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Comment on "A hidden chaotic attractor in the classical Lorenz system"

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ABSTRACT

In this short communication, we comment on the recent report of a hidden attractor in the classical Lorenz system. We contend that the reported system gives instead a chaotic transient whose duration approaches infinity at a critical value of the parameters. We caution others who are searching for hidden attractors to consider carefully the possibility that the attractor is instead a transient chaotic set.

In a recent article, Munmuangsaen and Srisuchinwong [1] reported the observation of a hidden chaotic attractor in the classical Lorenz system [2]

$$\begin{aligned}\dot{x} &= a(y - x)\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz\end{aligned} \tag{1}$$

for parameters in the vicinity of $(a, r, b) = (4, 29, 2)$ and initial conditions $(x_0, y_0, z_0) = (5, 5, 5)$. However, upon further examination, it appears that this apparent attractor is a long-duration chaotic transient as indicated by the plot of $x(t)$ in Fig. 1 in which the orbit abruptly attracts to one of the stable equilibria at $(x, y, z) = (\pm 7.483315, \pm 7.483315, 28)$ after a time the order of $\tau \sim 1 \times 10^4$ (about ten thousand cycles of the orbit). At the same time, the finite-time largest Lyapunov exponent drops abruptly from about 0.67 to -0.0618, the latter value in agreement with the real part of the largest eigenvalue of the corresponding equilibrium.

The geometric mean duration of the transient τ increases as r is increased, approaching infinity at $r \approx 29.2725$ according to $\tau \approx 200/(29.2725 - r)^{3.8}$ as shown by the least squares fit in Fig. 2. For $r > 29.2725$, the chaotic attractor is self-excited and can be found using initial conditions in the vicinity of the unstable saddle node at the origin and coexists with the two stable foci.

At $r = 29.2725$, there is a boundary crises [3,4] where the strange attractor collides with the basin boundary that separates it from the basin of the two stable equilibria as shown in Fig. 3. At this point, all three basins intersect the saddle point at the origin, shown as a small open black circle in the figure, and the strange attractor, shown in cross section as black lines, is destroyed. The three attractors coexist over the interval $29.2725 < r < 36$. At $r = 36$ the two equilibria undergo a Hopf bifurcation, leaving only the single globally attracting strange attractor for larger values of r. These calculations were done using a fourth-order Runge–Kutta integrator with adaptive step size.

Similar behavior was reported in 1979 by Yorke and Yorke [5] for $(a,\ r,\ b) \approx (10,\ 24.06,\ 8/3)$, along with the mechanism responsible for the transient, although there is no linear transformation of the variables that would indicate that the two regimes are equivalent. They estimated that the power law dependence of τ has a slope between 3.5 and 4.0, which agrees with our estimate of 3.8. The basins of attraction for the two cases are similar [6]. Thus it remains an interesting and open question whether there is a sustained hidden chaotic attractor in the classical Lorenz system.

The same kind of long-duration chaotic transient occurs in the Rabinovich system [7] with $(a, r, b) = (-0.5, 6.8, 0.99)$ where the orbit appears to lie on a hidden strange attractor but abruptly goes to one of the equilibria after a time of $\tau \sim 5 \times 10^7$.

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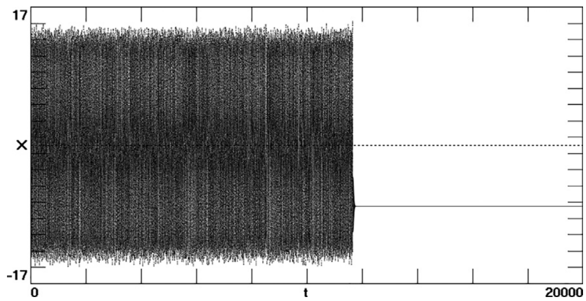

Fig. 1. Transient chaos in the Lorenz system for $(a, r, b) = (4, 29, 2)$.

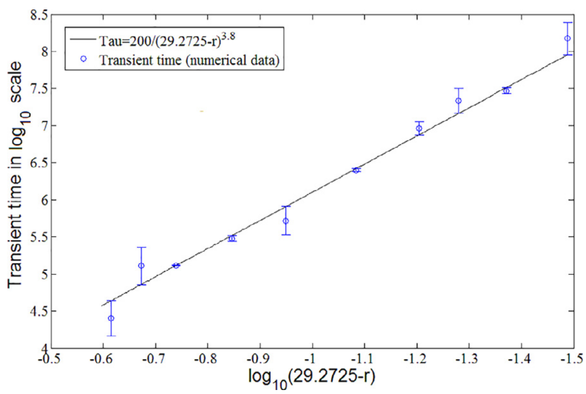

Fig. 2. Geometric mean duration of the chaotic transient as a function of r.

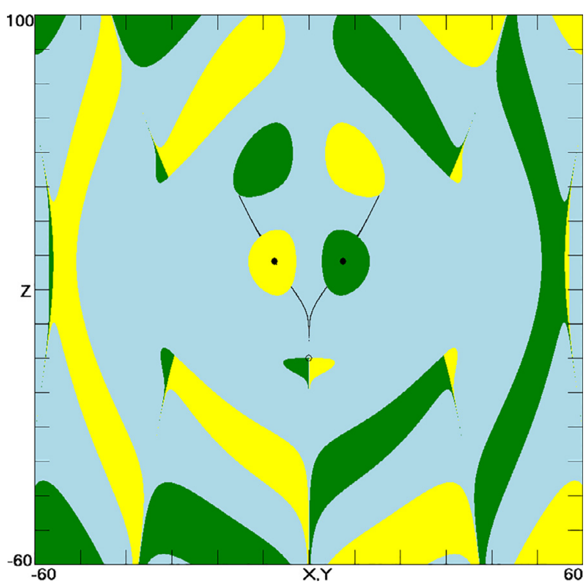

Fig. 3. Basins of attraction in the $x = y$ plane for the Lorenz system with $(a, r, b) = (4, 29.2725, 2)$ where a boundary crisis occurs.

We caution others who are looking for examples of hidden attractors to consider carefully the possibility that the attractor is actually a long-duration “transient chaotic set” as defined by Kuznetsov et al. [7], a term that we prefer to calling it a “transient attractor”. However, in a practical application, a hidden transient chaotic set may be just as problematic as a hidden attractor.

A necessary condition for the existence of a hidden attractor is that the orbit remain on it for a very long time such as $t > 1 \times 10^8$, but a sufficient condition requires searching a neighborhood of parameter space for transient solutions and then determining how the duration of the transient scales with parameters [8]. If very careful tuning of the parameters is required to obtain the hidden attractor, it is likely to be a long-duration transient. On the other hand, it is possible to have a false transient if the step size of the integrator is too large and the attractor comes very close to its basin boundary, as often happens, or if an initial condition is chosen too close to the basin boundary, which is sometimes a complicated fractal.

Finally, we remark on the incongruity of the term “hidden attractor” for cases in which the basin of the attractor is the entirety of the state space except for a set of measure zero representing the unstable periodic orbits, for which there are examples [9]. In fact, there are even hidden attractors in which every initial condition is arbitrarily close to the attractor [10]. Such attractors may be hidden, but they are not hidden very well!

References

[1] Munmuangsaen B, Srisuchinwong B. A hidden chaotic attractor in the classical Lorenz system. Chaos Solitons Fractals 2018;107:61–6.
[2] Lorenz EN. Deterministic nonperiodic flow. J Atmos Sci 1963;20:130–41.
[3] Grebogi C, Ott E, Yorke JA. Chaotic attractors in crisis. Phys Rev Lett 1982;48:1507–10.
[4] Grebogi C, Ott E, Yorke JA. Crises, sudden changes in chaotic attractors and transient chaos. Physica D 1983;13:261–8.
[5] Yorke JA, Yorke ED. Metastable chaos: the transition to sustained chaotic behavior in the Lorenz model. J Stat Phys 1979;21:263–77.
[6] Xiong A, Sprott JC, Lyu J, Wang X. 3D printing – the basins of tristability in the Lorenz system. Int J Bifurcation Chaos Appl Sci Eng 2017;27(1750128):1–5.
[7] Kuznetsov NV, Leonov GA, Mokaev TN, Prasad A, Shrimali MD. Finite-time Lyapunov dimension and hidden attractor in the Rabinovich system. Nonlinear Dyn. 2018. https://doi.org/10.1007/s11071-018-4054-z.
[8] Grebogi C, Ott E, Romeiras F, Yorke JA. Critical exponents for crisis-induced intermittency. Phys Rev A 1987;36:4365–80.
[9] Li C, Sprott JC, Thio W, Zhu H. A new piecewise linear hyperchaotic circuit. IEEE Trans Circuits Syst II, Exp Briefs 2014;61:977–81.
[10] Sprott JC, Hoover WG. Harmonic oscillators with nonlinear damping. Int J Bifurcation Chaos Appl Sci Eng 2017;27(1730037):1–19.