OPTIMIZATION OF F-THETA PUMPING

J.C. SPROTT

PLP 1084

June 1991

Plasma Studies

University of Wisconsin

Optimization of F-Theta Pumping

J. C. Sprott

I. Introduction

A difficulty with current drive by F-Theta pumping in an RFP is that when sufficiently high oscillating voltages are applied to the gaps to provide the desired rectified dc voltage to sustain the plasma, reversal may be lost during a portion of the cycle if F swings positive. This loss of reversal will likely raise the plasma resistance, producing an effect opposite to the one sought. Here we calculate using the Polynomial Function Model (PFM)[1] the variation of F that accompanies steady-state F-Theta pumping and the conditions under which this variation is mimimized. Conceptual design examples are provided for MST running both as an RFP and as a tokamak.

II. Polynomial Function Model

Reference 1 [Eq. (43)] considers a case in which the toroidal and poloidal gap voltages are given by

$$V_t = v_t \cos \omega t$$

$$V_p = v_p \cos(\omega t - \phi)$$

The condition for maintaining a plasma with resistance R_p at current I_p is given [Eq. (49)] by

$$I_p R_p = 9 v_t v_p \Theta^2 \sin\phi \;/\; (5+10\Theta^2)\mu_o a \omega I_p$$

where a is the minor radius. Theta is given by

$$\Theta = \mu_o I_p a \;/\; 2\Phi$$

and Φ is the toroidal magnetic flux. The flux and plasma current have superimposed on their average values oscillatory parts [Eq. (44)] given respectively by

$$\delta\Phi = v_t \sin \omega t \;/\; \omega$$

and

$$\delta I_p = v_p \sin(\omega t - \phi)/\omega L - A v_t \sin(\omega t)/\omega L$$

where [Eq. (26)]

$$L = 9\mu_o R_o(2+3\Theta^2) \;/\; 8(6+\Theta^2)$$

and [Eq. (28)]

$$A = 4R_o\Theta(1-\Theta^2) \;/\; a(8+3\Theta^3)$$

and R_o is the major radius. The F-theta relation for the PFM is given approximately [Eq. (17)] by

$$F = 1 - \Theta^2/2$$

There are a number of approximations in the above equations which are discussed in reference 1.

III. Perturbation Analysis

The quantity we want to calculate is the magnitude of the oscillation in F, which we will denote by $|\delta F|$. The calculation is carried out to first order in the amplitude of the oscillations which are assumed small compared to the equilibrium values. The result is

$$|\delta F| = \Theta[(5+10\Theta^2)\mu_o aR_p(C^2/\alpha+\alpha-2C\cos\phi)/\omega\sin\phi]^{\frac{1}{2}}/3L$$

where

$$\alpha = v_p \;/\; v_t$$

and

$$C = A + 2L\Theta/\mu_o a$$

The first significant new result is that there is an optimum ratio (α) for the oscillating voltages on the two gaps that mimimizes $|\delta F|$, determined by setting $d|\delta F|/d\alpha$ equal to zero, and this optimum ratio is a function only of theta and the aspect ratio:

$$\alpha = C = 4R_o\Theta(1-\Theta^2)/a(8+3\Theta^3) + 9R_o\Theta(2+3\Theta^2)/4a(6+\Theta^2)$$

For MST ($R_o/a = 1.5/0.52$) at $\Theta = 1.6$, the result is about $\alpha =$ 10.3. For an RFP, most of the oscillating voltage should thus be applied to the poloidal gap. The quantity $a\alpha/R_o$ is plotted versus Θ in Fig. 1. For the optimum value of α, the expression for $|\delta F|$ simplifies somewhat:

$$|\delta F| = (2\Theta/3L)[C(5+10\Theta^2)\mu_o aR_p/\omega\sin\phi]^{\frac{1}{2}}\sin(\phi/2)$$

We can now ask what is the optimum ϕ for smallest $|\delta F|$. Fig. 2 shows a graph of $\sin(\phi/2)/[\sin\phi]^{\frac{1}{2}}$ which represents the phase-dependent part of the above expression. It rises monotonically from zero at $\phi =$ 0 to infinity at $\phi = 180°$. There is no real solution for $180° < \phi < 360°$ since that represents negative heating. With a phase angle of zero, F does not vary. This case corresponds to the plasma current and toroidal flux oscillating in phase so that theta (and hence F) remain constant. However, with ϕ small, the oscillating voltages required become very large, which poses technical complications in addition to the plasma physics issues involved with large oscillations of I_p and Φ. It is probably best to keep ϕ near 90° where the voltage requirements are minimum, in which case the variation of F simplifies further to:

$$|\delta F| = [R_p/\omega\mu_o R_o]^{\frac{1}{2}}\, G(\Theta)$$

where $G(\Theta)$ is a dimensionless function of theta given by

$$G(\Theta) = [16\Theta(6+\Theta^2)/27(2+3\Theta^2)]\{[2\Theta(1-\Theta^2)/(8+3\Theta^3)+9\Theta(2+3\Theta^2)/8(6+\Theta^2)](5+10\Theta^2)\}^{\frac{1}{2}}$$

with a value of about 6.2 at $\Theta = 1.6$ and whose graph is shown in Fig. 3.

Thus minimizing $|\delta F|$ amounts to minimizing R_p/ω since Θ is constrained to a narrow range for RFP operation and $\mu_o R_o$ is a constant with units of inductance and equal to 1.885 μH for MST. Thus we require low plasma resistance and high frequency. The frequency is limited by the requirement that field-line reconnection occur faster than the time scale of the oscillation so that the plasma relaxes to its preferred F-Θ value. As a design example, take $\omega = 2\pi \times 1000$ Hz and R_p = 10V / 500 kA. Then $|\delta F|$ is 0.25. Thus we might operate with F = -0.25 and swing from F = 0 to -0.5. These values are consistent with $\Theta = 1.6$ according to the PFM. The voltages required are

$$v_p = Cv_t = (I_p/3\Theta)[\mu_o a\omega R_p C(5+10\Theta^2)/\sin\phi]^{\frac{1}{2}}$$

which for this case with $\phi = 90°$ is about v_p = 530 volts and v_t = 51 volts. The ac impedances of the circuits are

$$Z_p = [-(A\omega L/\alpha)\sin\phi + j\omega L(1-(A/\alpha)\cos\phi)] \;/\; [1+A^2/\alpha^2-2(A/\alpha)\cos\phi]$$

and

$$Z_t = (\omega La/R_o)(\alpha\Theta\sin\phi+Bj) \;/\; (\alpha^2\Theta^2\sin^2\phi+B^2)$$

where

$$B = L(2+\Theta^2)/\mu_o a + A\Theta - \alpha\Theta\cos\phi$$

For $\alpha = C$ and the other parameters as above, the numerical values for MST are

$$Z_p = 2.04 + 14.8j \text{ m}\Omega$$

and

$$Z_t = 0.18 + 0.16j \text{ m}\Omega$$

IV. Numerical Analysis

In order to test many aspects of the predictions including algebraic errors and the neglect of higher order terms, the calculation was repeated numerically by successive iteration of the following system of equations:

$$\text{Initialize } I_p,\ \Phi,\ t$$

$$\Theta = \mu_o I_p a \ / \ 2\Phi$$

$$L = 9\mu_o R_o(2+3\Theta^2) \ / \ 8(6+\Theta^2)$$

$$A = 4R_o\Theta(1-\Theta^2) \ / \ a(8+3\Theta^3)$$

$$d\Phi/dt = v_t \cos \omega t$$

$$dI_p/dt = [v_p\cos(\omega t-\phi) - Av_t\cos(\omega t) - I_pR_p] \ / \ L$$

The equations were solved using the forward Euler method with Δt = 0.156 μs. The results are practically indistinguishable from the solution with Δt = 1.56 μs. Initial values at t = 0 were taken as I_p = 500 kA and Φ = 0.102 Wb. Other parameters used were R_o = 1.5 m, a = 0.52 m, $\omega = 2\pi$ x 1000 Hz, ϕ = 90°, and R_p = 20 $\mu\Omega$. Two cases were examined, one with $v_p = v_t = 0$ and a second with v_p = 530 V and v_t = 51 V. The second case corresponds to the one for which the perturbation analysis predicts a constant average plasma current and a field reversal parameter that barely remains negative. The result is shown in Fig. 4. The numerical result is slightly more optimistic than the perturbation analysis, presumably because of higher order terms that were neglected in the perturbation analysis. Note that when averaged over a cycle, the toroidal flux Φ is constant, and so if the plasma current decays, theta will decrease and F will become more positive as seen in the figure. The numerical code has also been used to corroborate various qualitative aspects of the perturbation analysis such as the dependence on ϕ, α, and Θ.

The parameters of the numerical calculation were varied slightly by trial-and-error to obtain a case in which the plasma current is constant with an average value of 500 kA. This required an initial plasma current of I_p = 471 kA and gap voltages of v_p = 440 V and V_t = 42 volts. These voltages are about 17% lower than predicted by the perturbation analysis. The result in Fig. 5 shows a constant plasma current and a field-reversal parameter that remains comfortably negative. The gap ac impedances deduced from the phase and amplitude of the numerically calculated δI_p and δI_t agree with the perturbation analysis to the order of 1%. The parameters for this case are taken as the MST F-Theta pumping design example (RFP) and are summarized in Table I.

Note that there is a 25% error in the power balance ($P_p + P_t \neq I_p^2R_p$) presumably resulting from the approximations used for the variation of L, A, and F with Θ. The Polynomial Function Model should conserve energy exactly since that is one of its input assumptions. This error is in a direction such as to offset the difference between the numerical calculation and the perturbation analysis. Thus the perturbation analysis may fortuitously be more accurate than it appears.

V. Tokamak Case

F-Theta pumping might also provide current drive in a tokamak. There is nothing in the preceding analysis that is special to an RFP. The only assumption is that there is a preferred current density profile that the plasma relaxes to on a time scale shorter than the period of the oscillation. The toroidal current density profile assumed by the PFM for a tokamak with $q = 2.5$ at $r = a$ is shown in Fig. 6. It is only slightly broader than what it typically observed in tokamak experiments. Thus the calculation follows through in the same fashion as above except for a change in terminology. In the tokamak case, there is little concern with the field reversal parameter F which is very close to unity. However, the safety factor q must remain comfortably high. The safety factor at $r = a$ is related to Θ by

$$q = a / R_o\Theta$$

For small theta, the F-theta curve is given more accurately by

$$F = 1 - 3\Theta^2/8$$

The quantity we want to calculate is the magnitude of the oscillation in q, which we will denote by $|\delta q|$. The calculation is carried out to first order in the amplitude of the oscillations which are assumed small compared to the equilibrium values. The result is

$$\delta q = 4q\delta F / 3\Theta^2$$

As with the RFP, there is an optimum ratio (α) for the oscillating voltages on the two gaps that mimimizes $|\delta q|$, and this optimum ratio is a function only of theta (or q) and the aspect ratio:

$$\alpha = C = 4R_o\Theta(1-\Theta^2)/a(8+3\Theta^3) + 9R_o\Theta(2+3\Theta^2)/4a(6+\Theta^2)$$

For MST ($R_o/a = 1.5/0.52$) at $q = 2.5$ ($\Theta = 0.1387$), the result is about $\alpha = 0.504$. This means that most of the voltage should be applied to the toroidal gap. As pointed out in reference 1, the poloidal gap can be connected to a passive element (i.e., it need not be driven directly) in the tokamak case.

As a design example, take $\langle B_t \rangle = 0.6$ T, $\omega = 2\pi \times 1000$ Hz and $R_p = 1V / 216$ kA. Then $|\delta q|$ is 0.42. Thus we might operate with $q = 2.5$ and swing from $q = 2.08$ to 2.92. The voltages required are $v_p = 116$ volts and $v_t = 230$ volts. The ac impedances of the circuits are

$$Z_p = -1.54 + 3.96j \ m\Omega$$

and

$$Z_t = 0.021 + 0.696j \ m\Omega$$

The negative real part of Z_p means that ohmic power exits the poloidal gap, and thus it can be connected to a passive element with an impedance $-Z_p$ (i.e., a series RC or parallel RL). All the power is fed in through the toroidal gap.

In order to test many aspects of the predictions including algebraic errors and the neglect of higher order terms, the calculation was repeated numerically as for the RFP case. Initial values at $t = 0$ were taken as $I_p = 216$ kA and $\Phi = 0.51$ Wb. Other parameters used were $R_o = 1.5$ m, $a = 0.52$ m, $\omega = 2\pi \times 1000$ Hz, $\phi = 90°$, and $R_p = 4.63$ $\mu\Omega$. Two cases were examined, one with $v_p = v_t = 0$ and a second with $v_p = 116$ V and $v_t = 230$ V. The second case corresponds to the one for which the perturbation analysis predicts a constant average plasma current and a q that oscillates ±0.42 about 2.5. The result is shown in Fig. 7. As with the RFP case, the numerical result is slightly more optimistic than the perturbation analysis, presumably because of higher order terms that were neglected in the perturbation analysis. The numerical code has also been used to corroborate various qualitative aspects of the perturbation analysis such as the dependence on ϕ, α, and q.

The parameters of the numerical calculation were varied slightly by trial-and-error to obtain a case in which the plasma current is constant with an average value of 216 kA. This required an initial plasma current of $I_p = 200$ kA and gap voltages of $v_p = 71$ V and $V_t = 140$ volts. These voltages are about 39% lower than predicted by the perturbation analysis. The result in Fig. 8 shows a constant plasma current and a safety factor that remains comfortably high. The gap ac impedances deduced from the phase and amplitude of the numerically calculated δI_p and δI_t agree with the perturbation analysis to the order of 1%. The parameters for this case are taken as the MST F-Theta pumping design example (tokamak) and are summarized in Table II.

VI. Conclusions

With proper optimization, F-Theta pumping appears possible (though marginal) in MST in both the RFP and tokamak modes provided reasonably low loop voltages are obtained and the plasma relaxes to a preferred current-density profile in a small fraction of the plasma L/R time ($\mu_o R_o/R_p$). The gap voltage ratios and phases need to be accurately maintained, and the rf power must be equivalent to the ohmic power that it replaces. To maintain a true steady state, the toroidal field needs to be appropriately power-crowbarred.

Reference

1. J. C. Sprott, Phys. Fluids 31, 2266 (1988).

Table I

MST F-Theta Pumping Design Example (RFP Case)

Quantity	Symbol	Value
Major radius	R_o =	1.5 m
Minor radius	a =	0.52 m
Characteristic inductance	$\mu_o R_o$ =	1.885 μH
Ratio of v_p/v_t	α =	10.3
Amplitude of poloidal gap voltage	v_p =	440 V
Amplitude of toroidal gap voltage	v_t =	42 V
Oscillation frequency	f =	1000 Hz
Phase angle	ϕ =	90°
Field reversal parameter	F =	-0.30
Amplitude of F oscillation	$\|\delta F\|$ =	0.21
Dimensionless δF	G =	6.2
Pinch parameter	Θ =	1.6
Amplitude of Θ oscillation	$\|\delta\Theta\|$ =	0.132
Safety factor (at r=a)	q =	-0.217
Amplitude of q oscillation	$\|\delta q\|$ =	0.018
Toroidal flux	Φ =	0.102 Wb
Amplitude of toroidal flux oscillation	$\|\delta\Phi\|$ =	0.0067 Wb
Average toroidal field	$\langle B_t \rangle$ =	0.12 T
Toroidal field at wall	B_{tw} =	-0.03 T
Poloidal field at wall	B_{pw} =	0.19 T
DC current in toroidal field circuit	I_t =	-230 kA
Amplitude of toroidal current oscillation	$\|\delta I_t\|$ =	180 kA
Plasma current	I_p =	500 kA
Amplitude of plasma current oscillation	$\|\delta I_p\|$ =	29 kA
Plasma resistive voltage	V_R =	10 V
Plasma resistance	R_p =	20 $\mu\Omega$
Plasma inductance	L =	2.4 μH
Plasma coupling coefficient	A =	-1.42
Poloidal gap ac impedance	Z_p =	2.04+14.8j mΩ
Toroidal gap ac impedance	Z_t =	0.18+0.16j mΩ
Poloidal gap input power	P_p =	874 kW
Toroidal gap input power	P_t =	2846 kW

Table II

MST F-Theta Pumping Design Example (Tokamak Case)

Quantity	Symbol	Value
Major radius	R_o =	1.5 m
Minor radius	a =	0.52 m
Characteristic inductance	$\mu_o R_o$ =	1.885 μH
Ratio of v_p/v_t	α =	0.504
Amplitude of poloidal gap voltage	v_p =	71 V
Amplitude of toroidal gap voltage	v_t =	140 V
Oscillation frequency	f =	1000 Hz
Phase angle	ϕ =	90°
Field reversal parameter	F =	0.99
Amplitude of F oscillation	$\|\delta F\|$ =	0.0015
Dimensionless δF	G =	0.162
Pinch parameter	Θ =	0.14
Amplitude of Θ oscillation	$\|\delta\Theta\|$ =	0.014
Safety factor (at r=a)	q =	2.5
Amplitude of q oscillation	$\|\delta q\|$ =	0.26
Toroidal flux	Φ =	0.51 Wb
Amplitude of toroidal flux oscillation	$\|\delta\Phi\|$ =	0.022 Wb
Average toroidal field	$\langle B_t \rangle$ =	0.6 T
Toroidal field at wall	B_{tw} =	0.59 T
Poloidal field at wall	B_{pw} =	0.083 T
DC current in toroidal field circuit	I_t =	4.47 MA
Amplitude of toroidal current oscillation	$\|\delta I_t\|$ =	200 kA
Plasma current	I_p =	216 kA
Amplitude of plasma current oscillation	$\|\delta I_p\|$ =	16 kA
Plasma resistive voltage	V_R =	1 V
Plasma resistance	R_p =	4.63 $\mu\Omega$
Plasma inductance	L =	0.725 μH
Plasma coupling coefficient	A =	0.196
Poloidal gap ac impedance	Z_p =	-1.54+3.96j mΩ
Toroidal gap ac impedance	Z_t =	0.021+0.696j mΩ
Poloidal gap input power	P_p =	-213 kW
Toroidal gap input power	P_t =	429 kW

Figure Captions

1. Ratio (α) of poloidal to toroidal oscillating voltages which optimize F-Theta pumping versus Θ for a given aspect ratio R_O/a.

2. A function proportional to the variation of F for a plasma held in steady state by F-Theta pumping versus the phase angle between the oscillating voltages on the poloidal and toroidal gaps. A small phase angle results in negligible variation of F but requires enormous gap voltages.

3. Dimensionless constant (G) used to predict the variation of F during F-Theta pumping versus Θ for voltages whose ratio is optimized and whose phase difference is maintained at $\phi = 90°$.

4. Plasma current (a) and field reversal parameter (b) resulting from numerical modeling of MST at an initial theta of 1.6 without F-Theta pumping (smooth curves) and with F-Theta pumping optimized according to the perturbation analysis (wiggly curves). The perturbation analysis overestimates the F-Theta pumping requirements (plasma current ramps up with pumping). Without F-Theta pumping, reversal is quickly lost because an ideal toroidal-field crowbar has been assumed, whereas the plasma current resistively decays.

5. Plasma current (a) and field reversal parameter (b) resulting from numerical modeling of MST at an initial theta of 1.6 without F-Theta pumping (smooth curves) and with F-Theta pumping with the oscillating gap voltages adjusted to maintain an average steady state (wiggly curves). Note that F remains comfortably negative, although its oscillation is large.

6. Plasma current-density profile predicted by the Polynomial Function Model for a tokamak with q = 2.5. The profile is somewhat broader than is generally observed in tokamak experiments but not wholly unrealistic.

7. Plasma current (a) and safety factor (b) resulting from numerical modeling of MST at an initial q of 2.5 without F-Theta pumping (smooth curves) and with F-Theta pumping optimized according to the perturbation analysis (wiggly curves). The perturbation analysis overestimates the F-Theta pumping requirements (plasma current ramps up with pumping). Without F-Theta pumping, q rises quickly because an ideal toroidal-field crowbar has been assumed, whereas the plasma current resistively decays.

8. Plasma current (a) and safety factor (b) resulting from numerical modeling of MST at an initial q of 2.5 without F-Theta pumping (smooth curves) and with F-Theta pumping with the oscillating gap voltages adjusted to maintain an average steady state (wiggly curves). Note that q remains comfortably above 2.0, although its oscillation is large.

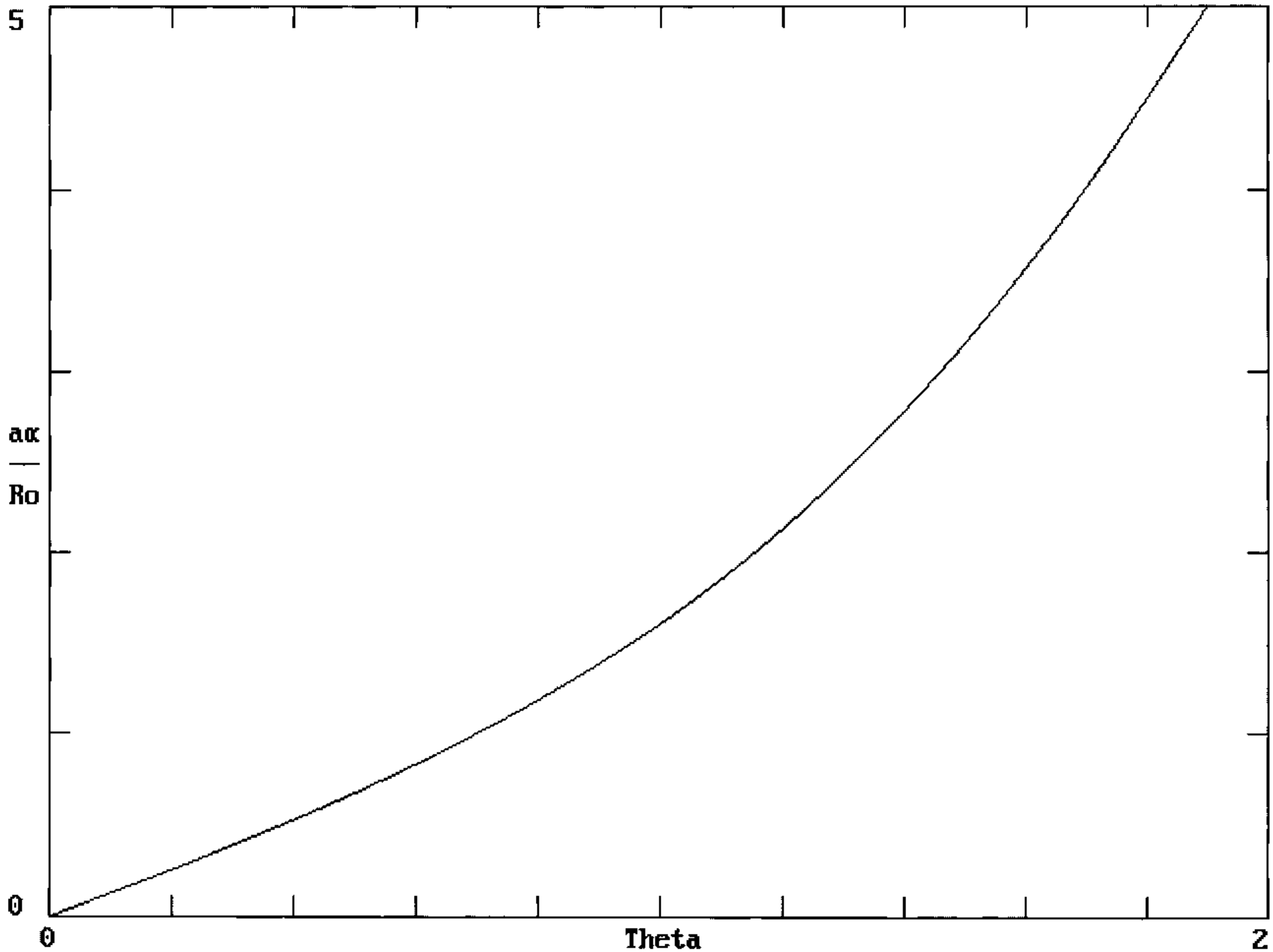


FIG. 1

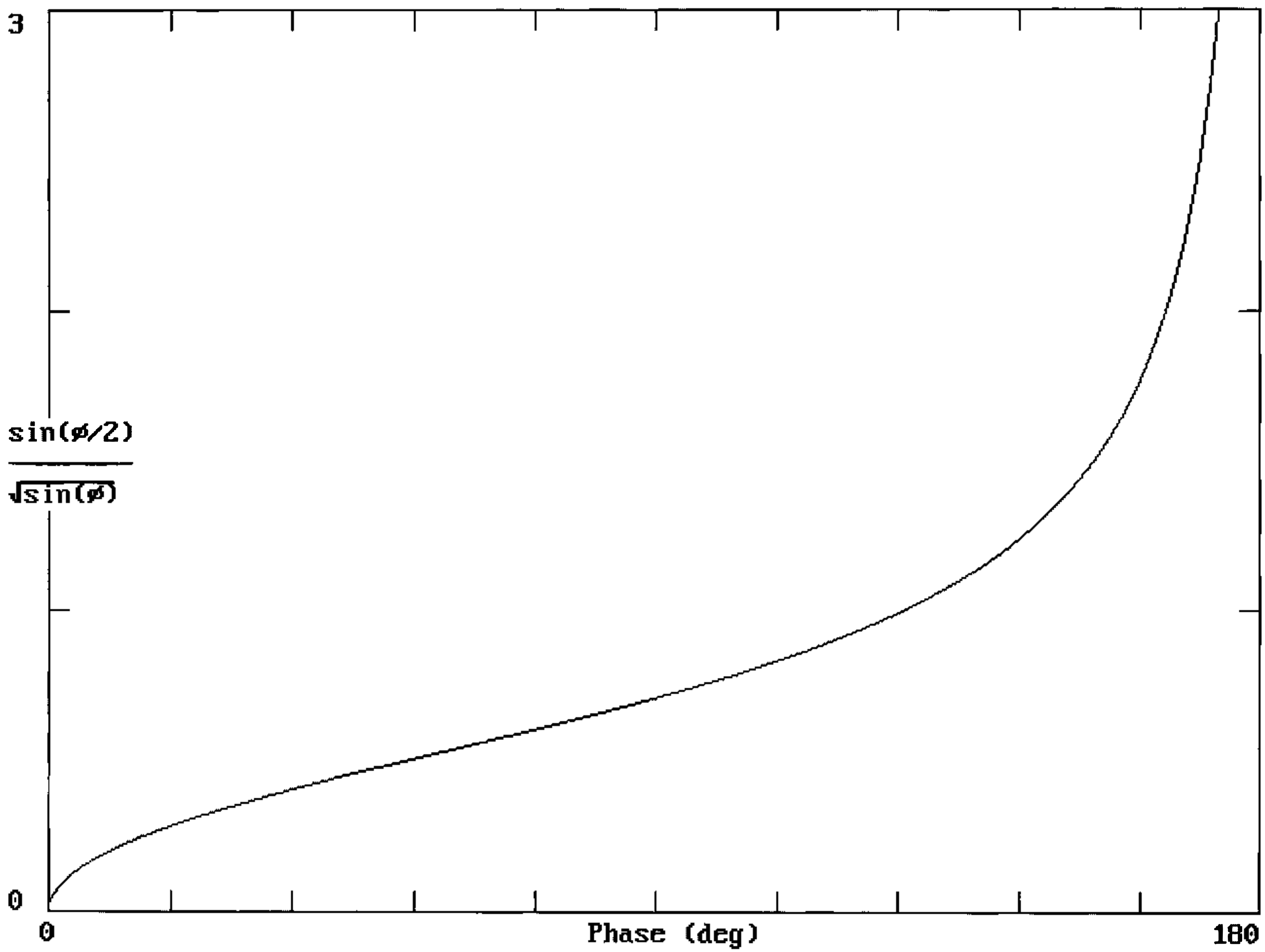


FIG. 2

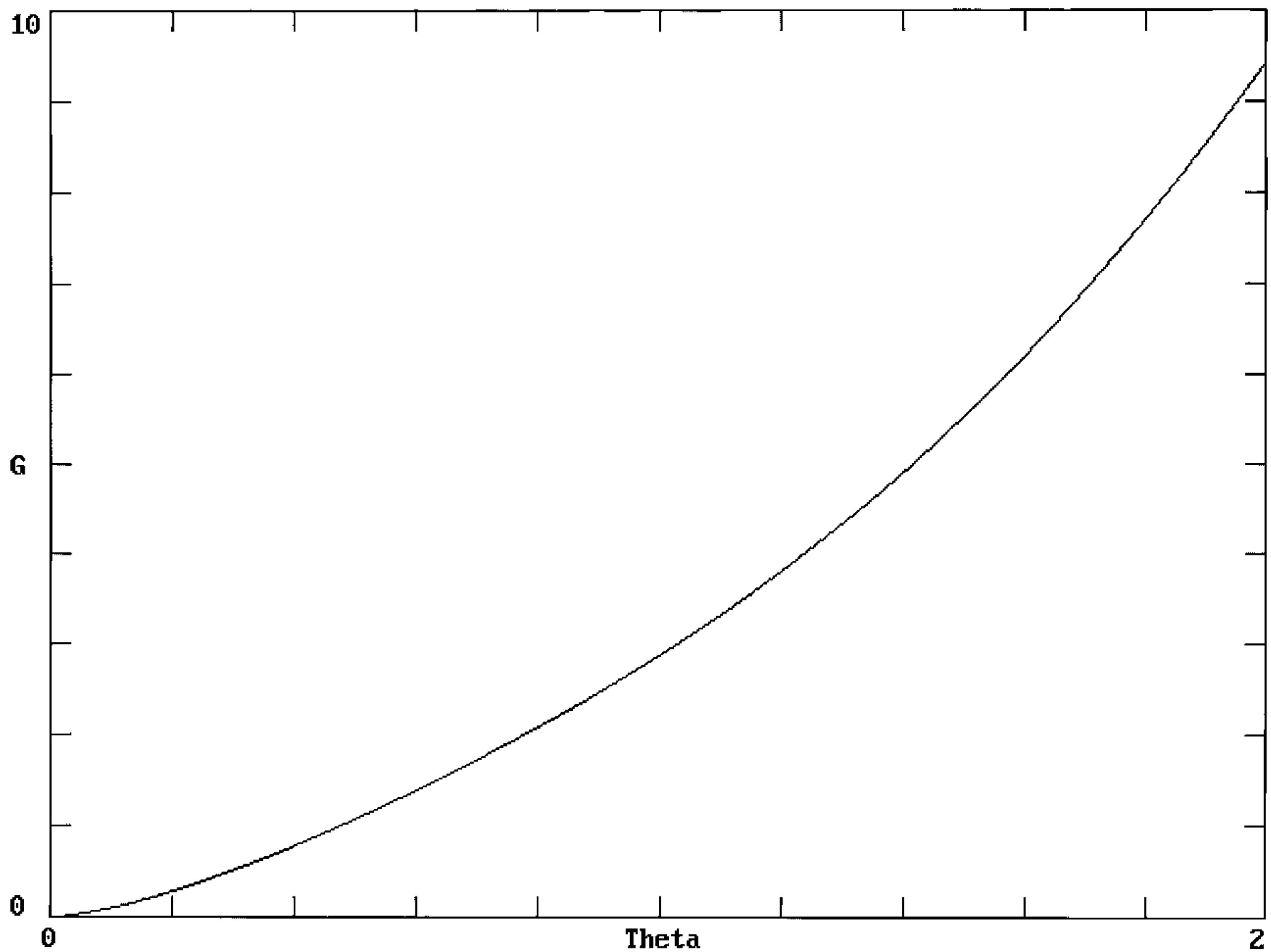
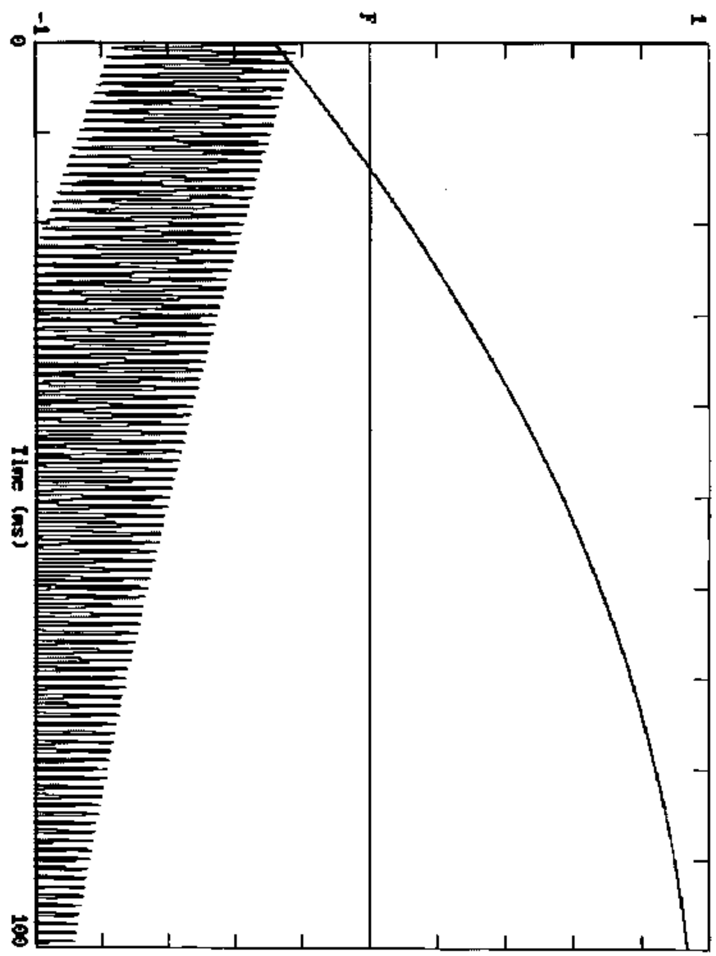
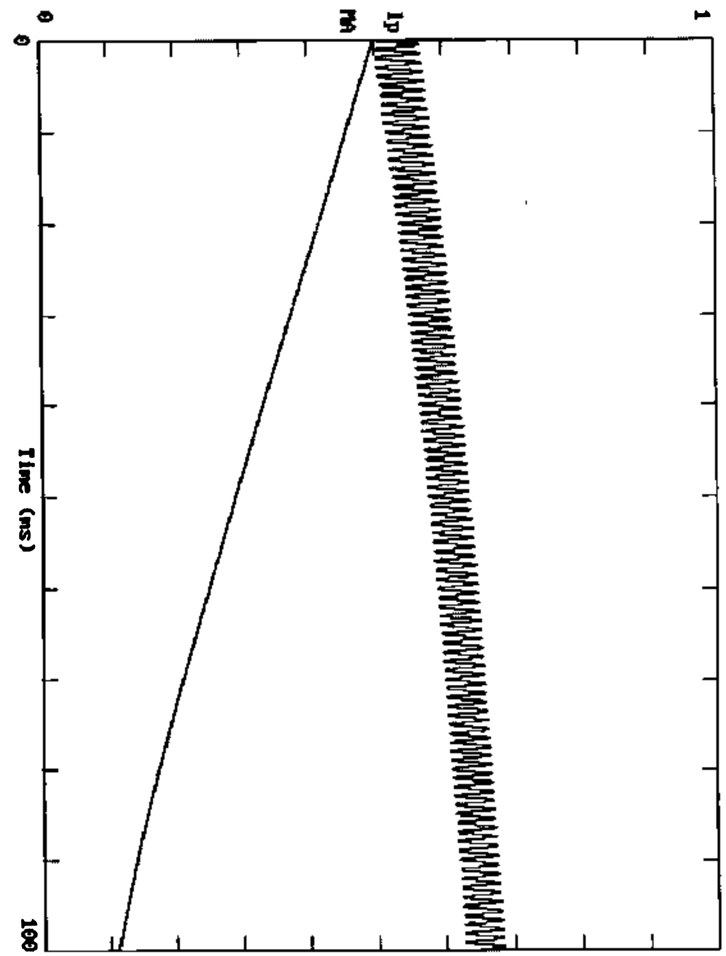


FIG. 3

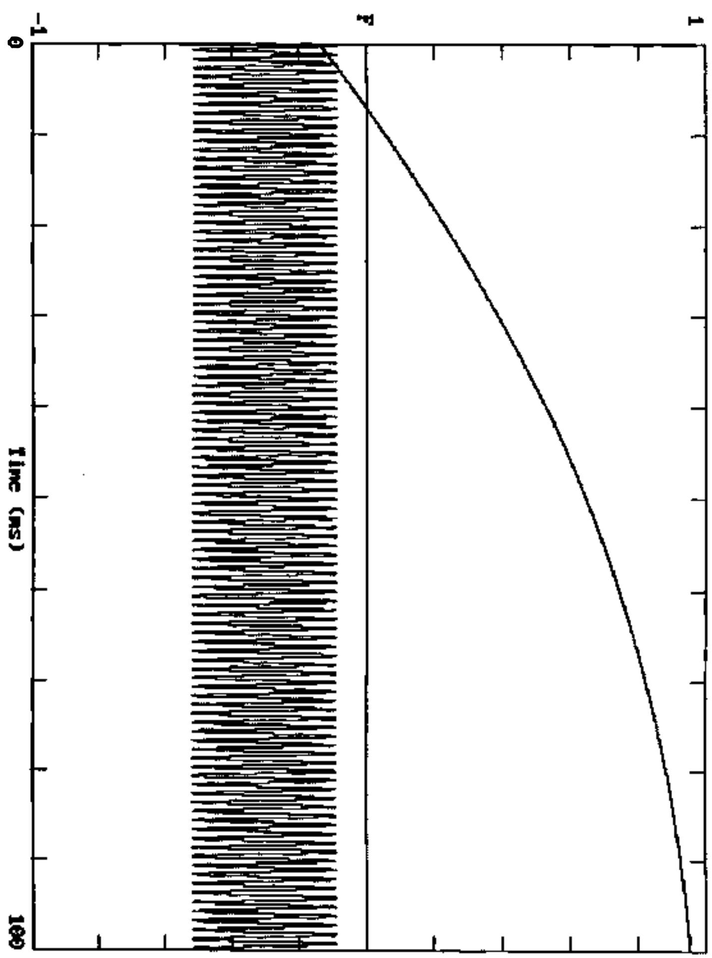


(a)

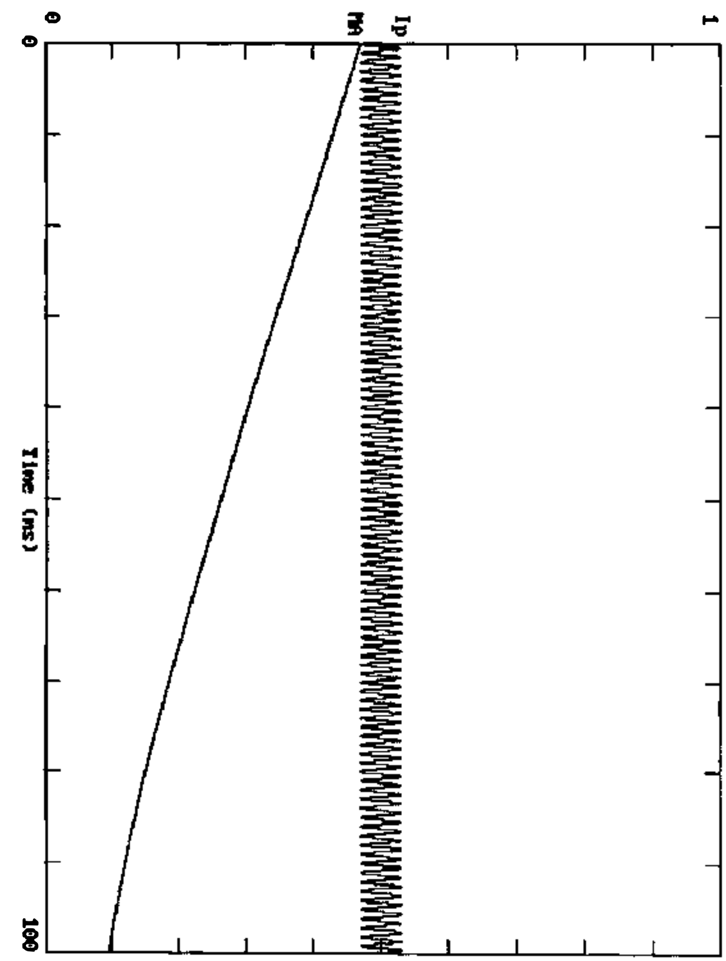


(b)

FIG. 4



(a)



(b)

FIG. 5

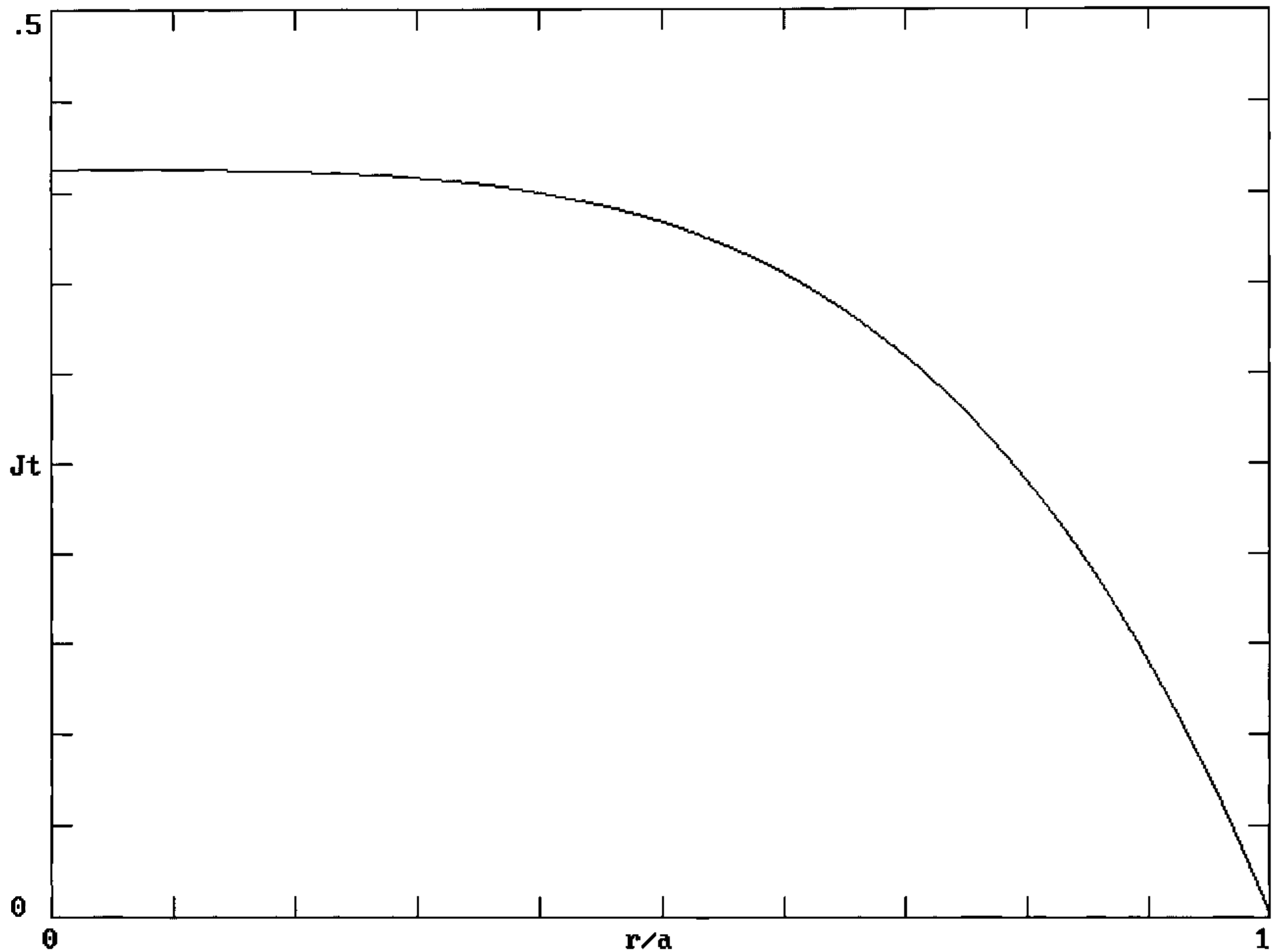
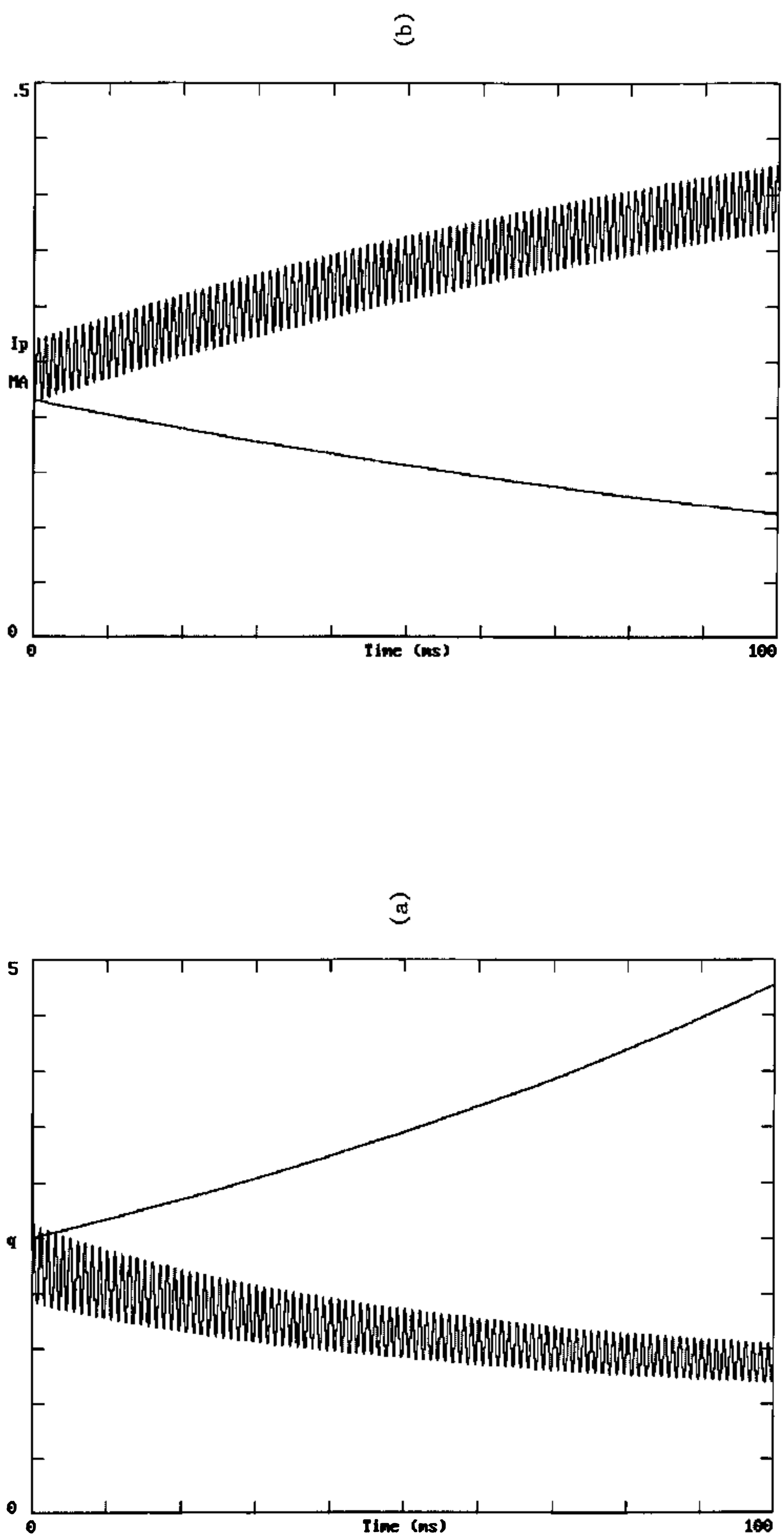
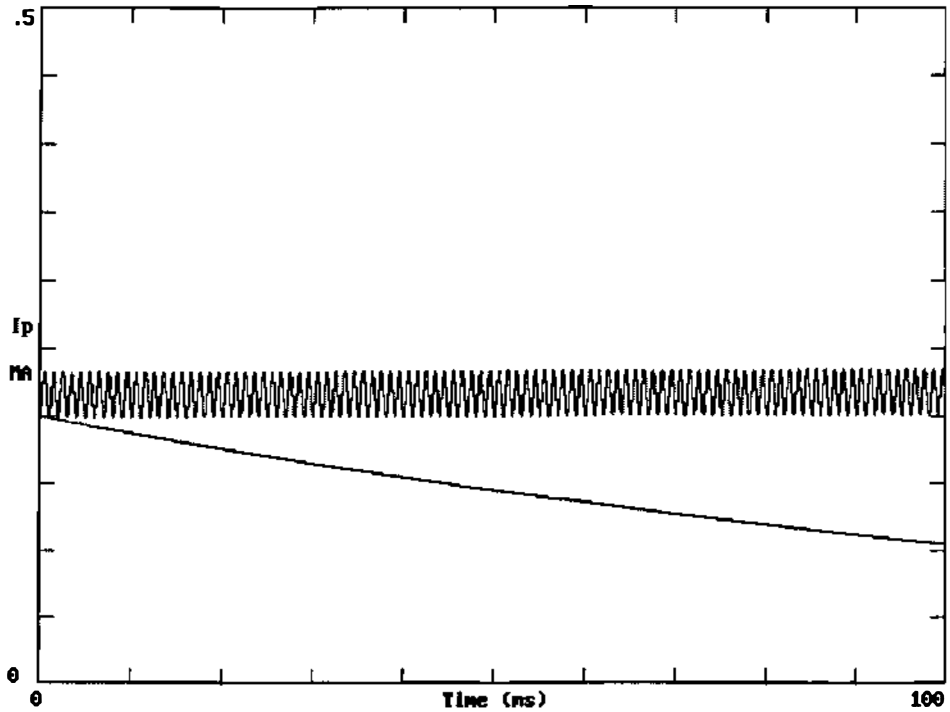


FIG. 6



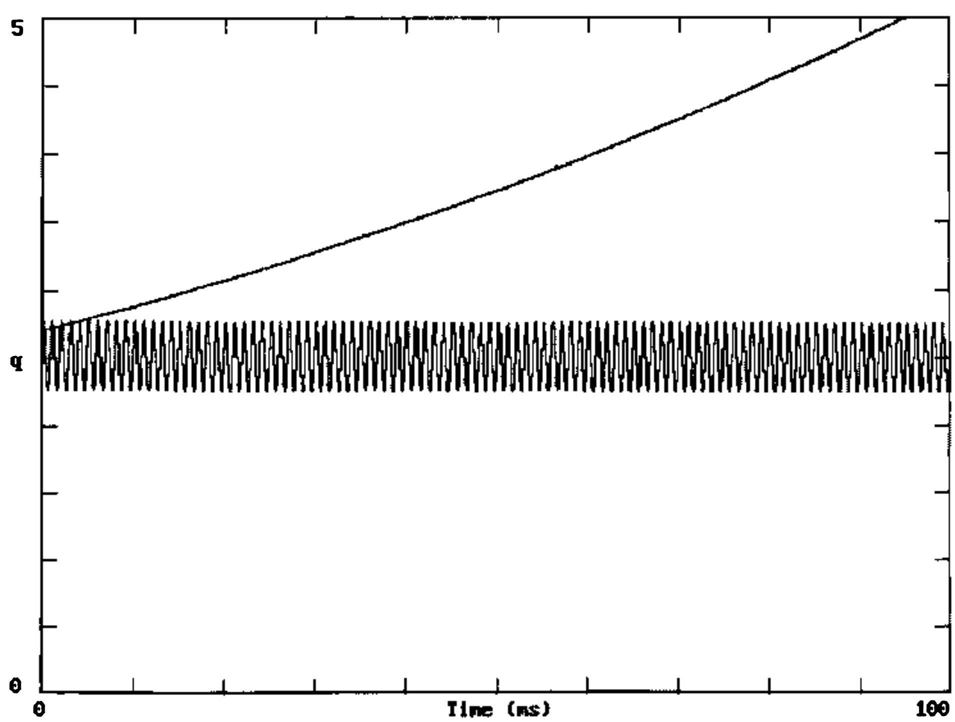
(b)
(a)

FIG. 7



(a)

FIG. 8



(b)