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Simple Chaotic Systems with Specific Analytical Solutions

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In this paper, a new structure of chaotic systems is proposed. There are many examples of differential equations with analytic solutions. Chaotic systems cannot be studied with the classical methods. However, in this paper we show that a system that has a simple analytical solution can also have a strange attractor. The main goal of this paper is to show examples of chaotic systems with a simple analytical solution that is unstable so that the chaotic orbit does not track it. We believe the structures presented here are new. Two categories of chaotic systems are described, and their dynamical properties are investigated. The proposed systems have analytic solutions that exist far from the equilibrium. Of course, all strange attractors are dense in unstable periodic orbits, but mostly the equations that describe these orbits are unknown and difficult to calculate. The analytical solutions provide examples where the orbits can be calculated despite their instability.

Keywords: Analytical solution; chaotic system; strange attractor.

1. Introduction

Chaotic systems and their properties present challenges in the study of dynamical systems [Dudkowski *et al.*, 2018]. There are many ambiguities in the creation of strange attractors. An important question is whether equilibrium points or other structural features influence the creation of chaotic solutions and their properties in a dynamical system. Attractors can be categorized as either self-excited or hidden [Dudkowski *et al.*, 2016; Leonov *et al.*, 2015; Jafari *et al.*, 2015]. Self-excited attractors can be found using initial conditions in the neighborhood of an unstable equilibrium, while hidden attractors are not related to any equilibrium [Sharma *et al.*, 2015; Danca & Kuznetsov, 2017; Wei *et al.*, 2017; Kuznetsov *et al.*, 2017; Kuznetsov *et al.*, 2018; Danca *et al.*, 2017]. Efforts have been made to find methods for the localization of hidden attractors [Kuznetsov *et al.*, 2011; Dudkowski *et al.*, 2015; Nazarimehr *et al.*, 2017a; Nazarimehr *et al.*, 2017b]. Hidden attractors have been found in real-world systems [Hosseini *et al.*, 2017]. Systems with no equilibrium [Wei, 2011; Wang & Chen, 2013; Wei *et al.*, 2013; Nazarimehr *et al.*, 2018], stable equilibrium [Wang & Chen, 2012; Wei *et al.*, 2014; Wei & Zhang, 2014], and curve of equilibria [Gotthans & Petržela, 2015; Barati *et al.*, 2016] belong to the category of hidden attractors. Many

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systems have been designed to have hidden chaotic attractors [Jafari & Sprott, 2013; Jafari *et al.*, 2013; Molaie *et al.*, 2013]. Those systems were designed with specific equilibria, and then the space of initial conditions and parameters were searched for chaotic solutions. Hidden attractors have been studied in many systems [Wei *et al.*, 2015a; Kuznetsov *et al.*, 2018; Wei *et al.*, 2016].

There are many classical methods to solve differential equations analytically [Zaitsev & Polyanin, 2002; Mattheij & Molenaar, 1996]. These methods can identify systems with a bounded exponential solution, which implies that the orbit is approaching an equilibrium point. Also the system can have a periodic solution in which the orbit approaches a limit cycle. However, there is no analytical method to calculate the chaotic solution of a dynamical system. This raises the question of whether the classical methods for solving differential equations can be trusted when there is a complex type of dynamics.

In this paper, we design two new categories of systems with well-known analytical solutions that also exhibit chaos, and we investigate the dynamical properties of these systems. In particular, we design some chaotic flows with the analytical solutions at Ae^{-t} and $A\sin(t)$, but the numerical solutions are attracted to a chaotic attractor instead of the analytical solution.

2. Proposed Systems

The proposed systems are based on the general structure given by the following quadratic jerk system:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= a_1x + a_2y + a_3z + a_4x^2 + a_5y^2\\ &\quad + a_6z^2 + a_7xy + a_8xz + a_9yz + a_{10}.\end{aligned} \tag{1}$$

The ten parameters a_i ($i = 1$ to 10) are chosen so that the solutions are either of the form $x = Ae^{-t}$ or $x = A\sin(t)$ with ($A \in R$). In the first case, the variable x of Eq. (1) is set to $x = Ae^{-t}$, which implies that $y = -Ae^{-t}$ and $z = Ae^{-t}$ from the first and second equalities of Eq. (1). In this paper, we focus on the jerk system. It is an easy and popular form to study chaotic attractors [Wei *et al.*, 2015b]. The necessary conditions for the parameters are then calculated from the third equality of Eq. (1), giving

$$\begin{aligned} a_1 - a_2 + a_3 &= -1,\\ a_4 + a_5 + a_6 - a_7 + a_8 - a_9 &= 0,\\ a_{10} &= 0.\end{aligned} \tag{2}$$

The same process is followed for the second case ($x = A\sin(t)$), giving

$$\begin{aligned} a_1 - a_3 &= 0,\\ a_2 &= -1,\\ a_4 + a_6 - a_8 + a_{10} &= 0,\\ a_5 + a_{10} &= 0,\\ a_7 - a_9 &= 0.\end{aligned} \tag{3}$$

More details of the proposed systems will be discussed in the following subsections.

2.1. *Exponential solution*

System (1) with the constraints of Eq. (2) has the analytical solution Ae^{-t}. Using a time-consuming computer search, some simple systems with this structure with chaotic attractors are found. These systems and some of their dynamical properties are given in Table 1. Since these systems have an analytical solution in the form of $(x, y, z) = (Ae^{-t}, -Ae^{-t}, Ae^{-t})$ and $\lim_{t\to\infty} Ae^{-t} = 0$, the origin is an equilibrium.

As an example, the equilibrium of system F_3 is also calculated from

$$\begin{aligned}\dot{x} = 0 &\Rightarrow y = 0,\\ \dot{y} = 0 &\Rightarrow z = 0,\\ \dot{z} = 0 &\Rightarrow x = 0.\end{aligned} \tag{4}$$

Thus the only equilibrium is at the origin. Stability analysis of this equilibrium shows that its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 0.5 + 2.179i$, $\lambda_3 = 0.5 - 2.179i$. Therefore, the origin is a saddle point with stable and unstable manifolds. As a better explanation, consider the parameter $A = 1$ as a special solution of the discussed systems. The stable manifold is an infinite line given by $x = -y = z$, and thus any initial condition in the form $(x_0, y_0, z_0) = (e^{-t_0}, -e^{-t_0}, e^{-t_0}) = (a, -a, a), \forall\, a \in \Re$ will exponentially approach the equilibrium. All other initial conditions, even ones arbitrarily close to the stable manifold will eventually go to the strange attractor. Furthermore, small numerical errors in the integration will take the orbit off the stable manifold to

Table 1. Three examples of chaotic systems with analytical solution $(x, y, z) = (Ae^{-t}, -Ae^{-t}, Ae^{-t})$.

Case	Equations	Parameters	Equilibrium	Eigenvalue	LEs	(x_0, y_0, z_0)
F_1	$\dot{x} = y$	$a = 4$	0	-1	0.0063	-4.3
	$\dot{y} = z$	$b = 11$	0	0.5+0.8660i	0	8.67
	$\dot{z} = -x + ay^2 + bxz + cyz$	$c = 7$	0	0.5−0.8660i	−1167.7946	−18.7
F_2	$\dot{x} = y$	$a = 0.7$	0	-1	0.0562	−1.36
	$\dot{y} = z$	$b = 0.7$	0	0.5+0.8660i	0	8.32
	$\dot{z} = -x + ax^2 - y^2 + bxy + xz$		0	0.5−0.8660i	−3.6310	3.91
F_3	$\dot{x} = y$	$a = -5$	0	-1	0.1126	−0.97
	$\dot{y} = z$	$b = -4$	0	0.5+2.179i	0	2.53
	$\dot{z} = ax + by - y^2 + xz$		0	0.5−2.179i	−1.5256	2.16

the strange attractor. It should be noted that the focus of this paper to propose new systems is on the jerk equations with complete feedback. The feedback loop can be incomplete in amplitude or polarity. Li *et al.* [2016] have proposed some chaotic flows with incomplete feedback in a jerk structure. Those structures can be used as the general structure of our proposed system and then chaotic attractors can be searched in those structures by considering the calculated conditions.

Figure 1 shows the numerical solution of system F_3 in magenta and the analytical solution in black. The numerical solution runs away from the analytical one after $T = 62$. All of the systems have this same behavior near the origin.

Note that the stable manifold is a set of measure zero totally surrounded by the basin of the chaotic attractor. Thus any small numerical error takes the orbit off the stable manifold to the chaotic attractor. This behavior is called Lyapunov instability [Lyapunov, 1992] and can be seen near the saddle points of a chaotic system (if it exists). Thus analytical methods for solving differential equations are unreliable in such cases. We emphasize that the analytic solution is valid arbitrarily far from the equilibrium, which is highly unusual for a stable manifold. Chaotic attractors of the three systems in Table 1 are shown in Fig. 2. Each row of the figure shows three cross-sections of each chaotic system. Dynamical behaviors of system F_3 are investigated using the bifurcation diagram of Fig. 3. The system has different periodic and chaotic attractors when changing the parameter a, and the positive largest Lyapunov exponent indicates that the solution is chaotic.

2.2. *Sinusoidal solution*

System (1) has a sinusoidal solution with the constraints of Eq. (3). Coexistence of limit cycles and chaotic attractors has been investigated in the numerical solution of flows [Li & Sprott, 2014a, 2014b, 2018]. In this paper, we study the coexistence of an analytical solution with a chaotic attractor. Using a systematic computer search, two chaotic systems in the specific form are obtained and they are given in Table 2. Since the analytical solutions of these systems are in the form $x = A\sin(t), \dot{x} = y = A\cos(t), \ddot{x} = z = -A\sin(t)$, their analytical attractors are cycles. By considering $A = 1$, any initial condition in the form $(\sin(t_0), \cos(t_0), -\sin(t_0))$ will result in the cyclic analytical solution. Figure 4 shows the comparison of analytical (black color) and numerical solutions (magenta color) for system F_5. Small numerical errors in the integration will make the two analytical and numerical solutions separated. In other words, the cyclic solutions are immersed in the basins of attraction of chaotic attractors and unbounded orbits. So their time series cannot remain on the analytical solution for a long time.

Lyapunov exponents and Kaplan–Yorke dimension of strange attractors of systems F_4 and F_5 are presented in Table 2. Strange attractors of these two systems are shown in Fig. 5. All strange attractors are dense in unstable periodic orbits, but mostly the equations that describe them are unknown. The proposed systems are some examples where the orbits can be calculated despite their instability.

Bifurcation diagram and largest Lyapunov exponent of system F_5 are shown in Fig. 6. The

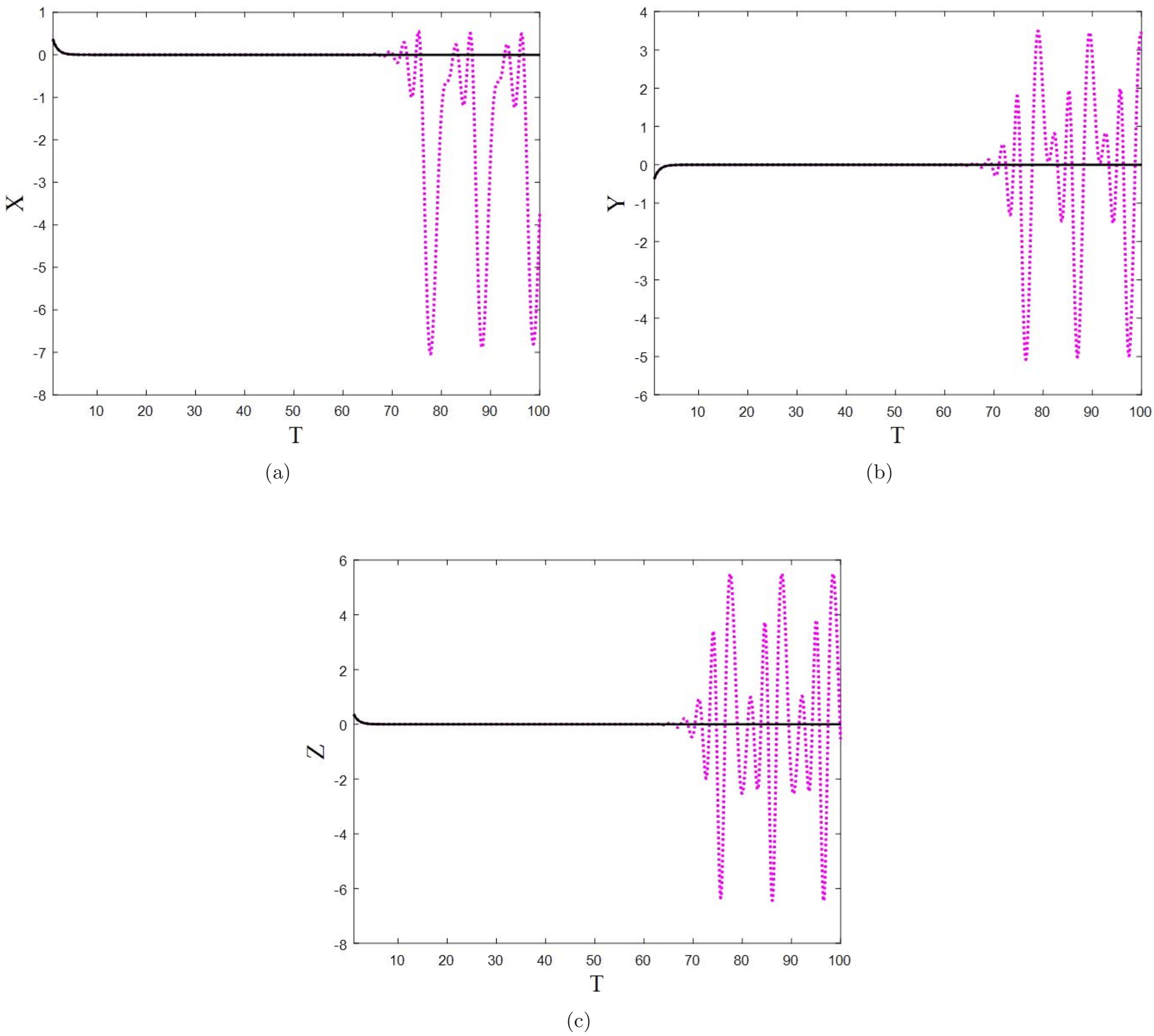


Fig. 1. Time series of the analytical solution $(x, y, z) = (e^{-t}, -e^{-t}, e^{-t})$ (black) and the numerical solution of system F_3 (magenta) with initial conditions $(1, -1, 1)$.

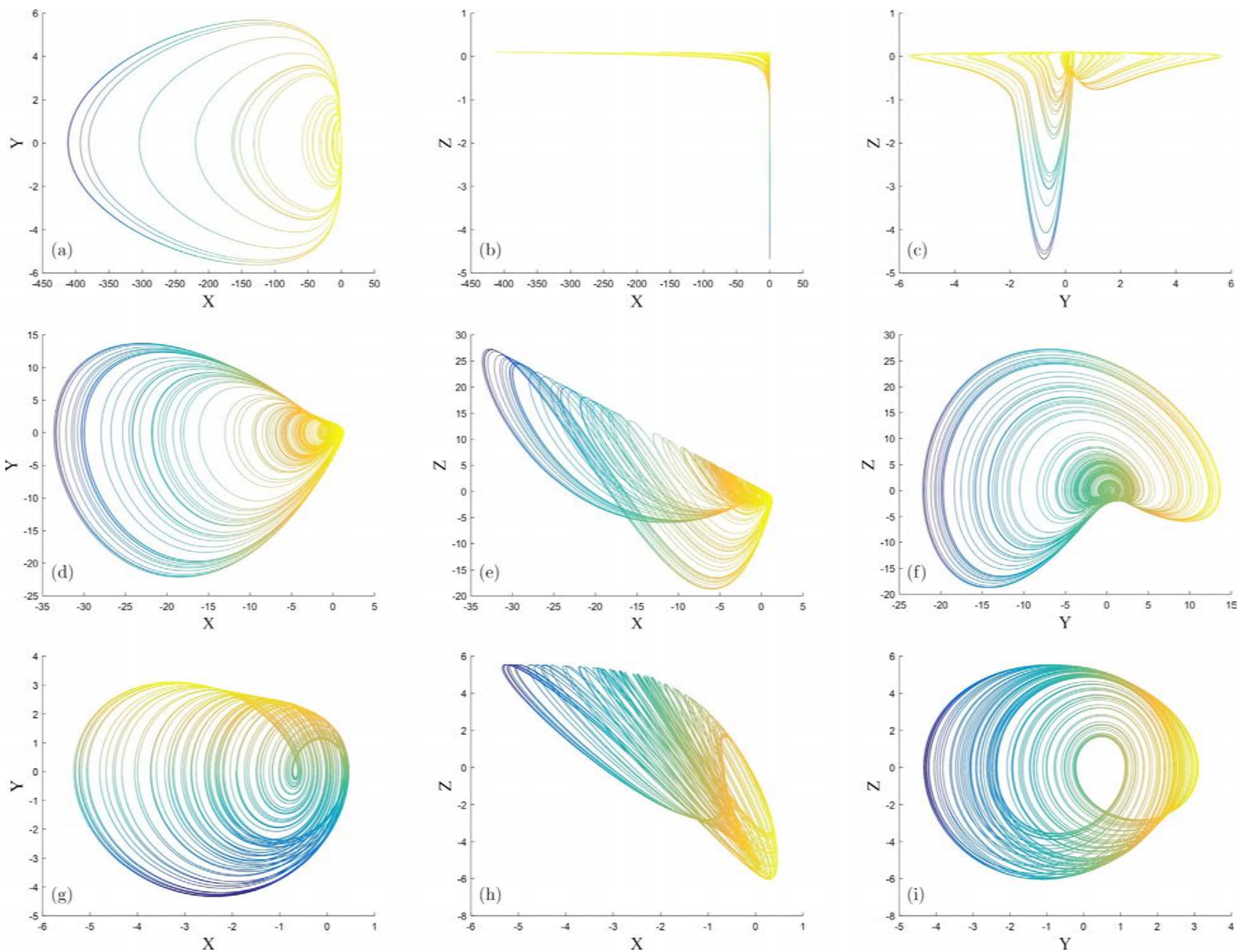
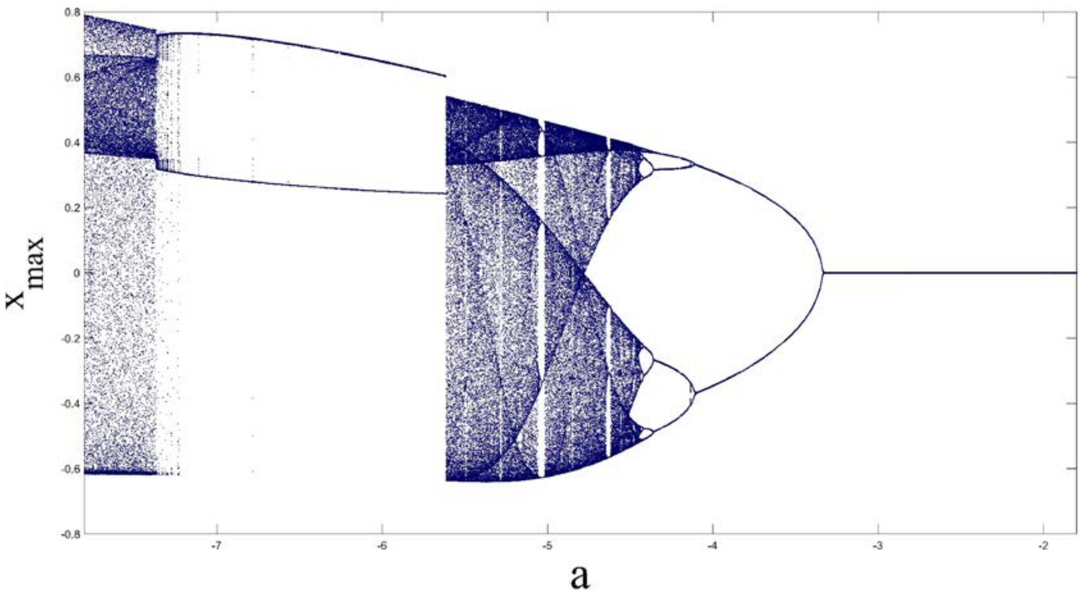
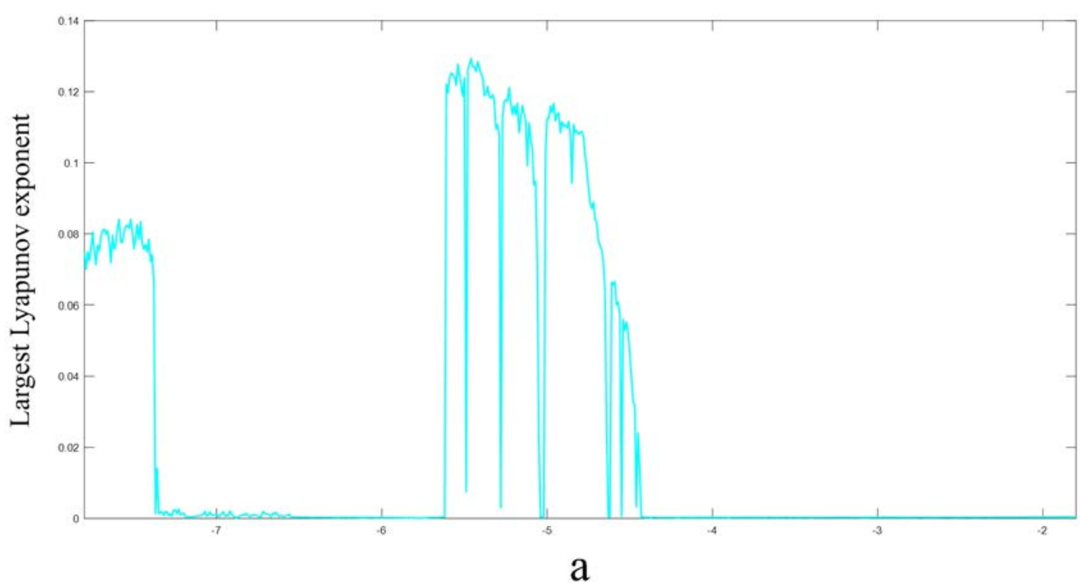


Fig. 2. Strange attractors of the proposed chaotic systems in Table 1. State space of system F_1 with initial conditions $(-4.3, 8.67, -18.7)$ in: (a) X–Y plane, (b) X–Z plane and (c) Y–Z plane. State space of system F_2 with initial conditions $(-1.36, 8.32, 3.91)$ in: (d) X–Y plane, (e) X–Z plane and (f) Y–Z plane. State space of system F_3 with initial conditions $(-0.97, 2.53, 2.16)$ in: (g) X–Y plane, (h) X–Z plane and (i) Y–Z plane.



(a)

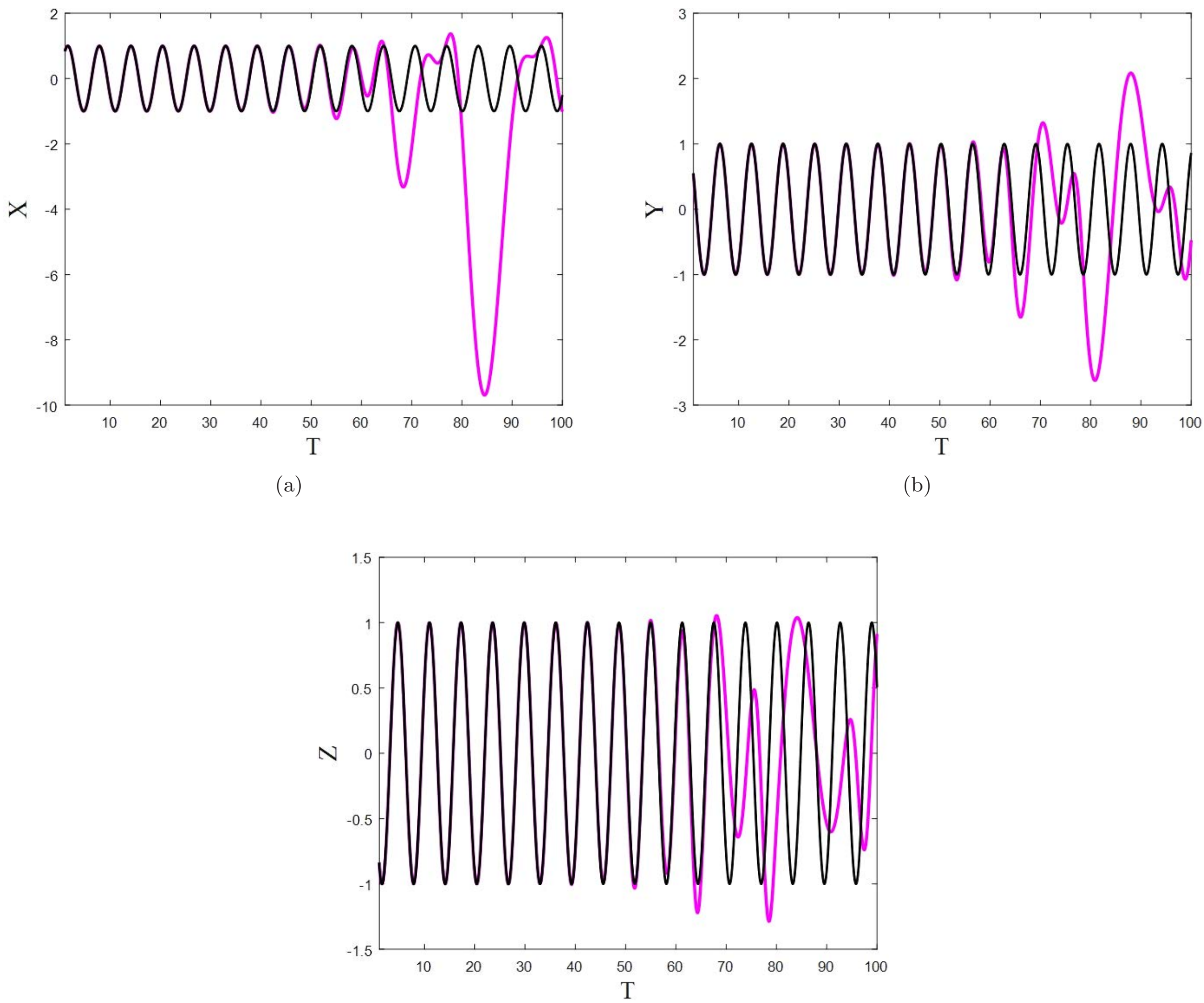


(b)

Fig. 3. (a) Bifurcation diagram and (b) largest Lyapunov exponent of system F_3 with respect to changing parameter a with $b = -4$ and constant initial conditions $(-0.97, 2.53, 2.16)$.

Table 2. Two chaotic systems with analytical solution $(x, y, z) = (A\sin(t), A\cos(t), -A\sin(t))$.

Case	Equations	Parameters	LEs	(x_0, y_0, z_0)
F_4	$\dot{x} = y$		0.1067	−3.97
	$\dot{y} = z$		0	−7.44
	$\dot{z} = -x - y - z - y^2 - z^2 + xy + yz + 1$		−1.1076	−0.07
F_5	$\dot{x} = y$	$a = -1.02$	0.0419	−7.87
	$\dot{y} = z$	$b = -1.02$	0	1.69
	$\dot{z} = ax - y + bz - y^2 + xz + 1$		−6.4681	−5.87



(c)

Fig. 4. Time series of the analytical solution $(x, y, z) = (\sin(t), \cos(t), -\sin(t))$ (black) and the numerical solution of system F_5 (magenta) with initial conditions $(0, 1, 0)$.

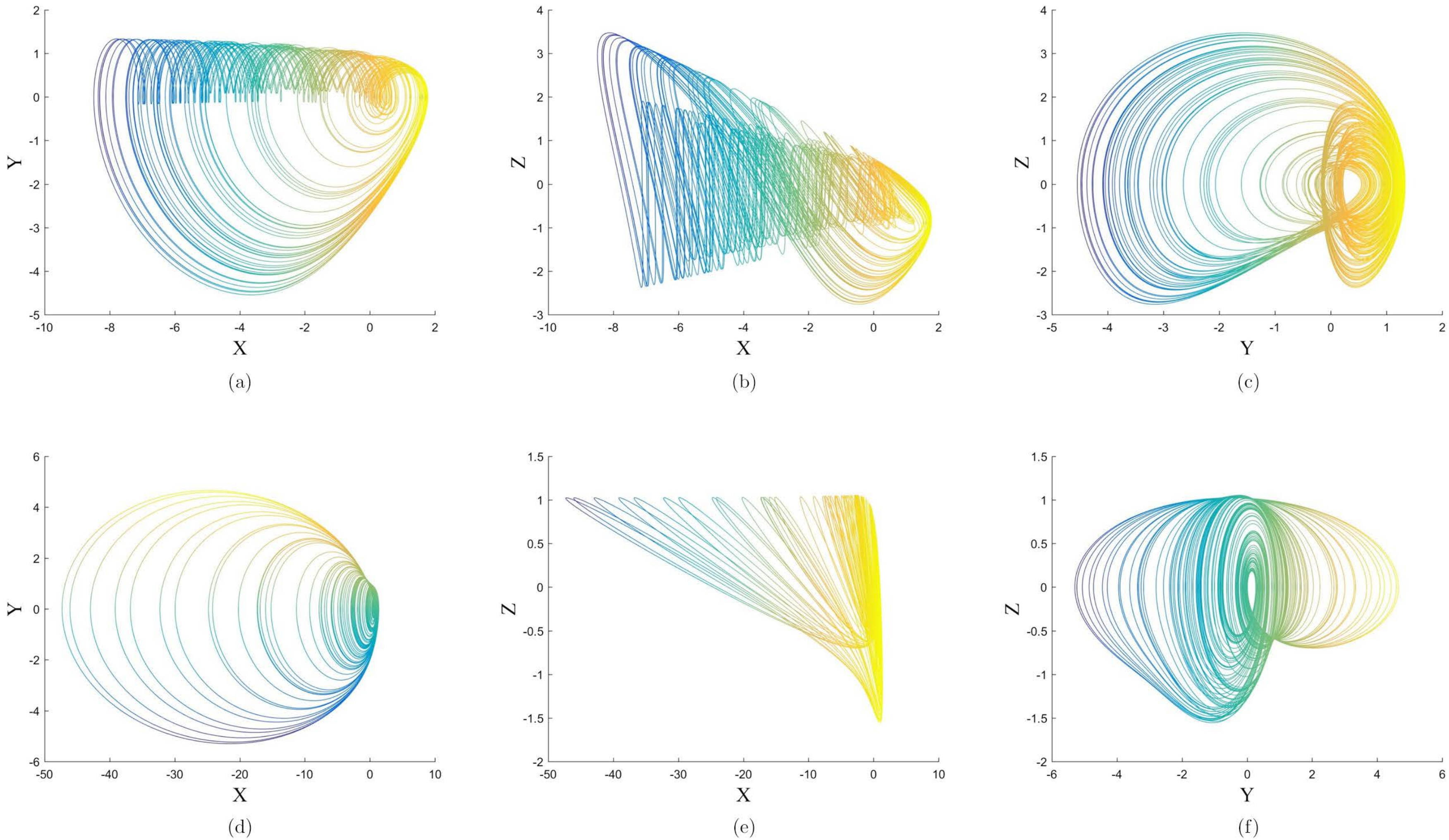
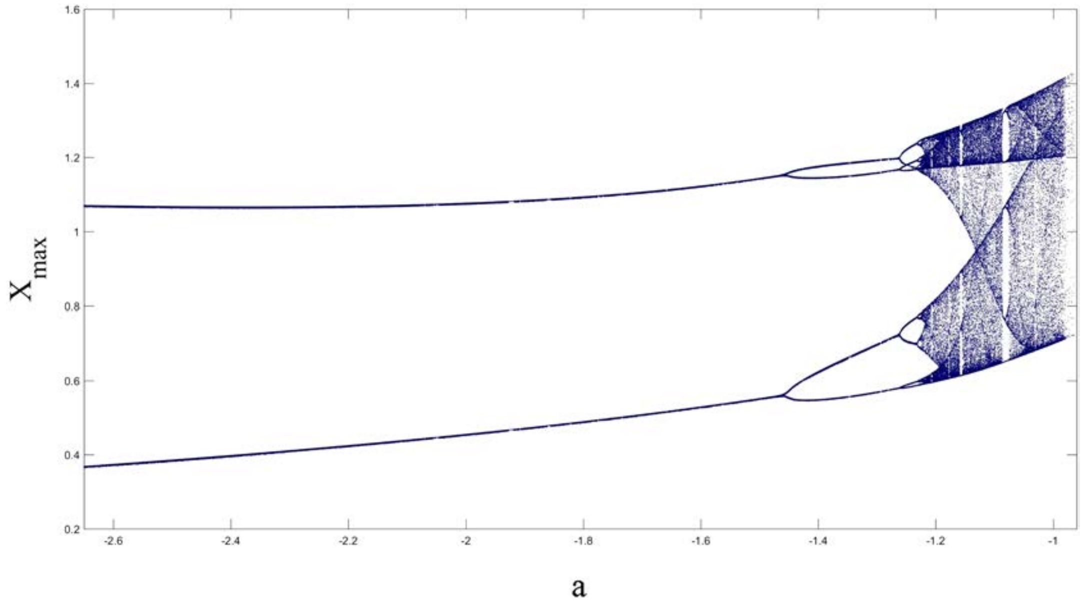
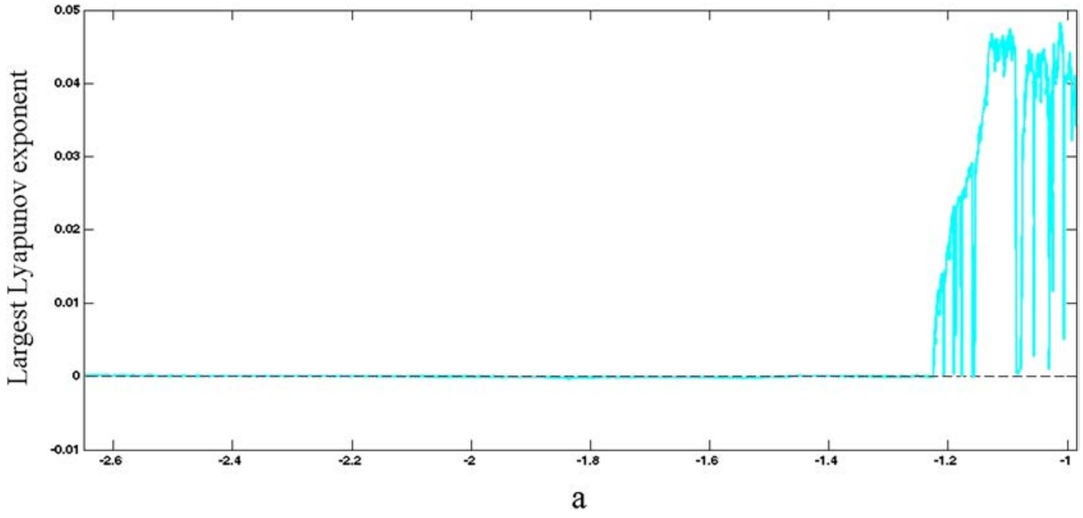


Fig. 5. Strange attractors of the proposed chaotic systems in Table 2. State space of system F_4 with initial conditions $(-3.97, -7.44, -0.07)$ in: (a) X–Y plane, (b) X–Z plane and (c) Y–Z plane. State space of system F_5 with initial conditions $(-7.87, 1.69, -5.87)$ in: (d) X–Y plane, (e) X–Z plane and (f) Y–Z plane.



(a)



(b)

Fig. 6. (a) Bifurcation diagram and (b) largest Lyapunov exponent of system F_5 with respect to changing parameter a and constant initial conditions $(-7.87, 1.69, -5.87)$.

system has a period-doubling route to chaos. Positive largest Lyapunov exponent in Fig. 6(b) proves the chaotic dynamics of system F_5.

3. Conclusion

In this paper, five new chaotic systems with special analytical solutions have been proposed. The proposed systems were in two categories. Three of the proposed systems had the analytical solution $(x, y, z) = (Ae^{-t}, -Ae^{-t}, Ae^{-t})$ and two of them had analytical solution $(A\sin(t), A\cos(t), -A\sin(t))$. Results show that the analytical solutions cannot attract time series of these systems for a long time. In other words, the analytical solution was immersed in the basins of attraction of chaotic attractors and unbounded orbits. Because of the numerical errors in the integration, the states of systems ran away from their analytical solution after a short time. As we know, chaotic systems with simple analytical solutions were proposed for the first time in this paper. The proposed systems were cases that analytic solutions exist far from the equilibrium. In addition, all strange attractors are dense in unstable periodic orbits, but mostly the equations that describe them are unknown. The analytical solutions showed some examples where the orbits can be calculated despite their instability.

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