Two Simplest Quadratic Chaotic Maps Without Equilibrium

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Two Simplest Quadratic Chaotic Maps Without Equilibrium

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Two simple chaotic maps without equilibria are proposed in this paper. All nonlinearities are quadratic and the functions of the right-hand side of the equations are continuous. The procedure of their design is explained and their dynamical properties such as return map, bifurcation diagram, Lyapunov exponents, and basin of attraction are investigated. These maps belong to the hidden attractor category which is a newly introduced category of dynamical system.

Keywords: Chaos; map; hidden attractors.

1. Introduction

Most known examples of chaotic flows have one or more saddle points. Such saddle points allow homoclinic and heteroclinic orbits and the prospect of rigorously proving the chaos when the Shilnikov condition is satisfied. One common way of locating attractors of a dynamical system is to choose the initial condition around its saddle points [Leonov *et al.*, 2011; Kuznetsov *et al.*, 2017]. Such attractors have been called "self-excited," and they are the most common type of dynamical systems described in the literature [Leonov *et al.*, 2012; Leonov & Kuznetsov, 2013; Leonov *et al.*, 2014].

Recently, new chaotic flows have been discovered that are not associated with a saddle point. These include the ones without any equilibrium points [Wei, 2011; Jafari *et al.*, 2013], with only stable equilibria [Wang & Chen, 2012; Molaie *et al.*, 2013], or with a line containing infinitely many equilibrium points [Gotthans & Petržela, 2015; Jafari & Sprott, 2013]. The attractors for such systems is called "hidden attractors" [Leonov *et al.*, 2015; Sharma *et al.*, 2015; Danca & Kuznetsov, 2017], and that is because it is hard to discover them and there is no systematic way to find initial conditions that lead to these attractors except by extensive numerical search. There are plenty of researches which are associated with designing and studying rare examples of simple chaotic flows with hidden attractors [Jafari *et al.*, 2018; Pham *et al.*, 2018;

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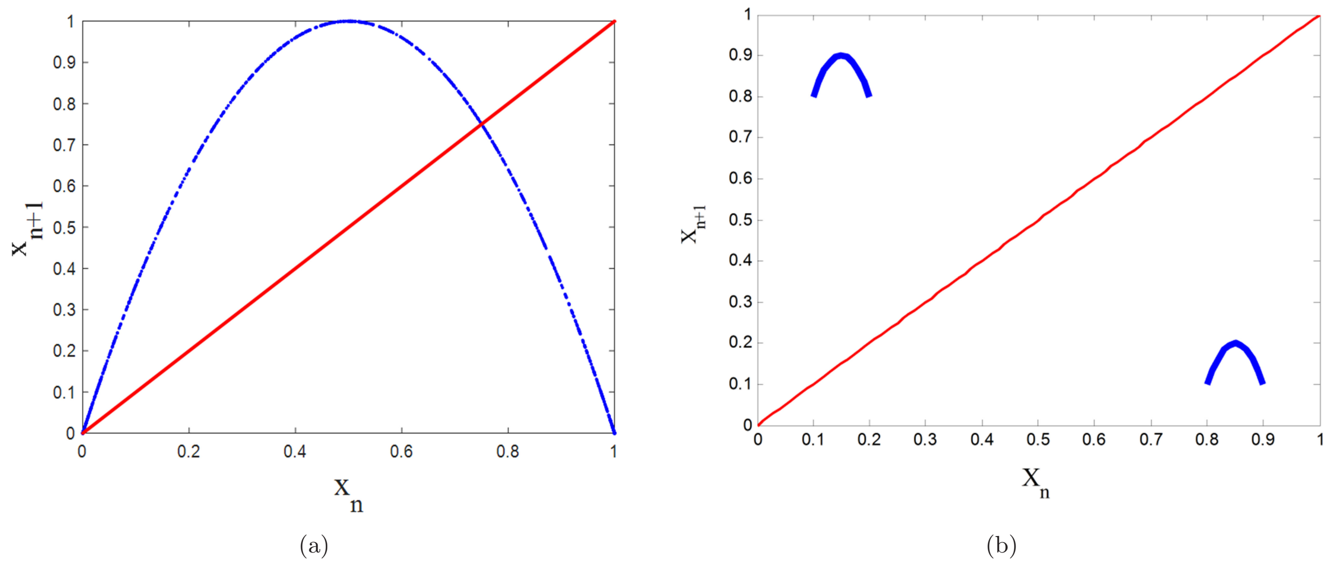


Fig. 1. (a) Logistic map (for $A = 4$) and (b) a chaotic discontinuous map proposed in [Jafari *et al.*, 2016a].

Tang *et al.*, 2018]. However, there is little knowledge about this topic in chaotic maps and discrete systems [Jiang *et al.*, 2016; Jafari *et al.*, 2016a].

In this paper, we consider two-dimensional (2D) chaotic maps with no equilibria which are categorized as a system with hidden attractors [Jafari *et al.*, 2016a]. The authors in [Jafari *et al.*, 2016a] have introduced some chaotic maps with hidden attractors for the first time. A different class of two-dimensional and three-dimensional maps with different kinds of equilibria has been introduced in [Jiang *et al.*, 2016].

It should be noted that, it is impossible to have a one-dimensional "continuous" chaotic map with no equilibria. The reason is that the domain and range of these maps should cover each other; therefore there would be at least one point where the return map in the state space meets the identity functions and this means there exists at least one equilibrium. One example is the known Logistic map [represented in Fig. 1(a)]. It should be noted that Jafari *et al.* [2016a] proposed a 1D chaotic map with no equilibria. However that map was "discontinuous" [Fig. 1(b)].

In this paper, we introduce two simple chaotic 2D No Equilibria Maps (NEM). It is easier to construct NEM with discontinuity in their equations, but they are of less interest [Sprott, 2010]. Thus we focus on quadratic maps with no discontinuity in the right-hand equations. The rest of the paper is organized as follows: Section 2 is about designing new simple 2D maps without equilibria. Some conventional investigations of chaotic maps such as plotting strange attractors, bifurcation diagram, and Lyapunov exponents diagram are done in Sec. 3. Finally, Sec. 4 is the conclusion.

2. New No Equilibrium Maps

In this part, we perform a systematic search to find 2D chaotic maps with no equilibria. We use our own custom software which is described in [Sprott, 2010]. The main point is to find the simplest no equilibrium chaotic maps with quadratic nonlinearities. We consider the following difference equations as a class of 2D maps.

$$\begin{aligned} x(n+1) &= y(n) + x(n), \\ y(n+1) &= F(x(n), y(n)) + y(n)[a_1 x(n) \\ &\quad + a_2 y(n) + a_3] + y(n), \end{aligned} \tag{1}$$

where a_1, a_2, and a_3 are real coefficients and $F(\cdot)$ is a continuous function that determines the type of system we want to design. In order to find the fixed points of System (1), the following conditions should be satisfied:

$$x(n) = y(n) + x(n) \to y(n) = 0, \tag{2}$$

$$\begin{aligned} y(n) &= F(x(n), y(n)) + y(n)[a_1 x(n) \\ &\quad + a_2 y(n) + a_3] + y(n) \\ &\to F(x(n), y(n)) = 0. \end{aligned} \tag{3}$$

Thus, to design the chaotic map with no equilibrium, we should just choose $F(x(n), y(n))$ in a way

Table 1. Simplest quadratic chaotic map with no equilibria.

Case	Map	Initial Condition	Parameter
NEM_1	$x(n+1) = y(n) + x(n)$ $y(n+1) = 0.1x(n)^2 + 0.1 + y(n)[-x(n) - by(n)] + y(n)$	$(1.7, -0.39)$	$b = 2$
NEM_2	$x(n+1) = y(n)$ $y(n+1) = -0.9y(n)^2 + cx(n)y(n) + 1$	$(1, 0.1)$	$c = 2$

that it cannot become zero. If we choose the function as second degree equation with $\Delta < 0$, this map cannot have any real equilibria:

$$\begin{aligned} &F(x(n), y(n)) = a_4x(n)^2 + a_5x(n) + a_6, \\ &\text{Condition:}\ a_5^2 - 4a_4a_6 < 0. \end{aligned} \tag{4}$$

Another simple structure which can be used in designing 2D maps without equilibria, is Hénon-like map. Thus we consider the following quadratic 2D map:

$$\begin{aligned} x(n+1) &= y(n), \\ y(n+1) &= a_1x(n) + a_2y(n) + a_3x(n)^2 \\ &\quad + a_4y(n)^2 + a_5x(n)y(n) + a_6. \end{aligned} \tag{5}$$

To find the equilibria of this map the following conditions are needed:

$$\begin{aligned} x(n) &= y(n), \\ y(n) &= a_1x(n) + a_2y(n) + a_3x(n)^2 \\ &\quad + a_4y(n)^2 + a_5x(n)y(n) + a_6. \end{aligned} \tag{6}$$

Thus, in order to find the equilibria, the following equation should be solved:

$$(a_3 + a_4 + a_5)x^2 + (a_1 + a_2 - 1)x + a_6 = 0. \tag{7}$$

To design a system without equilibria, it is enough to find the coefficients of Eq. (7) which satisfy $\Delta < 0$. As a result, the condition to find the map without equilibria is Eq. (8):

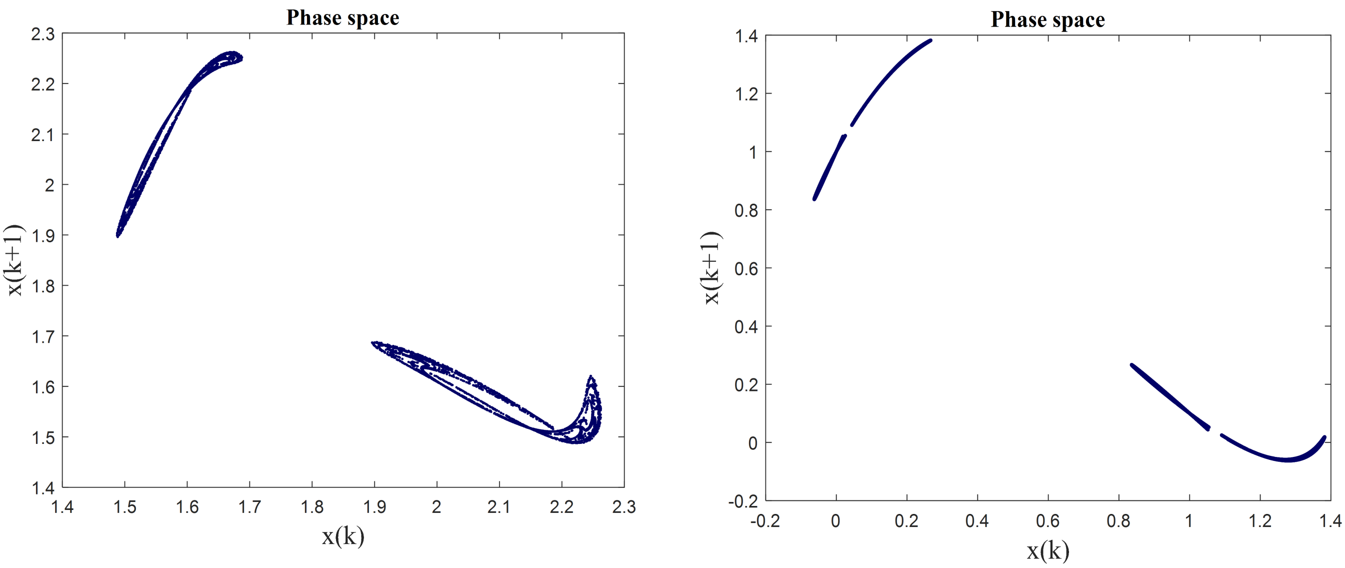
$$\begin{aligned} x(n) &= y(n), \\ y(n) &= a_1x(n) + a_2y(n) + a_3x(n)^2 \\ &\quad + a_4y(n)^2 + a_5x(n)y(n) + a_6, \\ \text{Condition:}\ &(a_1 + a_2 - 1)^2 \\ &\quad - 4a_6(a_3 + a_4 + a_5) < 0. \end{aligned} \tag{8}$$

We performed a systematic search to find chaotic solutions in Systems (1) and (5) considering the constraints calculated in the other above equations. Our search was based on the methods described in [Sprott, 2010] (this method was first used in [Sprott, 1994], and have been used in many other researches such as [Jafari *et al.*, 2013; Molaie *et al.*, 2013; Jafari & Sprott, 2013; Barati *et al.*, 2016; Sprott *et al.*, 2015; Jafari *et al.*, 2016b; Jafari *et al.*, 2016c]). We used our own custom software. Our objective was to find the algebraically simplest cases which cannot be further reduced by the removal of terms without destroying the chaos. So we did an exhaustive computer search considering many thousands of combinations of the coefficients and initial conditions subject to the constraints, seeking cases for which the largest Lyapunov exponent is greater than 0.001. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" in [Sprott, 2010], by which we mean that as many coefficients as possible are set to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits.

The simplest cases obtained from the search procedure (which may be the simplest possible NEMs) are listed in Table 1.

3. Dynamical Analysis

In this section, we perform routine dynamical analysis of the designed maps. Strange attractors of the cases in Table 1 are plotted in Fig. 2. Bifurcation, Lyapunov exponent and Kaplan–Yorke diagrams show different behaviors of the system when changing its parameters. For NEM_1, Fig. 3(a) shows the bifurcation diagram with the change of the parameter a, Fig. 3(b) is Kaplan–Yorke diagram, and Fig. 3(c) represents the Lyapunov exponents diagram which is derived using the Wolf's algorithm [Wolf *et al.*, 1985]. The same plots for NEM_2 can be seen in Fig. 4. It can be seen there exists common period-doubling route to chaos. Figure 5 represents the basin of attraction of NEM_1. It is clear that both cases in Table 1 belong to the systems with hidden attractors, because there is no equilibrium which can have intersection with the basin



(a) (b)

Fig. 2. Strange attractors of the cases in Table 1: (a) NEM_1 and (b) NEM_2. Note that the two separated sets of points belong to one attractor consisting of two parts.

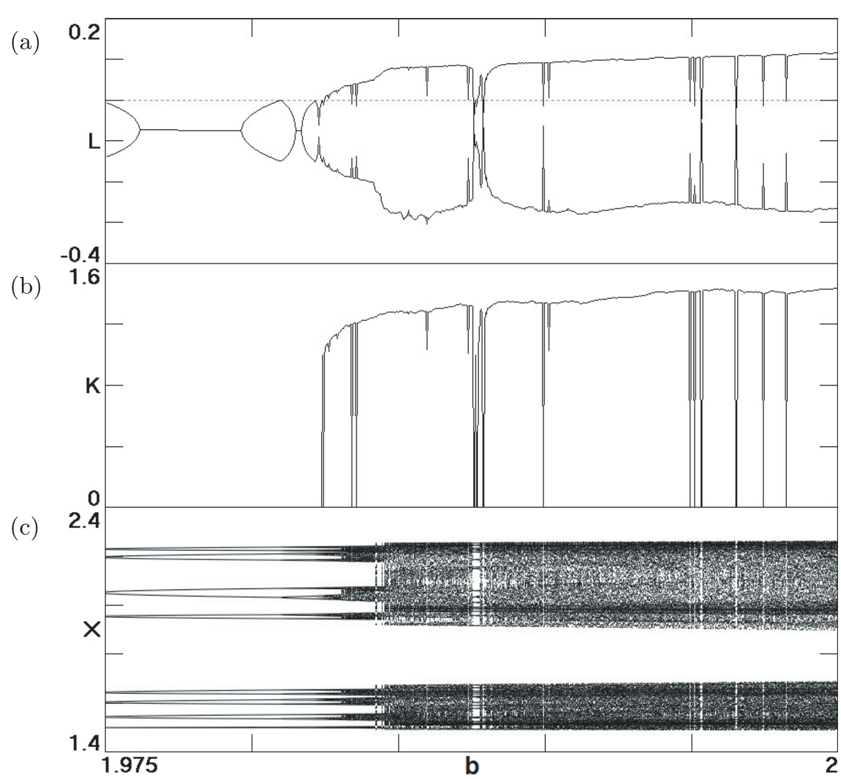


Fig. 3. (a) Lyapunov exponents diagram, (b) Kaplan–Yorke diagram and (c) bifurcation diagram of the NEM_1 with respect to the parameter b.

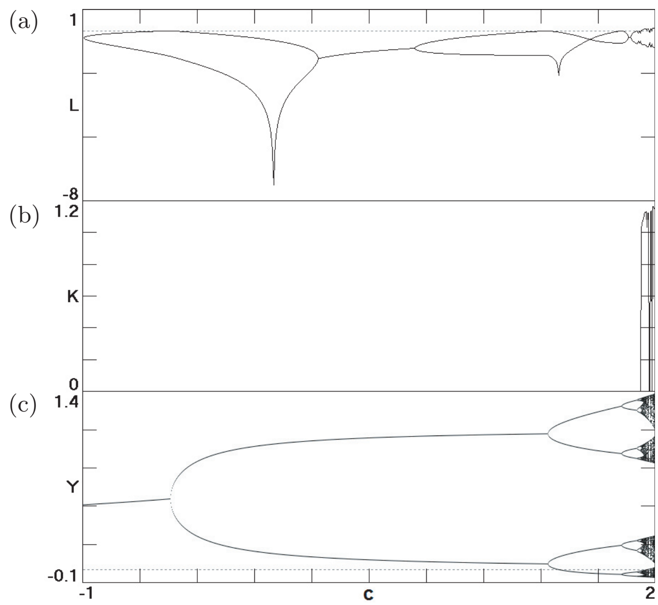


Fig. 4. (a) Lyapunov exponents diagram, (b) Kaplan–Yorke diagram and (c) bifurcation diagram of the NEM_2 with respect to the parameter c.

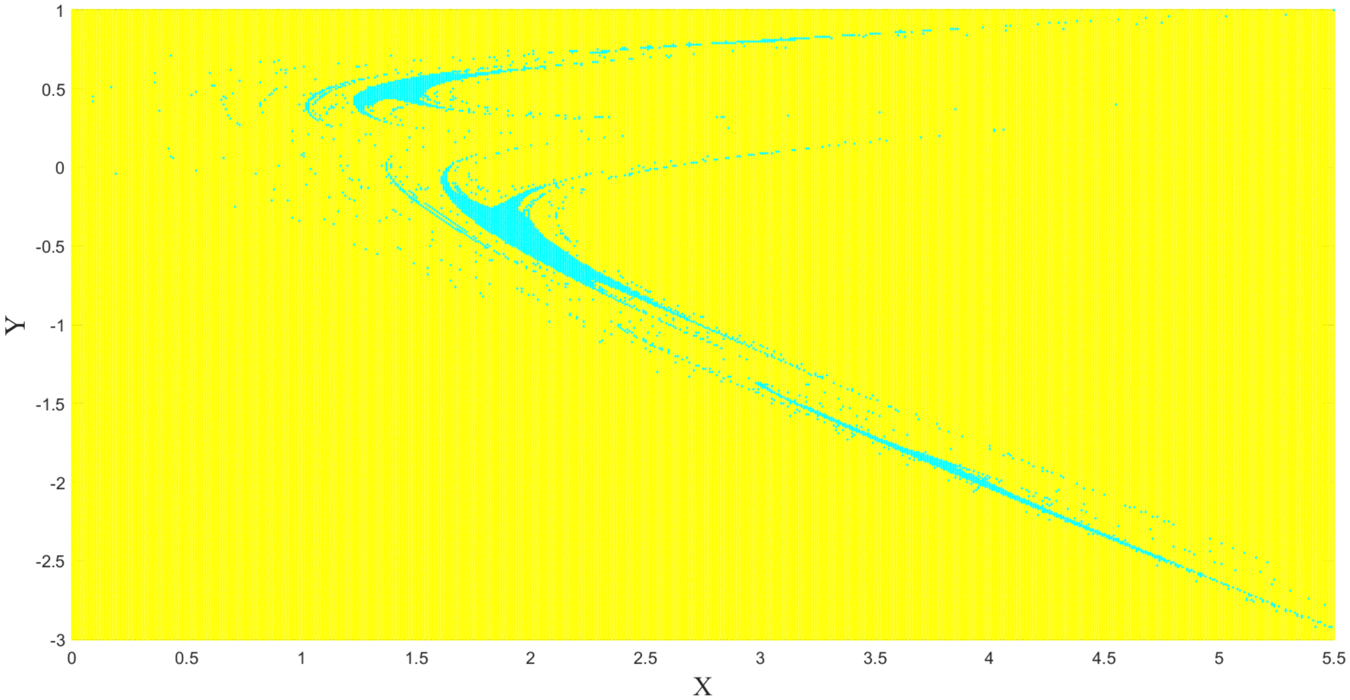


Fig. 5. Basin of attraction of the NEM_1. Initial conditions in the light blue region lead to the chaotic attractor, and unbounded regions are shown in yellow.

of strange attractor. As shown, initial conditions in the light blue region lead to the chaotic attractor, and unbounded regions are shown in yellow.

4. Conclusion

In this paper, we have tried to design the simplest two-dimensional chaotic maps with no equilibria. Such maps belong to the hidden attractors' category which is a newly introduced category of dynamical systems. First, we gave a detailed explanation about the procedure of designing such systems with quadratic nonlinearities and continuous right-hand side functions. Then some dynamical analysis was investigated by the help of plotting return map, bifurcation diagram, Lyapunov exponents diagram, and basin of attraction.

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