), \ldots, f_N(X)\big)$, here $X = \big(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N\big)$, the original system $\dot{X} = F(X)$ can be transformed to its conditional version by changing the term kx_{i_0} in $f_{j_0}\big(x_1, x_2, \ldots, x_{i_0}, \ldots, x_N\big) = h_{j_0}\big(x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N\big) + kx_{i_0}$to $H\left(x_{i_0}\right)$ (here $H\left(x_{i_0} + c_0\right) = -H\left(x_{i_0}\right)$).

Theorem 2.1 *The following dynamical system can be transformed into a system with conditional reflection symmetry with respect to the z dimension by introducing a nonmonotonic function $F(x)$.*

$$\begin{cases} \dot{x} = a_1 y^2 + a_2 z^2 + a_3 y + a_9, \\ \dot{y} = a_4 y^2 + a_5 z^2 + a_6 y + a_{10}, \\ \dot{z} = a_7 z + a_8 yz + a_{11} x, \end{cases} \tag{1}$$

Proof Let $x \to x + c, y \to y, z \to z$ so that a new constant $a_{11}c$ is added to the last equation without altering the other two. According to Definition 2.1, system (1) is variable-boostable. Let $x \to x, y \to y, z \to -z$ so that the first two dimensions of Eq. (1) remain the same, while the polarity balance in the dimension z is destroyed unless the offset boosting of the variable x generates a polarity reversal. Let $x \to F(x)$, since $F(x)$ is nonmonotonic, if $x = u + c$ makes $F(x) = F(u + c) = -F(u)$, the variable substitution $x \to u + c, y \to y, z \to -z$ will restore the following equation,

$$\begin{cases} \dot{x} = a_1 y^2 + a_2 z^2 + a_3 y + a_9, \\ \dot{y} = a_4 y^2 + a_5 z^2 + a_6 y + a_{10}, \\ \dot{z} = a_7 z + a_8 yz + a_{11} F(x), \end{cases} \tag{2}$$

which proves that system (1) can be transformed into system (2) with conditional reflection symmetry by a special nonmonotonic function $a_{11}F(x)$. □

For the same reason, the following Eq. (3) can be transformed into a system with conditional rotational symmetry since the polarity damage from the transformation $x \to x, y \to -y, z \to -z$ in the z dimension can be restored by a special nonmonotonic function $a_9 F(x)$.

$$\begin{cases} \dot{x} = a_1 y^2 + a_2 z^2 + a_3 yz + a_8, \\ \dot{y} = a_4 y + a_5 z, \\ \dot{z} = a_6 y + a_7 z + a_9 x, \end{cases} \tag{3}$$

The above procedure gives a method for constructing chaotic systems with conditional symmetry. If we find chaotic flows from Eq. (1) or (3), a nonmonotonic function such as an absolute value function or a trigonometric function can be introduced to give conditional symmetry since the offset boosting of the variable x may result in a polarity reversal that satisfies the symmetric transformation as shown in Fig. 1. Success of constructing conditional symmetry is guaranteed if the dynamic is preserved under the transformation $a_{11}x \to -a_{11}x$ in Eq. (1) or $a_9 x \to -a_9 x$ in Eq. (3) before introducing the nonmonotonic function.

3 Examples of conditional symmetry

3.1 Conditional reflection symmetry

An exhaustive computer search of Eq. (1) reveals many chaotic cases, one simple example of which is given by

$$\begin{cases} \dot{x} = y^2 - 0.4z^2, \\ \dot{y} = -z^2 - 1.75y + 3, \\ \dot{z} = yz + x, \end{cases} \tag{4}$$

whose strange attractor is shown in Fig. 2 with Lyapunov exponents (0.1191, 0, −1.2500) and Kaplan–Yorke dimension of 2.0953. System (4) has four equilibrium points $P_1 = (3.5576, -1.5, 2.3717)$, $P_2 = (-3.5576, -1.5, -2.3717)$, $P_3 = (-1.0119, 0.8, 1.2649)$, $P_4 = (1.0119, 0.8, -1.2649)$ with eigenvalues $\lambda_1 = (0.6029, -1.9265 \pm 3.7926i)$, $\lambda_2 = (-1.1397, -1.0551 \pm 2.9085i)$, $\lambda_3 = (-1.5526, 0.3013 \pm 1.9123i)$, and $\lambda_4 = (1.4116, -1.1808 \pm 1.6515i)$, respectively, two of which are saddle-foci of index 1, one is a saddle-focus of index 2, and one is a stable focus. An initial condition in the vicinity of P_2 attracts to the point attractor at P_2. Since the system is asymmetric, the equilibria are not symmetric and have different stability. Since the transformation $x \to x, y \to y, z \to -z$ required for reflection symmetry requires also a sign change of the variable x in the z dimension, a nonmonotonic function can be introduced in the x term to give conditional reflection symmetry.

On the basis of Theorem 2.1, we can construct a conditional reflection symmetric system by replacing the variable x with its absolute value and then offset boosting it with an arbitrary constant c in the negative

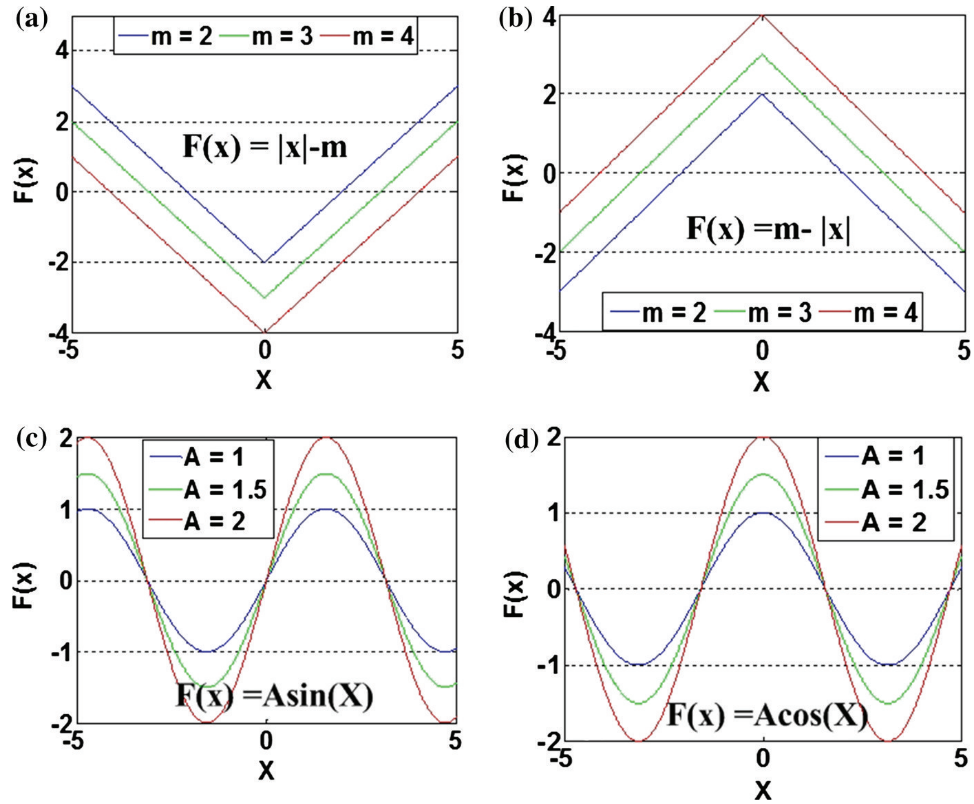


Fig. 1 Examples of offset-boosting generating polarity reversal with the variable x

direction, where a polarity reversal of the variable x can be obtained. The constant c should not be too large in case that the negative velocity makes the state variable converge to a point. A medium value like $c = -3$ is good for polarity reversal,

$$\begin{cases} \dot{x} = y^2 - 0.4z^2, \\ \dot{y} = -z^2 - 1.75y + 3, \\ \dot{z} = yz + (|x| - 3), \end{cases} \tag{5}$$

System (5) has an additional group of equilibria with the same stability as above, which are $P_{11} = (6.5576, -1.5, 2.3717)$, $P_{12} = (1.9881, 0.8, 1.2649)$, $P_{13} = (4.0119, 0.8, -1.2649)$ with the same eigenvalues of P_1, P_3, and P_4, respectively, $P_{21} = (-6.5576, -1.5, 2.3717)$, $P_{22} = (-4.0119, 0.8, -1.2649)$, $P_{23} = (-1.9881, 0.8, 1.2649)$ with the same eigenvalues of P_2, P_3, and P_4, respectively. The equilibrium point P_{21} is also a stable focus, and therefore it coexists with the other strange attractors.

The conditional symmetry of Eq. (5) is obtained by offset boosting the variable x. Since the offset boosting of the variable x in system (5) can cause a polarity reversal: $|u + c| - 3 = 3 - |x|$, it completes the conditional symmetry transformation $x \to u+c$, $y \to v$, $z \to -w$ and generates coexisting reflection symmetric attractors (Fig. 3). The basins of attraction for the coexisting attractors are shown in Fig. 4, indicating that the symmetric attractors lie in corresponding asymmetric basins. Note that although the strange attractors coexist with a stable focus, they are not hidden according to the concept of hidden attractor [35–40]. Each basin of attraction for the strange attractor contains an unstable equilibrium point.

From Theorem 2.1, we can also construct conditional symmetry based on Eq. (4) by introducing other nonmonotonic functions $F(x)$, such as $F_2(x) = 3 - |x|$, $F_3(x) = 1.5\sin(x)$, and $F_4(x) = 1.5\cos(x)$. Since the trigonometric functions $\sin(x)$ and $\cos(x)$ are also periodic, the attractor basin will be correspondingly periodic giving two groups of infinitely many duplication of the attractors.

3.2 Conditional rotational symmetry

An exhaustive computer search of Eq. (3) revealed a simple example of conditional rotational symmetry based on the chaotic system

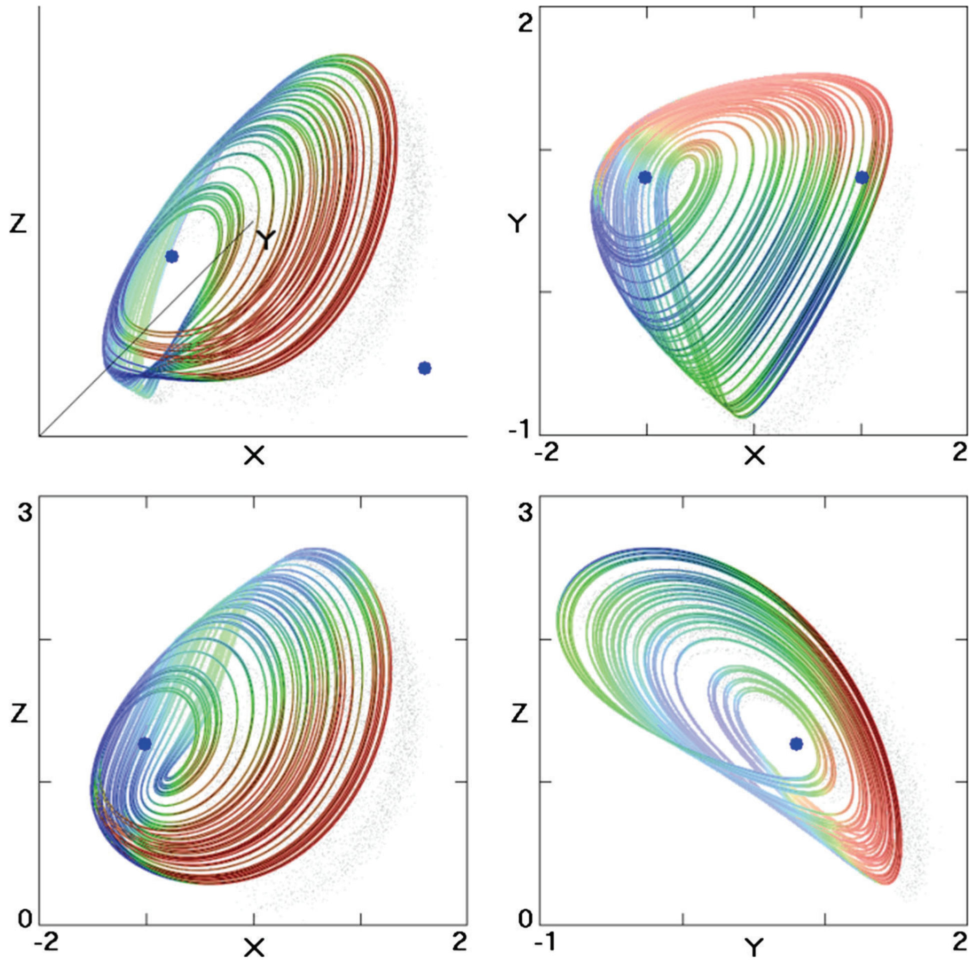


Fig. 2 Four views of the strange attractor in Eq. (4) with initial conditions (1, −1, 1). The *colors* indicate the value of the local largest Lyapunov exponent with positive values in *red* and negative values in *blue*. The equilibrium points are shown as *blue dots*. (Color figure online)

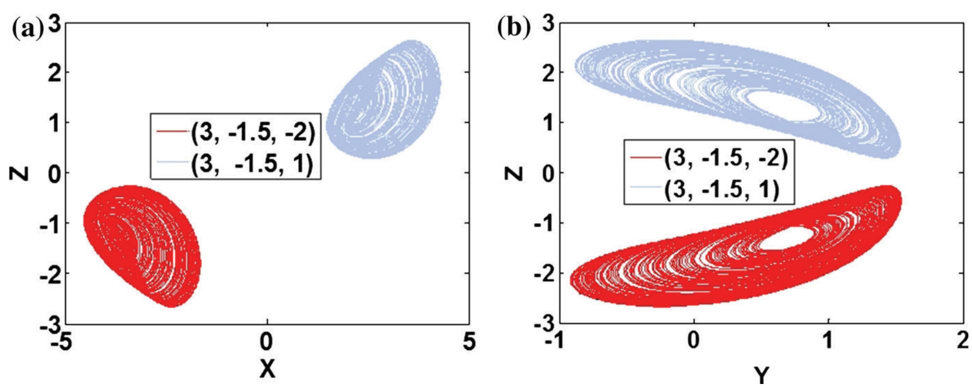


Fig. 3 Coexisting conditional reflection symmetric strange attractors of Eq. (5)

$$\begin{cases} \dot{x} = y^2 - 1.22, \\ \dot{y} = 8.48z, \\ \dot{z} = -y - z + x, \end{cases} \tag{6}$$

whose strange attractor is shown in Fig. 5, with corresponding Lyapunov exponents (0.2335, 0, −1.2335) and Kaplan–Yorke dimension of 2.1893. The asymmetric system (6) has two equilibrium points $P_1 = (1.1045, 1.1045, 0)$ and $P_2 = (-1.1045, -1.1045, 0)$ with different eigenvalues $\lambda_1 = (1.5211, -1.2606 \pm 3.2751\text{i})$ and $\lambda_2 = (-1.8590, 0.4295 \pm 3.1452\text{i})$, which are saddle-foci of index 1 and index 2, respectively.

According to Theorem 2.1, a rotational symmetric system can be obtained by introducing a special non-monotonic function of the variable x. Because polarity reversal of the function can be realized from offset boosting, the revised system can satisfy polarity bal-

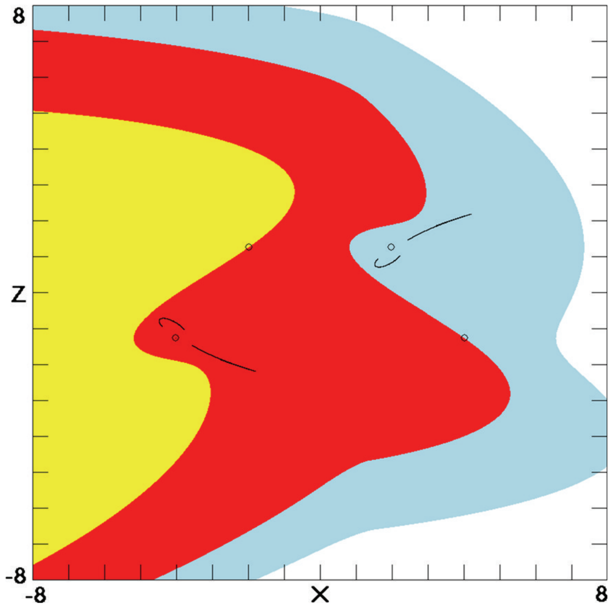


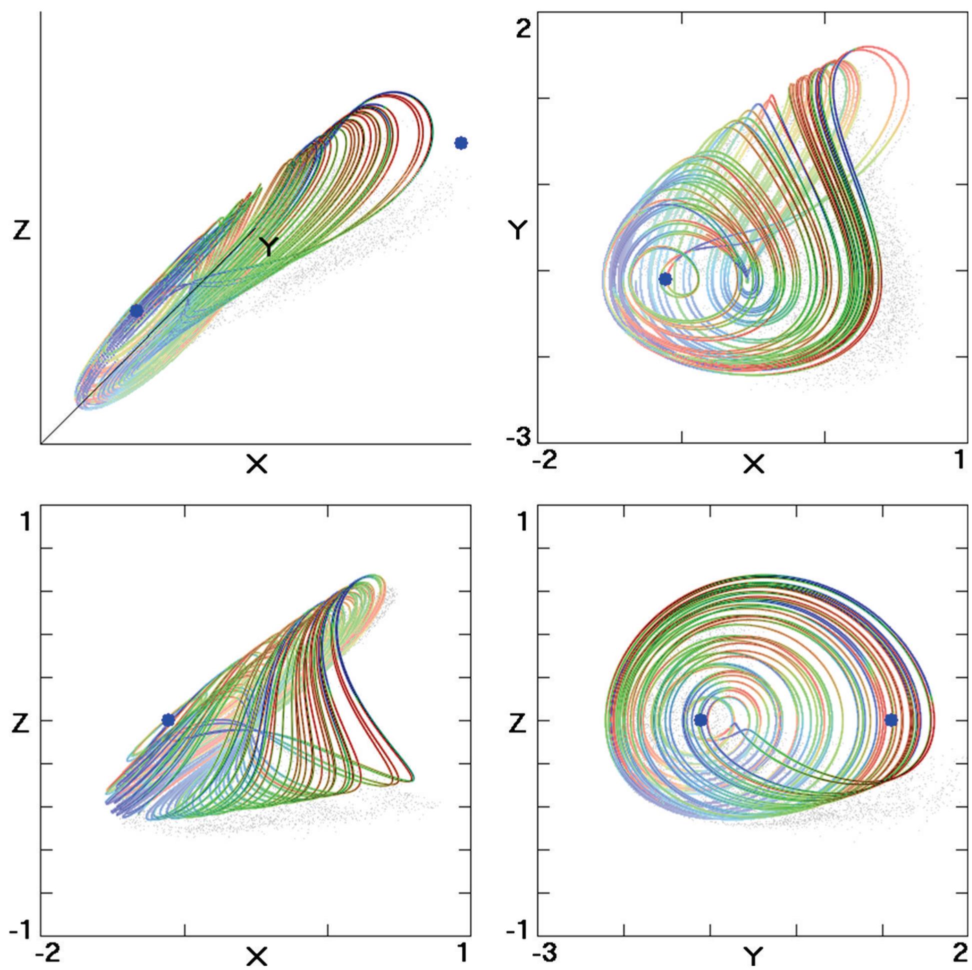
Fig. 4 Basins of attraction for coexisting reflection symmetric attractors in the plane $y = -1.5$. The strange attractor resides in the *red* and *light blue* basins, respectively, while the *yellow* is the basin of the point attractor at P_{21}. (Color figure online)

ance required for the symmetry transformation. Also taking the absolute value function, for example, the following Eq. (7) can achieve conditional rotational symmetry and give coexisting symmetric attractors according to the y-axis and z-axis, as shown in Fig. 6.

$$\begin{cases} \dot{x} = y^2 - 1.22, \\ \dot{y} = 8.48z, \\ \dot{z} = -y - z + (|x| - 3), \end{cases} \tag{7}$$

Similarly, system (7) has an additional group of equilibria with the same stability when an absolute value function is introduced, which are $P_{11} = (4.1045, 1.1045, 0)$, $P_{12} = (1.8955, -1.1045, 0)$, $P_{21} = (-1.8955, -1.1045, 0)$, $P_{22} = (-4.1045, 1.1045, 0)$. The conditional symmetry of Eq. (7) can be achieved by offset boosting variable x since system (6) is a variable-boostable system. When $|u + c| - 3 = 3 - |x|$, the transformation $x \to u + c$, $y \to -v$, $z \to -w$ in Eq. (7) is subject to the same governing equation and therefore generates coexisting rotational symmetric attrac-

Fig. 5 Four views of the strange attractor in Eq. (6) with initial conditions $(-0.5, 1, 0.5)$. The equilibrium points are shown as *blue dots*



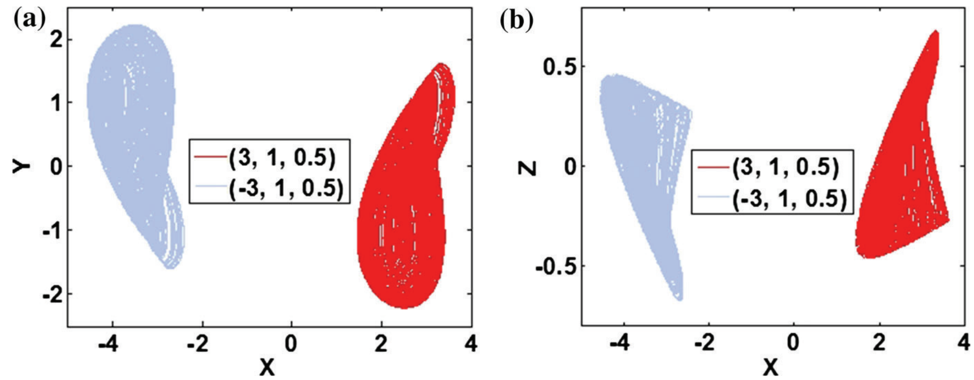


Fig. 6 Coexisting conditional rotational symmetric strange attractors of Eq. (7)

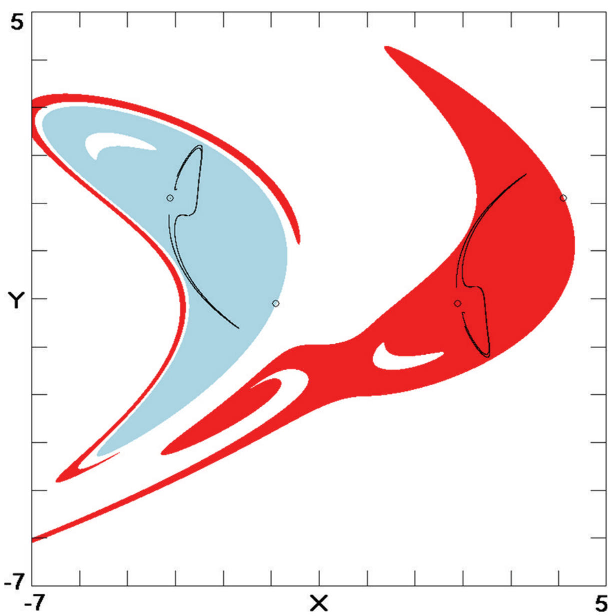


Fig. 7 Basins of attraction for coexisting rotational symmetric attractors in the plane $z = 0$

tors. The basin of attraction for the coexisting attractors in Fig. 7 shows that the symmetric attractors lie in corresponding asymmetric basins.

By comparison, we give another relative regime of hidden symmetry, which means that the original system is variable-boostable system and has a symmetric structure, but a new introduced constant that hides the symmetry. An example is the variable-boostable system $\dot{x} - ayz$, $\dot{y} = 1$, $\dot{z} = x + yz$, which is symmetric with respect to a 180^o rotation about the y-axis. This can be shown by the coordinate transformation $(x, y, z) \rightarrow (-x, y, -z)$. A new introduced constant c in the z dimension will hide the symmetry since the new equation does not satisfy the condition of the above-mentioned rotational coordinate transformation. However, this symmetry is preserved according to a different location of the x-axis.

4 Discussion and conclusions

It is common to find multistability in symmetric systems, and some fragments of the original attractor can remain when the symmetry is broken. However, for some dynamical systems, the coexisting symmetric attractor does not locate near the origin but lies far away in a certain coordinate. This special class of symmetry, called a conditional symmetry, is associated with offset boosting and usually generates coexisting symmetric attractors at different locations of the boostable variable. By searching special variable-boostable systems with incomplete symmetry, two regimes of conditional symmetry were constructed. Basins of attraction show that the coexisting symmetric attractors lie in respective asymmetric regions but with symmetric cross sections.

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