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Offset Boosting for Breeding Conditional Symmetry

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Symmetry is usually prevented by the broken balance in polarity. If the offset boosting returns the balance of polarity when some of the variables have their polarity reversed, the corresponding system becomes conditionally symmetric and in turn produces coexisting attractors with that type of symmetry. In this paper, offset boosting in one dimension or in two dimensions in a 3D system is made for producing conditional symmetry, where the symmetric pair of coexisting attractors exist from one-dimensional or two-dimensional offset boosting, which is identified by the basin of attraction. The polarity revision from offset boosting provides a general method

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for constructing chaotic systems with conditional symmetry. Circuit implementation based on FPGA verifies the coexisting attractors with conditional symmetry.

Keywords: Conditional symmetry; offset boosting; coexisting attractors.

1. Introduction

Multistability has attracted great interest in nonlinear dynamics studies for its potential value or threats in engineering. It exists in physics [Arecchi *et al.*, 1982], chemistry [Thomson & Gunawardena, 2009], biology [Laurent & Kellershohn, 1999], neural networks [Zeng *et al.*, 2010], and even in ice sheets [Robinson *et al.*, 2012]. Symmetric chaotic systems are a source of coexisting symmetric pairs of attractors [Lai & Chen, 2016; Li *et al.*, 2015a; Li & Sprott, 2014; Bao *et al.*, 2016; Li *et al.*, 2015b; Li *et al.*, 2018a; Zhou *et al.*, 2017]. Meanwhile, those ordinary differential equations with conditional symmetry can produce offset boosted symmetric attractors in an asymmetric structure [Li *et al.*, 2016, 2017; Li *et al.*, 2018b]. Conditional symmetry represents a special type of hindered symmetry where the polarity balance is broken by the polarity reversal of some variables but is consequently satisfied by their offset boosting. A system with conditional symmetry belongs to a unique structure, and the exploration of this type of system is helpful to understand where the coexisting attractors are located within the space of initial conditions. In [Li *et al.*, 2017], a chaotic system with conditional symmetry was constructed from a rigid variable-boostable system [Li & Sprott, 2016] where polarity imbalance is prevented by an isolated variable. However, the broken polarity balance can be recovered by a generalized offset boosting, and correspondingly, the system retains its conditional symmetry. Therefore, in this paper, we follow this procedure and develop new regimes of chaotic systems with conditional symmetry from a more flexible offset boosting. To understand the mechanism for conditional symmetry, we offer the following remarks:

Remark 1.1. For a dynamical system $\dot{X} = F(X)$ $(X = (x_1, x_2, \ldots, x_N))$, if there exists a variable substitution: $u_i = x_i, u_j = x_j + d_j$, (here $1 \leq j \leq N, i \in \{1, 2, \ldots, N\}\backslash\{j\}$) satisfying $\dot{U} = F(U, d_j), (U = (u_1, u_2, \ldots, u_N))$, then the variable x_j in system $\dot{X} = F(X)$ exhibits *offset boosting*, which means that the average of the variable x_j is boosted by the newly introduced constant d_j. If the variable x_j is the signal from an electrical circuit, the signal is offset boosted by the addition of a direct component d_j. Specifically, if $u_i = x_i, u_j = x_j + d_j$ only introduces a separate constant d_j in one dimension on the right-hand side of the equations, then the system can be regarded as a *variable-boostable system* [Li & Sprott, 2016].

Remark 1.2. Suppose in a dynamical system $\dot{X} = F(X)$ $(X = (x_1, x_2, \ldots, x_N))$, if $x_i \to -x_i$, then the variable x_i is *polarity reversed*. If the polarity reversal is taken in some variables $(x_{i_1}, x_{i_2}, \ldots, x_{i_h}, \ldots, x_{i_k})$ (here $1 \leq i_1, \ldots, i_h, \ldots, i_k \leq N$) subject to the same governing equation in which case the *polarity balance* must remain on both sides of the equation $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$ since $d(-x_{ih}) = -d(x_{ih})$ may give a polarity reversal on the left-hand side of $\dot{X}$ while the corresponding right-hand side of $\dot{X}$ needs to return $-f_{ih}(X)$ to balance the revised polarity on the left-hand side.

Remark 1.3. For a specified system with some dynamical properties, any transformation should agree with the basic law of *polarity balance*. A symmetric system can retain its polarity balance when some of the variables are *polarity reversed*. Moreover, in a dynamical system, the *polarity imbalance* can be induced by the polarity reversal of any of the variables and can also be induced by the offset boosting of any of the variables since the derivative of $-x_{ih}$ gives a definite negative polarity on the left-hand side, while the offset boosting of the variable may give a negative sign on the right-hand side, although it does not change the polarity of the left-hand side of the differential equation.

Remark 1.4. When some of the variables are *polarity reversed*, a symmetric dynamical system can retain its polarity balance while an asymmetric one loses its balance of polarity. Meanwhile, the polarity balance can be restored in some specific asymmetric systems when some of the variables are offset boosted.

Remark 1.5. For a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))(X = (x_1, x_2, \ldots, x_N))$, if there exists a variable substitution including polarity reversal and offset boosting such as $u_{i_1} = -x_{i_1}$, $u_{i_2} = -x_{i_2}, \ldots, u_{i_k} = -x_{i_k}$, $u_{j_1} = x_{j_1} + d_{j_1}$, $u_{j_2} = x_{j_2} + d_{j_2}, \ldots, u_{j_l} = x_{j_l} + d_{j_l}$, $u_i = x_i$, (here $1 \le i_1, \ldots, i_k \le N, 1 \le j_1, \ldots, j_l \le N$, $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ are not identical, $i \in \{1, 2, \ldots, N\} \backslash \{i_1, \ldots, i_k, j_1, \ldots, j_l\}$), the derived system will retain its balance of polarity on the two sides of the equation and will satisfy $\dot{U} = F(U)(U = (u_1, u_2, \ldots, u_N))$. The system $\dot{X} = F(X)$ $(X = (x_1, x_2, \ldots, x_N))$ can be defined as one of l-dimensional *conditional symmetry* since the polarity balance needs an l-dimensional offset boosting [Li *et al.*, 2017]. Specifically, for a three-dimensional dynamical system, $\dot{X} = F(X)(X = (x_1, x_2, x_3))$, there only exist conditional rotational symmetry in one dimension and conditional reflection symmetry in one dimension or two dimension.

2. Offset Boosting for Breeding Conditional Symmetry

Considering that conditional symmetry can be easily obtained from a general structure with offset boosting in addition to rigid variable-boostable systems [Li *et al.*, 2017; Li & Sprott, 2016], we here design four additional cases from searching for chaotic systems with conditional symmetry. For example, to obtain a conditional reflection-symmetric system, suppose there is a polarity reversal in the variable x_i. This will revise the polarity of the dimension of x_i on the left-hand side of $\dot{x}_i$, and in turn require adjustment in the polarity of the terms on the right-hand side of $\dot{x}_i$ to get $-f_i(X)$. Besides this, all the other dimensions retain polarity balance since $-x_i$ may introduce a minus sign on the right-hand side, which in turn requires a new minus sign from offset boosting to cancel it. That is to say, to keep the polarity balance of a polynomial equation $f_j(X)$, it is necessary to revise the equation to produce similar dynamics by introducing new functions in $f_j(x_1, x_2, \ldots, -x_i, \ldots, F_{j_1}(x_{j_1}), F_{j_2}(x_{j_2}), \ldots, F_{j_l}(x_{j_l}), \ldots, x_N)$ $(1 \le j_1, \ldots, j_l \le N$, $j_1, \ldots, j_l$ are not identical, and l is odd) for variables $(x_{j_1}, x_{j_2}, \ldots, x_{j_l})$ which admit offset boosting giving a new minus sign to cancel the one from $-x_i$ by $F_{j_m}(x_{j_m} + d_{j_m}) = -F_{j_m}(x_{j_m})$ $(j_1 \le j_m \le j_l)$. Note that offset boosting does not produce a minus sign on the left-hand side of the equation since $d(x_{j_m} + d_{j_m}) = d(x_{j_m})$. A jerk system is a simple structure to pass the polarity which consequently provides an easy way to consider polarity balance. In the following, we show some examples mainly from jerk structures and show how the polarity imbalance is restored by a suitable offset boosting.

Theorem 2.1. *The following jerk system can be transformed into a system with conditional reflection symmetry with respect to the x dimension by introducing a nonmonotonic function $F(y)$* [*Li et al.*, *2017*]:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a_1 xy + a_2 x^2 + a_3 y^2 + a_4 z + a_5 z^2 + a_6. \end{cases} \tag{1}$$

Proof. If $x \to -x$, $y \to y$, $z \to z$, the polarity balance of the first and last dimensions of Eq. (1) is destroyed because of the polarity invariance in the variables y and z, while the polarity balance in the second dimension $\dot{y}$ is preserved. However, offset boosting of the variable y can produce a polarity reversal and restore the polarity balance in the first and last dimensions if a suitable function is introduced. Suppose $y \to F(y)$, since $F(y)$ is nonmonotonic, if $y = u + c$ makes $F(y) = F(u+c) = -F(u)$, the variable substitution $x \to -x$, $y \to u + c$, $z \to z$ will restore the polarity balance in the equation

$$\begin{cases} \dot{x} = F(y), \\ \dot{y} = z, \\ \dot{z} = a_1 x F(y) + a_2 x^2 + a_3 (F(y))^2 \\ \qquad + a_4 z + a_5 z^2 + a_6, \end{cases} \tag{2}$$

which proves that system (1) can be transformed into system (2) with conditional reflection symmetry by a special nonmonotonic function $F(y)$. For the same reason, the jerk equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 xz \\ \qquad + a_5 yz + a_6 z^2 + a_7, \end{cases} \tag{3}$$

can also be transformed into a system with conditional rotational symmetry since the polarity imbalance from the transformation $x \to -x$, $y \to -y$,

$z \to z$ in the y and z dimensions can be recovered by offset boosting of the variable z from a special nonmonotonic operation $F(z)$. The above jerk equations give a new structure for developing chaotic systems with conditional symmetry that is different from the variable-boostable system [Li & Sprott, 2016]. We can transform chaotic flows with the structure of Eqs. (1) or (3) to have conditional symmetry if an absolute value function or a trigonometric function [Li *et al.*, 2017] is introduced to give conditional symmetry since the offset boosting may result in a polarity reversal that satisfies the symmetric transformation.

More generally, an asymmetric 3D system can recover its polarity balance from 2D offset boosting. We conclude that the dynamical system

$$\begin{cases} \dot{x} = a_1 z^2 + a_2 xz + a_3 yz + a_4, \\ \dot{y} = a_5 z^2 + a_6 xz + a_7 yz + a_8, \\ \dot{z} = a_9 z + a_{10} x + a_{11} y, \end{cases} \tag{4}$$

can also be transformed into a system with conditional reflection symmetry with respect to the z dimension from a 2D offset boosting by introducing nonmonotonic functions $F(x)$ and $G(y)$ in the right-hand side of the equations. Suppose $x \to x + c_1, y \to y + c_2, z \to -z$, if $u = x + c_1$ makes $F(u) = F(x + c_1) = -F(x)$, $v = y + c_2$ makes $G(v) = G(y + c_2) = -G(y)$, all the polarity balance can be restored in the revised system.

A jerk structure such as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = xz, \\ \dot{z} = a_1 xy + a_2 xz + a_3 x^2 + a_4 y^2 \\ \qquad + a_5 z^2 + a_6 yz + a_7, \end{cases} \tag{5}$$

can also provide such a case with conditional symmetry from 2D offset boosting. Here the polarity reversal of one variable restores its balance by an offset boosting from one of the other variables. Moreover, the polarity reversal of one variable also changes the polarity balance in the second dimension and consequently needs an extra offset boosting from the third variable. Here the polarity imbalance caused by the variable x in the first and second dimensions can be restored by offset boosting of the variables y and z when introducing nonmonotonic functions $F(y)$ and $G(z)$ in the right-hand side of the equation. ■

3. New Cases of Conditional Symmetry

Considering the above four cases, Eqs. (1) and (3) can be transformed to have conditional symmetry by 1D offset boosting, while Eqs. (4) and (5) can be revised to have conditional symmetry by 2D offset boosting. Equations (1) and (3) are different from the cases reported in reference [Li *et al.*, 2017] where a single 1D offset boosting is in the third dimension. Here the new conditional symmetric system is from a jerk structure where the reflection or rotational symmetry is broken by a neighboring variable, but the polarity balance can be restored by a general offset boosting. In fact, HJ5 is a system with conditional rotational symmetry according to the variables y and z [Li *et al.*, 2016] where offset boosting in the variable x restores the polarity balance for achieving symmetry. Equations (4) and (5) are new cases of conditional reflection systems where 2D offset boosting is necessary for restoring polarity balance. The original system is normally an asymmetric chaotic system (ACS) as listed in Table 1, and the corresponding conditional symmetric systems (CSS) show coexisting attractors.

The strange attractors in an asymmetric system as shown in Fig. 1 are all asymmetric. Symmetric pairs of coexisting attractors in the conditional symmetric versions are shown in Figs. 2–5. Figure 2 shows the coexisting attractors in CSS1 induced by 1D offset boosting in the y dimension, while Fig. 3 shows that the coexisting symmetric attractors in the plane $x = 0$ need an offset boosting in the z dimension. Two symmetric attractors in the plane $z = 0$ in system CSS3 demand offset boosting in the x and y dimensions as indicated in Fig. 4. In Fig. 5, we see two symmetric attractors in the plane $x = 0$ in system CSS4 requiring offset boosting in the y and z dimensions. As predicted, the coexisting attractors reside separately in the phase space with different offset gaps in one or two dimensions as identified by the basins of attraction shown in Fig. 6. Here the black lines indicate the cross-section of the coexisting attractors. As predicted by the above analysis, Figs. 6(a) and 6(b) clearly show the coexisting symmetric attractors in the plane $x = 0$ with the condition of offset boosting in the corresponding y and z dimensions. Figures 6(c) and 6(d) also agree with the 2D offset boosting in the dimensions x–y and y–z, respectively, for restoring the conditional symmetry. Note that the basins for each attractor are asymmetric and the area of the basins

Table 1. Asymmetric chaotic systems (ACS) and their conditional symmetric versions (CSS).

Systems	Equations	Parameters	Equilibria	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
ACS1	$\dot{x} = y$, $\dot{y} = z$, $\dot{z} = -x^2 - az + by^2 + 1$,	$a = 2.6$, $b = 2$	$(1, 0, 0)$ $(-1, 0, 0)$	$-2.8468, 0.1234 \pm 0.8290i$; $-2.1786, -1.1917, 0.7703$	0.0551, 0, -2.6551	2.0208	5, 1, 5
CSS1	$\dot{x} = F(y)$, $\dot{y} = z$, $\dot{z} = -x^2 - az + b(F(y))^2 + 1$, $F(y) = \lvert y\rvert - 4$	$a = 2.6$, $b = 2$	$(\pm 1, \pm 4, 0)$ $(\pm 1, \mp 4, 0)$	$-2.8468, 0.1234 \pm 0.8290i$; $-2.1786, -1.1917, 0.7703$	0.0463, 0, -2.6463	2.0175	0.5, 4, -1
ACS2	$\dot{x} = y$, $\dot{y} = z$, $\dot{z} = x^2 - ay^2 + bxy + xz$,	$a = 1.24$, $b = 1$	$(0, 0, 0)$	$0, 0, 0$	0.0671, 0, -1.2683	2.0529	$-1, -2, -1$
CSS2	$\dot{x} = y$, $\dot{y} = F(z)$, $\dot{z} = x^2 - ay^2 + bxy + xF(z)$ $F(z) = \lvert z\rvert - 8$	$a = 1.24$, $b = 1$	$(0, 0, \pm 8)$	$0, 0, 0$	0.0645, 0, -1.2582	2.0513	$4, 0.8, -2$
ACS3 (VB6)	$\dot{x} = 1 - yz$, $\dot{y} = az^2 - yz$, $\dot{z} = x$,	$a = 0.22$,	$(0, \pm 0.4690, \pm 2.1320)$	$-2.0350, 0.0865 \pm 0.9275i$ $-0.8941, 1.2837, 1.7425$	0.0717, 0, -1.6646	2.0431	$-1, 1, -1$
CSS3	$\dot{x} = 1 - G(y)z$, $\dot{y} = az^2 - G(y)z$, $\dot{z} = F(x)$, $F(x) = \lvert x\rvert - 3$ $G(y) = \lvert y\rvert - 5$	$a = 0.22$,	$(3, 5.4690, 2.1320)$ $(3, 4.5310, -2.1320)$ $(3, -5.4690, 2.1320)$ $(3, -4.5310, -2.1320)$ $(-3, 5.4690, 2.1320)$ $(-3, 4.5310, -2.1320)$ $(-3, -4.5310, -2.1320)$ $(-3, -5.4690, 2.1320)$	$-2.0350, 0.0865 \pm 0.9275i$; $-0.8941, 1.2837, 1.7425$; $2.3050, -0.0865 \pm 0.9275i$; $-1.7425, -1.2837, 0.8941$; $-1.7425, -1.2837, 0.8941$; $2.3050, -0.0865 \pm 0.9275i$; $-2.0350, 0.0865 \pm 0.9275i$; $-0.8941, 1.2837, 1.7425$	0.0729, 0, -1.6732	2.0436	$-1, 1, -1$
ACS4	$\dot{x} = y$, $\dot{y} = xz$, $\dot{z} = -axy - bxz - x^2 + y^2$,	$a = 3$, $b = 1.2$	$(0, 0, z)$	$0, \sqrt{z}, \sqrt{-z}$	0.0589, 0, -0.2636	2.2234	0, 1, 2
CSS4	$\dot{x} = F(y)$, $\dot{y} = xG(z)$, $\dot{z} = -axF(y) - bxG(z) - x^2 + (F(y)^2)$, $F(y) = \lvert y\rvert - 5$ $G(z) = \lvert z\rvert - 5$	$a = 3$, $b = 1.2$	$(0, \pm 5, z)$	$0, \sqrt{\lvert z\rvert - 5}, -\sqrt{\lvert z\rvert - 5}$ $0, \sqrt{5 - \lvert z\rvert}, -\sqrt{5 - \lvert z\rvert}$	0.0506, 0, -0.2904	2.1735	$0, -6, -6$

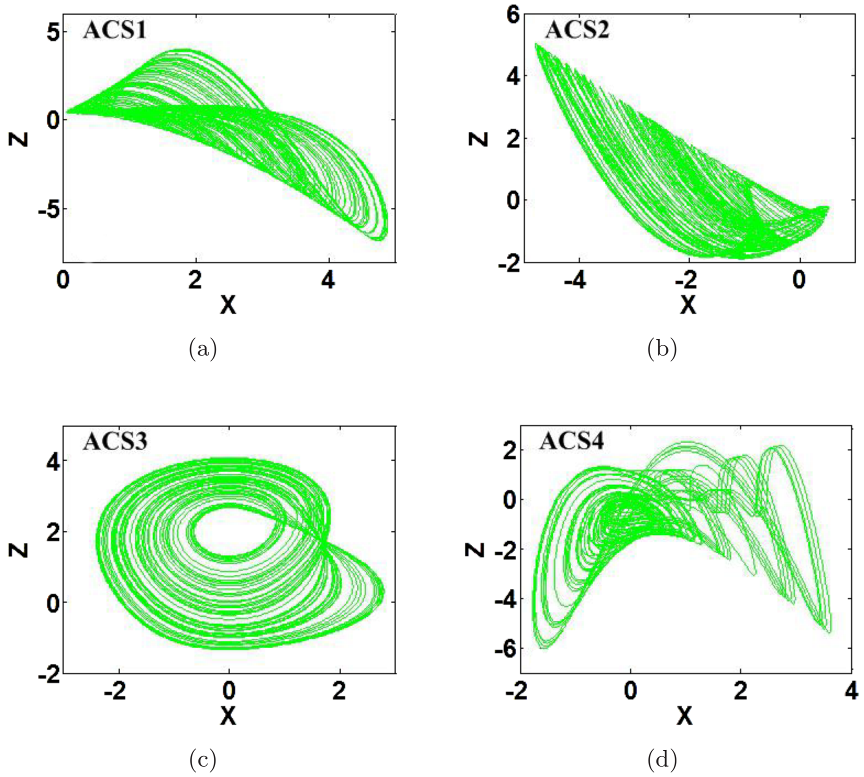

Fig. 1. Strange attractors in the original asymmetric chaotic systems: (a) ACS1, (b) ACS2, (c) ACS3 and (d) ACS4.

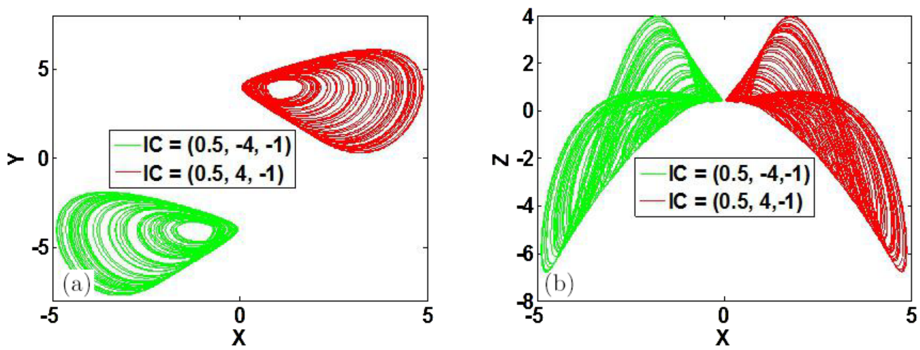

Fig. 2. Coexisting attractors in CSS1 induced by 1D offset boosting in the y dimension.

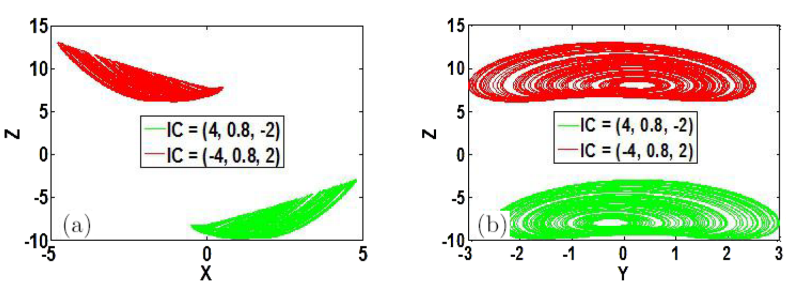

Fig. 3. Coexisting attractors in CSS2 induced by 1D offset boosting in the z dimension.

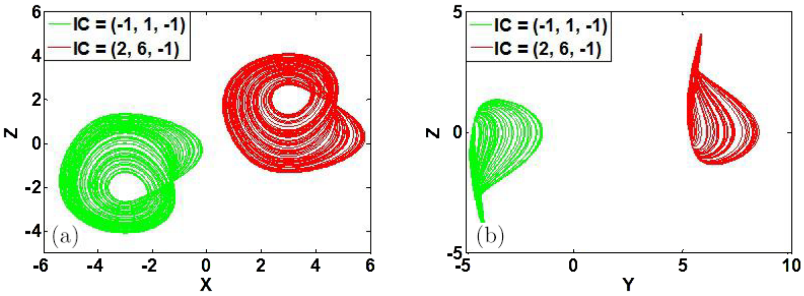

Fig. 4. Coexisting attractors in CSS3 induced by 2D offset boosting in the x and y dimensions.

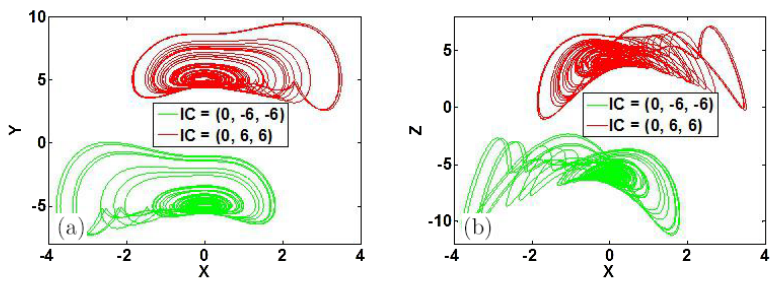

Fig. 5. Coexisting attractors in CSS4 induced by 2D offset boosting in the y and z dimensions.

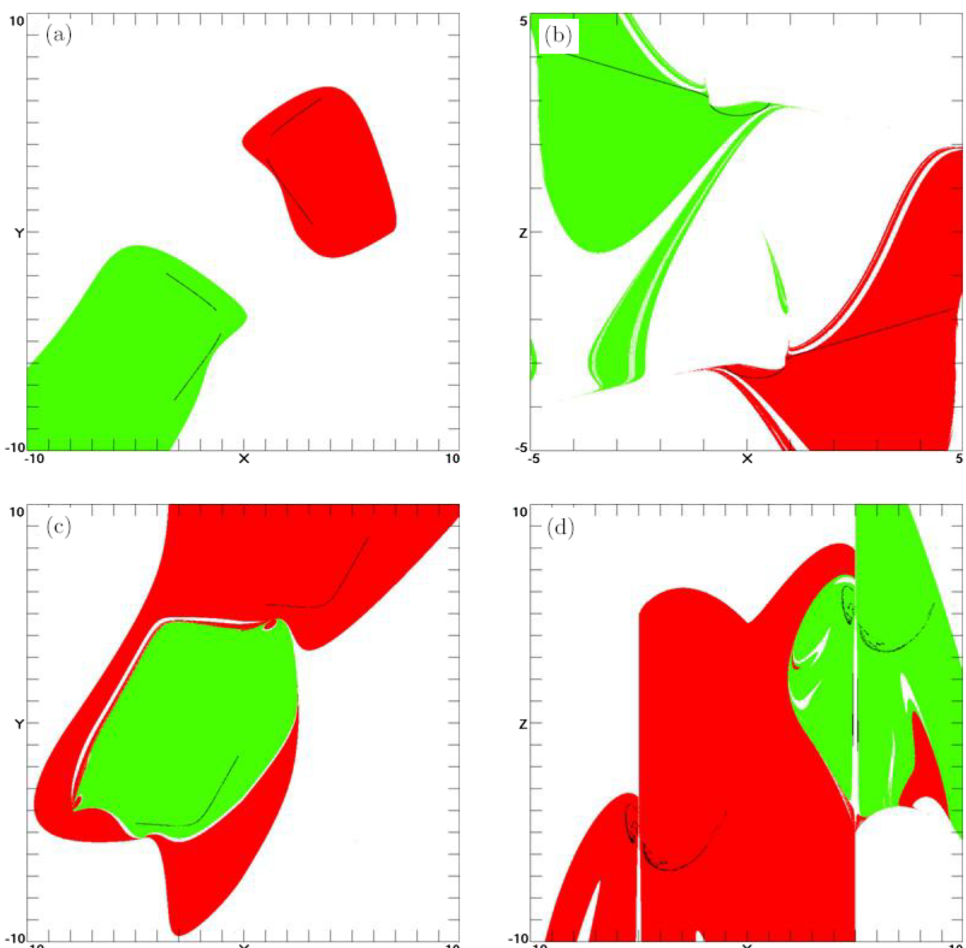

Fig. 6. Basins of attraction: (a) $z = 0$ for CSS1, (b) $y = 0$ for CSS2, (c) $z = 0$ for CSS3 and (d) $x = 0$ for CSS4.

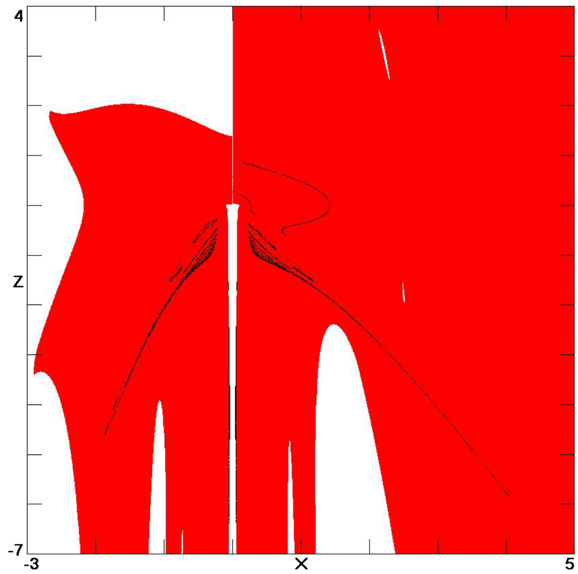

Fig. 7. Basins of attraction for ACS4 with $a = 3$, $b = 1.2$ when $y = 0$.

for the coexisting attractors are generally unequal. In Figs. 6(c) and 6(d), there are white bands near the boundary between the basins of attraction, which indicate that there is a transient that takes the orbit far from either attractor before settling back to one or the other.

The chaotic system ACS4 is special with a line of equilibrium points [Li *et al.*, 2015c; Jafari & Sprott, 2013; Bao *et al.*, 2018a]. Note that the line of equilibria $(0, 0, z)$ has different stability according to the value of z, and so initial conditions near the equilibria may lead to different phase trajectories. When $a = 3$, $b = 1.2$, the basin of attraction [Liu & Pang, 2011; Sprott & Xiong, 2015] for ASC4 as shown in Fig. 7, shows that this system is a case with hidden oscillations [Leonov *et al.*, 2011; Jafari *et al.*, 2015; Wei *et al.*, 2014; Kapitaniak & Leonov, 2015; Dudkowski *et al.*, 2016] since the initial condition in the neighborhood of the line does not

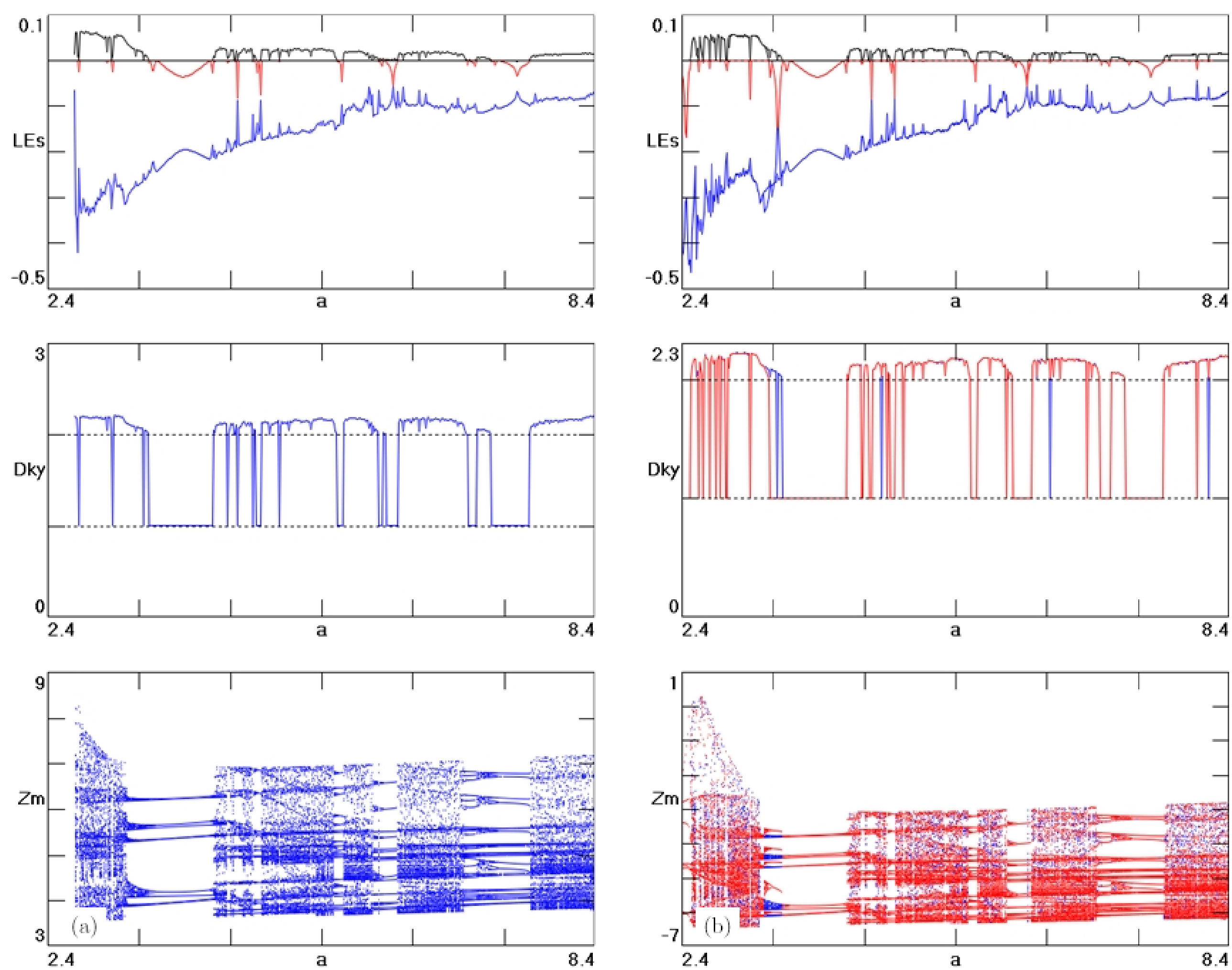

Fig. 8. Bifurcation diagrams of CSS4 when $b = 1.2$ while a varies in $[2.4, 8.4]$.

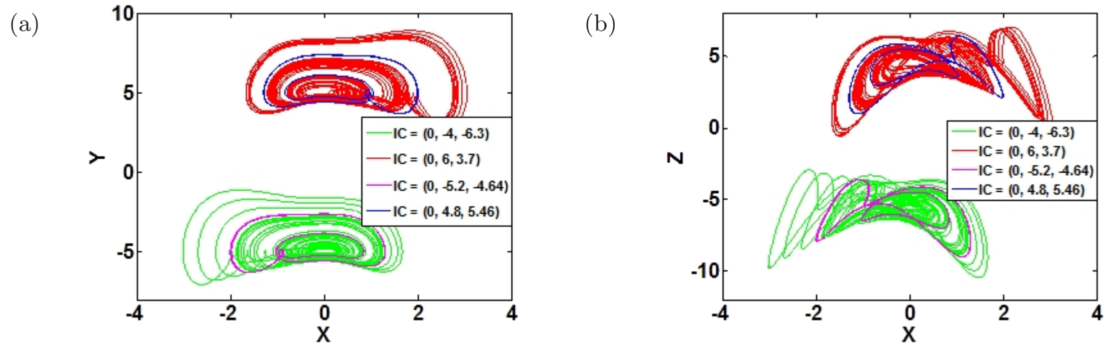

Fig. 9. Four coexisting attractors in CSS4 when $a = 3.4$, $b = 1.2$.

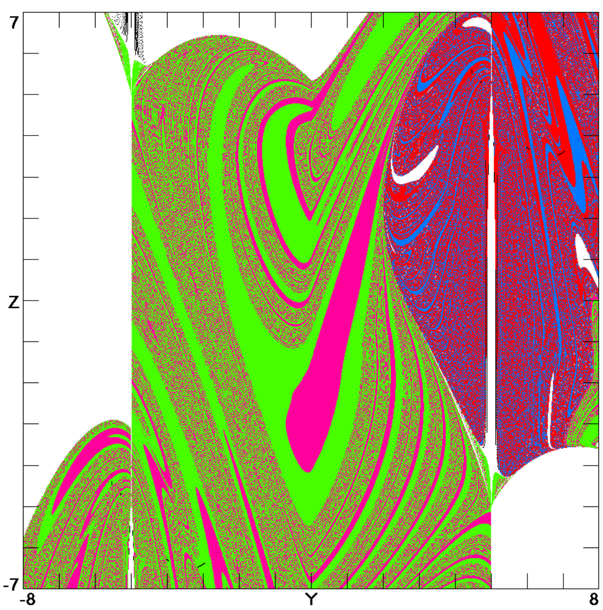

Fig. 10. Basins of attraction for CSS4 with $a = 3.4$, $b = 1.2$ when $x = 0$.

guarantee finding the chaotic oscillation. In fact, a white band close to $x = 0$ in Fig. 7 shows that there is an unbounded solution in system ACS4. For the version of conditional symmetry CSS4, when the parameter a varies in $[2.4, 8.4]$ under two types of initial condition, the system is mainly chaotic with diminishing Lyapunov exponents and has interspersed periodic windows as shown in Fig. 8. The plot uses 500 values of a, with each calculated for a time of 2.3e3. The parameter a is scanned upward and downward as a convergence check and for checking the evidence of hysteresis. The bifurcation diagrams look similar but slightly different especially when the parameter a is close to 3.4. Further exploration also shows that even when $a = 3$, two coexisting attractors of conditional symmetry do not share exactly the same Lyapunov exponents. The positive attractor has relatively higher Lyapunov exponents $(0.0550, 0, -0.2504)$ and Kaplan–Yorke dimension 2.2196 except for a shift in Z_m. Figure 8(b) shows the hysteresis in system CSS4, which is a sufficient condition for multistability but not a necessary condition. It is interesting that one of the bifurcation plots shows hysteresis while the other does not, which is different from those symmetric systems. When $a = 3.4$, system CSS4 has two pairs of coexisting attractors with conditional symmetry, as shown in Fig. 9. The basins in the $x = 0$ plane for those two pairs of attractors are shown in Fig. 10. Unlike the system of conditional reflection symmetry [Li *et al.*, 2017], here the basins of limit cycle

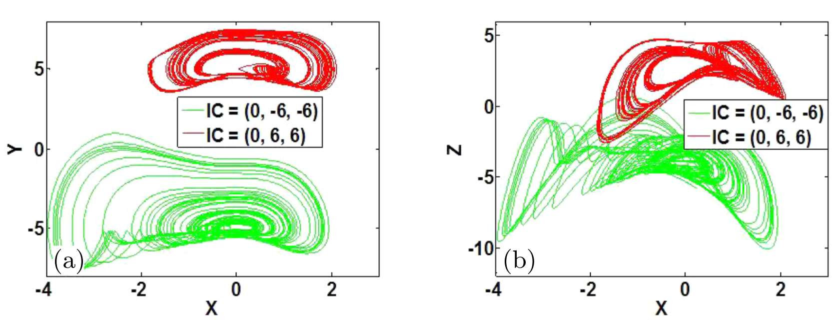

Fig. 11. Asymmetric coexisting attractors in CSS4 induced by 2D offset boosting in the y and z dimensions with $F(y) = |y| - 5$, and $G(z) = |z| - 3.3$.

are embedded in the ones of strange attractor rather than being outside any of the basins of attraction of conditional symmetry.

Note that a suitable threshold in the nonmonotonic operation $F(\cdot)$ is necessary for restoring the polarity balance under offset boosting. For $F(x) = |x| - a$, to obtain $F(x+2a) = |x+2a| - a = -F(x)$ (or $F(x-2a) = |x-2a| - a = -F(x)$), the variable x should be in the region $[-2a, 0]$ (or $[0, 2a]$) ($a \geq 0$). Modifying the threshold can give coexisting asymmetric attractors [Li *et al.*, 2016; Bao *et al.*, 2018b; Chen *et al.*, 2017], as shown in Fig. 11.

4. Circuit Implementation Based on FPGA

Chaotic circuits can be realized based on analog technology [Wang *et al.*, 2017; Zhou *et al.*, 2016, 2018]. However, in the following, we give the results from circuit implementation based on FPGA. The core board ALINX FPGA CYCLONE IV with FPGA chip EP4CE15F17C8 is applied here for numerical computation. Firstly, the continuous nonlinear system is discretized using the Euler algorithm:

$$\begin{cases} \dfrac{x_1(n+1) - x_1(n)}{\Delta T} \\ \qquad = f_1(x_1(n), x_2(n), \ldots, x_N(n)), \\ \dfrac{x_2(n+1) - x_2(n)}{\Delta T} \\ \qquad = f_2(x_1(n), x_2(n), \ldots, x_N(n)), \\ \qquad \vdots \\ \dfrac{x_N(n+1) - x_N(n)}{\Delta T} \\ \qquad = f_N(x_1(n), x_2(n), \ldots, x_N(n)). \end{cases} \tag{6}$$

Taking the system CSS1 for example,

$$\begin{cases} x(n+1) = x(n) + (|y(n)| - 4)\Delta T, \\ y(n+1) = y(n) + z(n)\Delta T, \\ z(n+1) = z(n) + (-x^2(n) - 2.6z(n) \\ \qquad\qquad + 2(|y(n)| - 4)^2 + 1)\Delta T. \end{cases} \tag{7}$$

Here ΔT is the time step of the discretization. For satisfying the accuracy, we select $\Delta T = 0.001$. Secondly, according to the standard of IEEE754, we use the calculation of 32-bit finite integer instead of floating-point operations. By the operations of multiplying, adding, or subtracting in the discrete model, the number for the next iteration can be obtained. The repeated iteration can achieve the process of discretization of the chaotic system. The absolute value operation only needs to set the first bit of the 32-bit integer to 0. For example, the RTL of the chaotic system CSS1 is shown in Fig. 12. The digital system consists of three modules, i.e. a control module, a computation module, and a transformation module. The control module deals with initial conditions, coefficients of the equations, and computation control based on the structure of the finite-state machine. Specifically $x(n+1), y(n+1), z(n+1)$, and the whole values are input into the computation module as time-share, and the results are read into the control module subsequently. The computation module contains an absolute value calculation block, a floating-point addition block, a floating-point multiplicative block, and so on, which are constructed based on the simplest general formula. In the transformation module, the floating-point results from the preceding calculation are transformed into 12-bit unsigned integers and input into a DAC for displaying the waveform on an oscilloscope. The FPGA chip EP4CE15 has 15 408 logic elements. To realize chaotic system CSS1, 7944 logic elements were occupied, which takes up 52% of the total resource. To restore the initial condition and corresponding coefficients, 148 523 elementary units were employed holding 29% of the total memory resource. Meanwhile, 42 multiplying units were used to represent 38% of the whole resource. Coexisting attractors can be observed on the oscilloscope by selecting different initial conditions, shown in Fig. 13. Note that offset boosting represents a shift of the average value of an alternating signal. In an oscillating circuit, this stands for the DC component in a signal and can be realized by introducing a source of direct current. Here in the system of conditional symmetry, offset boosting of a system variable is realized from the initial condition, which becomes involved with the polarity reversal.

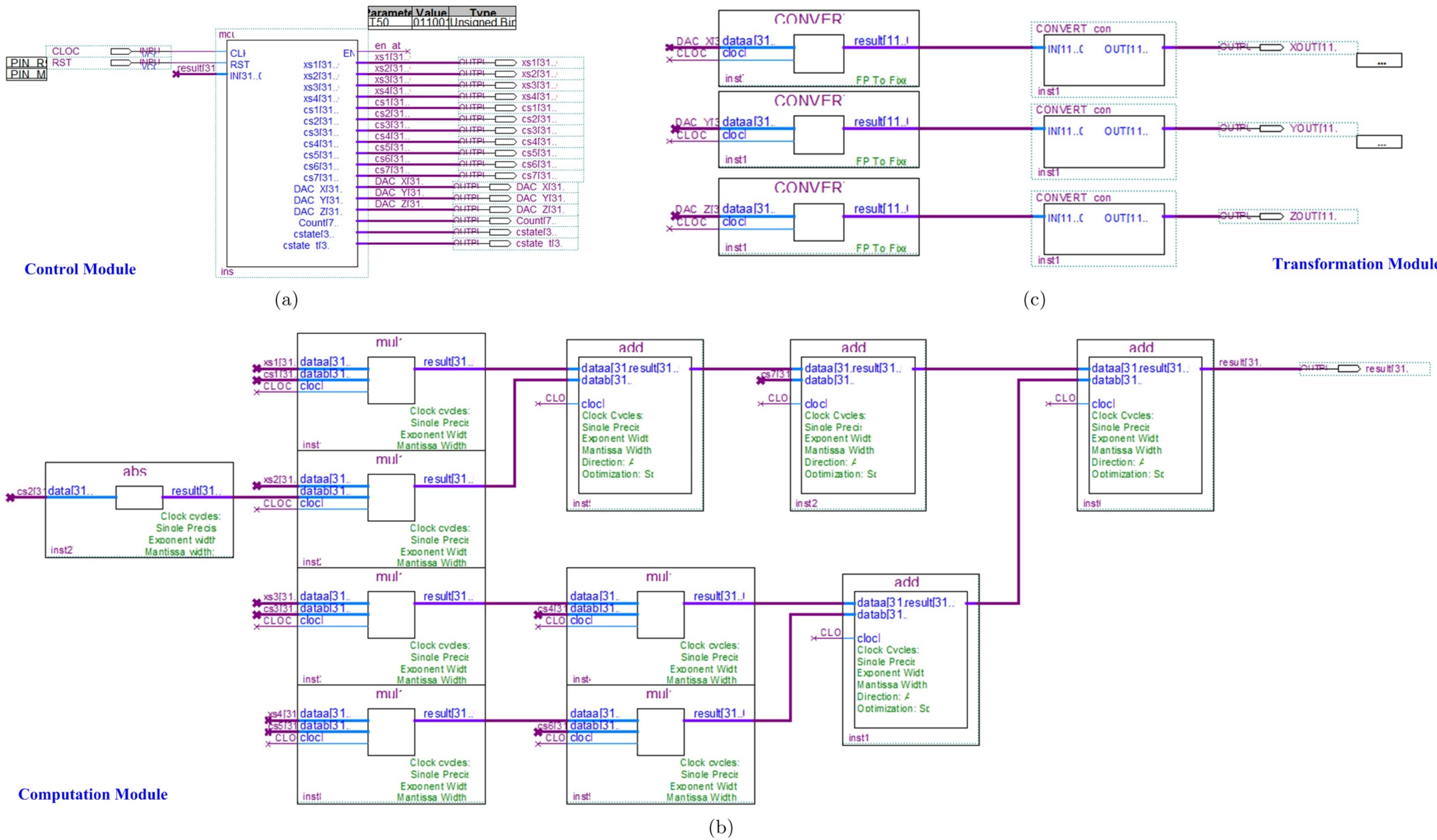

Fig. 12. Top-layer of RTL of CSS1.

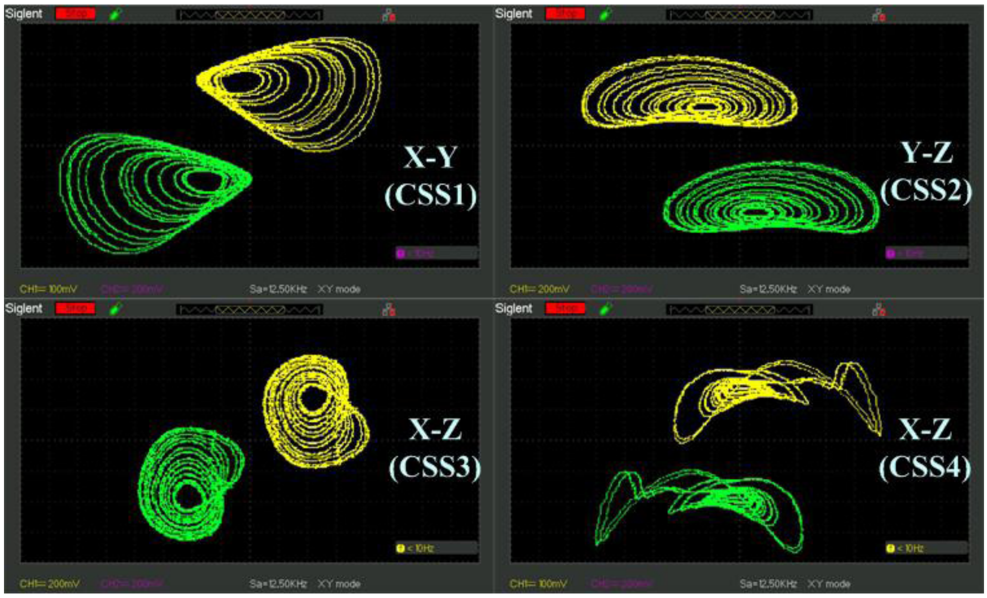

Fig. 13. Coexisting attractors in the oscilloscope obtained from FPGA.

5. Discussion and Conclusions

Symmetric pairs of attractors can be located close to the origin or far from it in phase space. Dynamical systems with conditional symmetry can produce such a special type of coexisting attractors. In such a conditional symmetric system, the polarity balance is destroyed by the symmetric transformation but restored by a further offset boosting. The introduction of nonmonotonic functions can restore the polarity balance by the offset boosting of some variables. Specifically, in a 3D system, 1D offset boosting can restore the reflection or rotation symmetry, and 2D offset boosting can restore the reflection symmetry. Chaotic conditional symmetric systems share an asymmetric structure that hides coexisting symmetric attractors. The basins of attraction show that the coexisting symmetric attractors lie in respective asymmetric regions but with symmetric cross-sections. In some circumstances, this special regime of a system provides a new approach for giving two sets of chaotic signals with opposite polarity in a physical system without requiring extra equipment for achieving polarity transformation.

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