International Journal of Bifurcation and Chaos, Vol. 27, No. 8 (2017) 1750128 (5 pages)

DOI: 10.1142/S0218127417501280

3D Printing — The Basins of Tristability in the Lorenz System

Anda Xiong and Julien C. Sprott
Physics Department, University of Wisconsin-Madison,
1150 University Avenue, Madison,
Wisconsin 53706, USA

Jingxuan Lyu and Xilu Wang
Mechanical Engineering Department,
University of Wisconsin-Madison,
1415 Engineering Drive, Madison,
Wisconsin 53706, USA

Received April 9, 2017

The famous Lorenz system is studied and analyzed for a particular set of parameters originally proposed by Lorenz. With those parameters, the system has a single globally attracting strange attractor, meaning that almost all initial conditions in its 3D state space approach the attractor as time advances. However, with a slight change in one of the parameters, the chaotic attractor coexists with a symmetric pair of stable equilibrium points, and the resulting tristable system has three intertwined basins of attraction. The advent of 3D printers now makes it possible to visualize the topology of such basins of attraction as the results presented here illustrate.

Keywords: Lorenz system; basins of attraction; tristability; parameter range; 3D printing.

1. Introduction

When the parameters of a system of ordinary differential equations are changed, bifurcations typically occur where the solution switches from chaotic to nonchaotic or where the attractor becomes a repellor. For systems that have attractors, when parameters change, the shapes of both the attractors and their basins of attraction will change correspondingly. It is relatively easy to visualize and describe the attractor and its basin when there is only a single attractor, but the situation becomes complicated when the system is multistable with two or more coexisting attractors, each with its own basin of attraction and often with a fractal boundary between the basins.

The Lorenz [1963] system as originally proposed is given by

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned} \tag{1}$$

with $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$.

With this set of parameters, the system has a single globally attracting attractor, meaning that almost all points in the space, except a zero measure set of points, will be attracted to it. Basins of attraction can be classified into four types [Sprott & Xiong, 2015] according to the portion of the state space that is occupied by the basin. According to this classification, the globally attracting Lorenz attractor is Class 1a, and such a basin is relatively rare among chaotic systems.

Yorke and Yorke [1979] long ago pointed out that when the parameter ρ is reduced from 28 to the range (24.06, 24.74) with $\sigma = 10$ and $\beta = 8/3$, the unstable equilibria at the center of the wings become stable while the strange attractor continues to exist. Thus the Lorenz system becomes tristable with three coexisting attractors, each with its basin of attraction. The basins for such a three-dimensional multistable state are intertwined and hard to visualize, and that is the main motivation for this work.

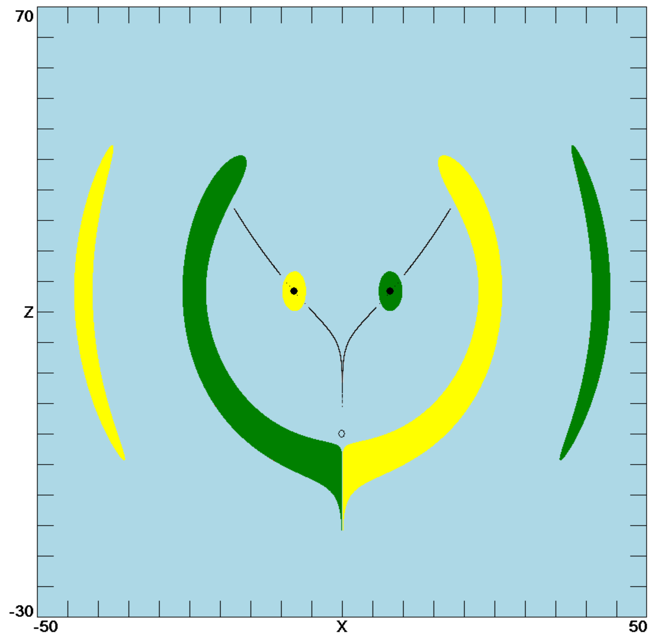

Fig. 2. Cross-section of three attractors (black) and their basins in the plane $x = y$ for $\rho = 24.4$. The unstable equilibrium point at the origin is shown as a small black circle.

2. Tristability of the Lorenz System

Basins of attraction for three-dimensional systems are usually examined in cross-section, and Figs. 1 and 2 show such plots for the Lorenz system with $(\sigma, \rho, \beta) = (10, 24.4, 8/3)$. Figure 1 is in the plane $z = 23.4$, and Fig. 2 is in the plane $x = y$, both chosen to pass through the stable equilibrium points. In the figures, the black lines represent the chaotic attractor, and the two black dots are the two stable equilibrium points. There is a third unstable equilibrium point at the origin, shown as a small open circle at the lower center of Fig. 2. The light blue area is the basin of the chaotic attractor, while the green and yellow areas are the basins of the respective stable equilibrium points. From the cross-sections, the basins are seemingly isolated, but a basin must be everywhere connected in space. Thus the basins must be connected in some kind of three-dimensional spiral structure since they are not connected in either plane shown in the figures.

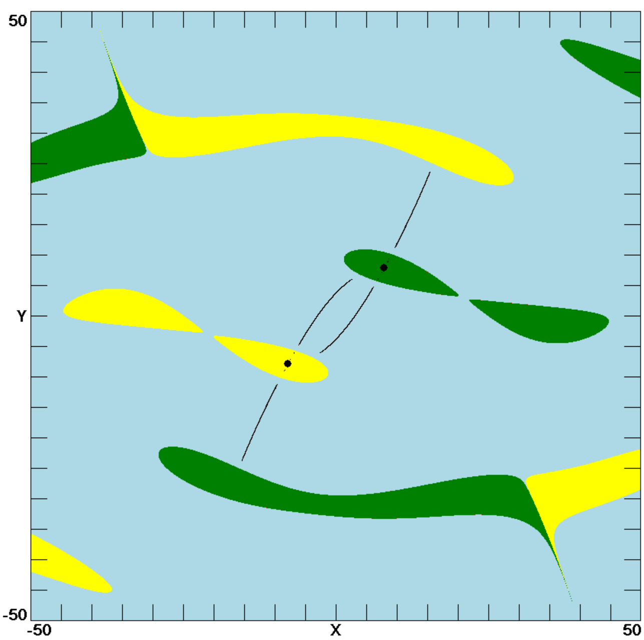

Fig. 1. Cross-section of three attractors (black) and their basins in the plane $z = 23.4$ for $\rho = 24.4$.

From the two cross-sections, several features of the basins can be observed. The three basins appear to fill the space, except for a set of measure zero, consisting of the basin boundaries and the unstable equilibrium point. This is expected since the Lorenz system had a globally attracting strange attractor before the parameter change. The basins of the two equilibrium points are symmetric and invariant under the transformation $(x, y, z) \rightarrow (-x, -y, z)$ as expected from the form of the equations. However, the two cross-sections are insufficient to visualize how the three basins are intertwined in three dimensions.

3. 3D Visualizing and 3D Printing

If a point in a given space as time advances moves toward and finally stays on an attractor, then that point belongs to a set of points called the basin of attraction of that attractor. While it may take a long time for the point in the basin to reach the attractor, a relatively shorter time is needed for a point to move to the neighborhood of the attractor and keep tending to it. The task can be reduced to two steps: first, calculating the basin, and second, visualizing its shape and form.

Currently there is no general method to determine the basin of attraction directly from the system of equations, and so it is necessary to test each point in state space as an initial condition and calculate numerically where it goes as time advances. Of course, it is impractical to test every point in a three-dimensional space, and so it is necessary to make some simplifications and approximations.

From Figs. 1 and 2, the basins of the two equilibrium points appear to be symmetric under the transformation $(x, y, z) \to (-x, -y, z)$ as can be simply proven. Call the basin a set B, and since all points in the basin go to the equilibrium point according to the equations, call the equilibrium point A. Then we have a function $f : A \to B$, which in this case is Eq. (1). We have $B = f^{-1}(A)$. Let T be the transformation sending (x, y, z) to $(-x, -y, z)$. Then the other equilibrium point which is symmetric to A is $T(A)$, and the basin B' of it is $B' = f^{-1} \circ T(A)$. Since functions f and T are commutative, $B' = f^{-1} \circ T(A) = T \circ f^{-1}(A) = T(B)$. Therefore, the symmetry of the two equilibrium points applies to their basins as well. Note that this proof is based on the existence of the two equilibrium points. Systems that are not rotationally symmetric will generally not have symmetric basins. Furthermore, the basin of the strange attractor is the complementary set to the basins of the two equilibrium points in the whole space, meaning that no points in the space will escape to infinity. This helps us to subdivide the regions. Studying the basin of one of the equilibrium points is sufficient to understand the entire system since the basin of other equilibrium point will be symmetric with it, and the basin of the strange attractor will be the remainder of the space.

Since the basins extend to infinity, it is necessary to limit the exploration to some finite portion of the space, here taken as −100 to 100 in each of the variables (x, y, z). Points in that subspace are

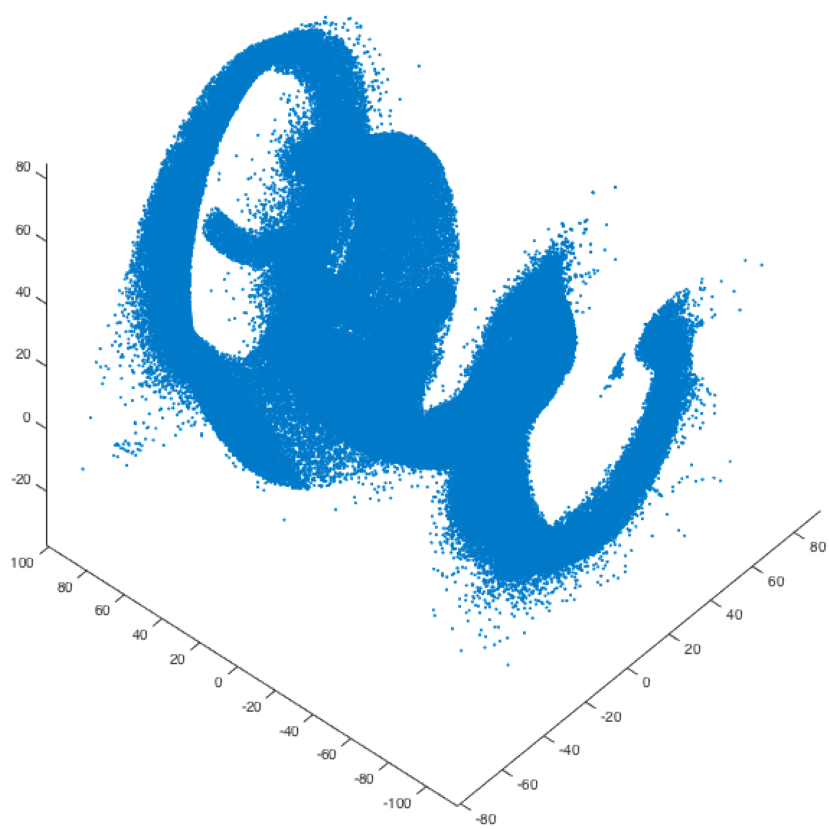

Fig. 3. Each of the examined points that belong to the basin is marked as a blue dot in space. Millions of discrete blue dots dispersed in the space give a rough shape of the basin. The space is (x, y, z) all from −100 to 100. This graph is plotted using MATLAB.

randomly chosen for testing. Taking each chosen point as an initial condition, the differential equations are solved with the fourth-order Runge–Kutta MATLAB ode45 code. Ode45 is an accessory code for solving differential equations with an adaptive step size chosen to limit the error at each step to less than 10^{-7}. If the trajectory of a given initial condition comes within a distance 2 of an equilibrium point, the initial condition is assumed to be in the basin of that point attractor. About five million points are tested, and Fig. 3 shows in a MATLAB window those points that are in the basin of one of the equilibrium points.

Since 3D printing requires the object to be solid and connected, the collection of isolated points is too dilute for actual printing. Therefore, a method was used to transfer the discrete points into a 3D printable solid. Since the basin extends to infinity in one direction, the rectangular space that contains the points was divided into $100 \times 50 \times 50$ boxes. If a box contains at least one point belonging to the basin, that box is counted. If a box does not contain a point, then it is ignored. This is a 3D generalization of the way images consisting of finitely many points are displayed in 2D on a computer screen by illuminating only those pixels that contain a point. We transfer the discrete point clouds into a 3D printable solid that is composed of small rectangular boxes. The boundary of the solid is then extracted by the "Marching Cube" method [Lorensen & Cline, 1987] as implemented in MATLAB through the function "isosurface()." The

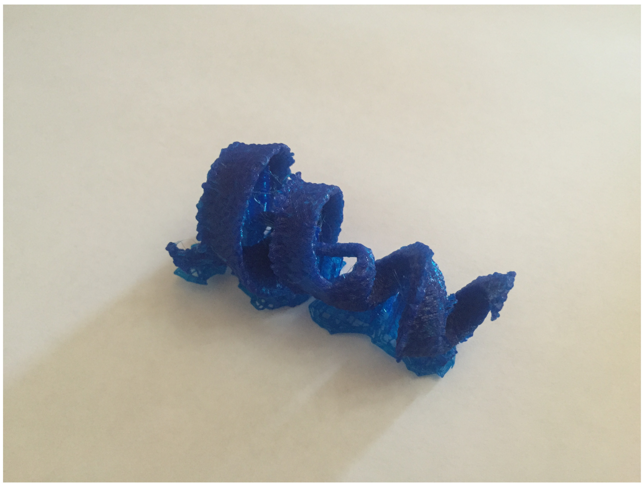

Fig. 5. A photograph of the basin of one of the stable equilibrium points after 3D printing. The actual object is small and light, and thus can be easily held and viewed from different angles.

extracted surface is represented by a 3D triangle mesh whose nodal and connectivity information is written to the output file via MATLAB functions following the standard STL format [Grimm, 2004]. The obtained STL file is then ready for 3D printing. By this method the shape of the basin becomes a bit rougher, but it still gives a good representation as shown in Fig. 4, which is produced by a program called "Meshlab" that can visualize files in STL format. Printing all the boxes that are included gives the 3D printed solid whose photograph is in Fig. 5. The basins of the two equilibrium points are twisted and entangled as shown by the photograph

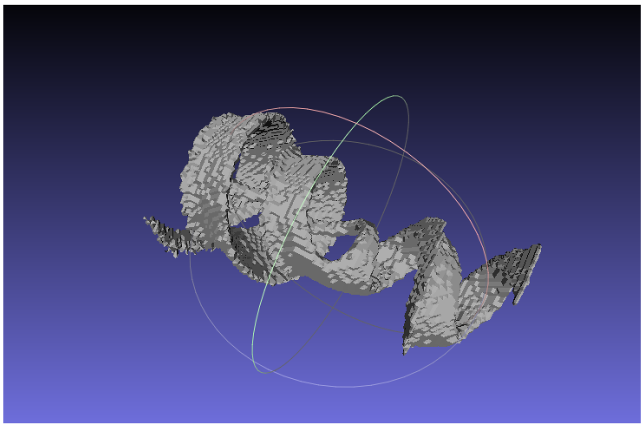

Fig. 4. The basin of attraction of one of the stable equilibrium points as solidified and displayed in Meshlab. The formerly discrete points have been transformed into a connected solid body that is 3D printable.

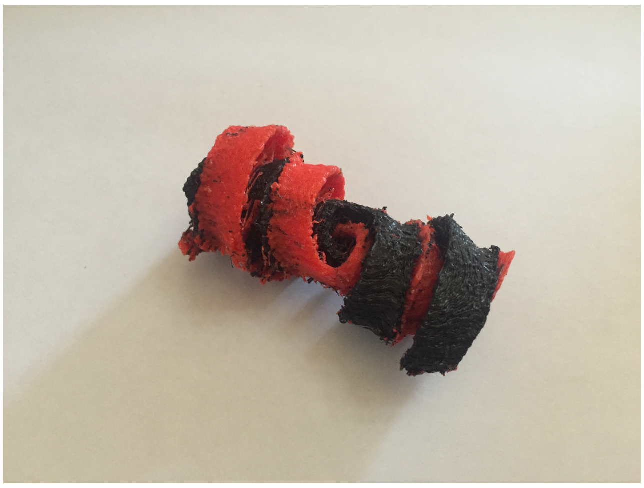

Fig. 6. A photograph of the basins of the two equilibrium points (red and black) after 3D printing. The basins are twisted and intertwined. The rest of the empty space is the basin of the chaotic attractor.

in Fig. 6. The rest of the "blank" space is the basin of the chaotic attractor.

4. Conclusion

The Lorenz system is one of the most famous chaotic systems, but its tristable form is not widely known. This paper describes a method for using a 3D printer to directly visualize the basins of attraction of this system. The method is easily generalized to other systems and provides a new means for visualizing basins of attraction for 3D flows. The method should be of interest to chaos researchers as well as artists and others who have never seen basins of attraction displayed in such a direct way.

References

Grimm, T. [2004] *User's Guide to Rapid Prototyping* (Society of Manufacturing Engineers), p. 55.

Lorenz, E. N. [1963] "Deterministic nonperiodic flows," *J. Atmos. Sci.* **20**, 130–141.

Lorensen, W. E. & Cline, H. E. [1987] "Marching cubes: A high resolution 3D surface construction algorithm," *ACM Comput. Graph.* **21**, 163–169.

Sprott, J. C. & Xiong, A. [2015] "Classifying and quantifying basins of attraction," *Chaos* **25**, 083101-1–7.

Yorke, J. & Yorke, E. [1979] "Metastable chaos: The transition to sustained chaotic behavior in the Lorenz model," *J. Stat. Phys.* **21**, 263–277.