A Single Particle Approach to Cyclotron Heating
in a Non-Uniform Magnetic Field

by

A. M. El Nadi

and

J. C. Sprott

September 1968

PLP 230

Plasma Studies
University of Wisconsin

In this paper the equation of motion of a charged particle in a particular non-uniform magnetic field is solved explicitly in order to determine the energy gained during one transit through a region of electron cyclotron resonance. A collection of non-interacting particles is then considered in order to estimate the heating rate. The approach is similar to that of Kuckes in a recent paper.[1]

We begin with the following assumptions:

1) Infinite magnetic field, linearly varying with distance:

$$B = B_0 \ (1 + \alpha z).$$

2) Linearly polarized electromagnetic wave propogating along the z-axis with wave number k and frequency ω.

3) Electric field along the x-axis with negligible magnetic field from the wave.

4) Particle with charge e and constant parallel velocity, v_z.

The equations of motion are

$$\frac{dv_x}{dt} = \frac{eE}{m} \sin \ (\omega t - kz + \phi) + \frac{e}{m} v_y B$$

and

$$\frac{dv_y}{dt} = - \frac{e}{m} v_x B.$$

The phase angle ϕ is introduced to allow us to assume the particle is at z = 0 at time t = 0, still with freedom to choose the plane z = 0. Multiplying the second equation of motion by i (= $\sqrt{-1}$) and adding the result to the first equation gives

$$\frac{d}{dt} (v_x + iv_y) = \frac{eE}{m} \sin \ (\omega t - kz + \phi) - i \frac{eB}{m} (v_x + iv_y). \qquad (1)$$

Now let $v \equiv v_x + iv_y$ where $|v| = v_\perp$ is the perpendicular velocity of the particle. Equation (1) then becomes

$$\frac{dv}{dt} + i\,\frac{eB}{m}\,v = \frac{eE}{m}\sin\,(\omega t - kz + \phi). \tag{2}$$

Because of the freedom of choosing the plane z = 0 (or in other words B_0), we can write

$$\omega_c = \frac{eB_0}{m} = \omega - kv_z$$

as a definition of B_0. Substituting B and z in equation (2) gives

$$\frac{dv}{dt} + i\,\frac{eB_0}{m}\,(1 + \alpha z)\;v = \frac{eE}{m}\sin\,(\omega t - kv_z + \phi)$$

or

$$\frac{dv}{dt} + i\omega_c(1 + \alpha v_z t)v = \frac{eE}{m}\sin\,(\omega_c t + \phi). \tag{3}$$

Equation (3) is the basic equation to be solved in this paper.

To simplify, let

$$v = Ve^{-i\omega_c(t + \alpha v_z t^2/2)}$$

so that equation (3) becomes

$$\frac{dV}{dt} = \frac{eE}{m}\,e^{i\omega_c(t + \alpha v_z t^2/2)}\,\sin(\omega_c t + \phi). \tag{4}$$

But

$$\sin\,(\omega_c t + \phi) = \frac{1}{2i}\,[e^{i(\omega_c t + \phi)} - e^{-i(\omega_c t + \phi)}],$$

so that equation (4) becomes

$$\frac{dV}{dt} = \frac{eE}{2mi}\,[e^{i(2\omega_c t + \alpha\omega_c v_z t^2/2 + \phi)} - e^{i(\alpha\omega_c v_z t^2/2 - \phi)}]. \tag{5}$$

A particle with zero perpendicular velocity at $t = -\infty$ will have at $t = +\infty$,

$$V(\infty) = \frac{eE}{2mi} \int_{-\infty}^{\infty} [e^{i(2\omega_c t + \alpha\omega_c v_z t^2/2 + \phi)} - e \quad c \; z \qquad . \tag{6}$$

By completing the square in the first term of the integrand,

$$\alpha\omega_c v_z t^2/2 + 2\omega_c t + \phi \quad \frac{\alpha\omega_c v_z}{2} (t + \frac{2}{\alpha v_z})^2 + \phi \quad \frac{2\omega}{\quad z}$$

equation (6) becomes a sum of two Fresnel integrals with the solution

$$V(\infty) = \frac{eE}{2mi} \sqrt{\frac{2\pi}{\alpha\omega_c v_z}}\, e^{i\pi/4} [e^{i(\phi - \frac{2\omega_c}{\alpha v_z})} - e^{-i\phi} \; ,$$

or

$$\lim_{t\to\infty} v = \frac{eE}{m} \sqrt{\frac{\pi}{2\alpha\omega_c v_z}}\, e^{-i\pi/4}\, e^{-i\omega_c(t + \alpha v_z t^2/2)} [e^{i(\phi - \frac{2\omega}{\alpha v_z}} - e \quad .$$

Since we are interested only in $|\lim_{t\to\infty} v| = v_\perp(\infty)$, we note that

$$|e^{i(\phi} \quad c \quad z - e \qquad c \quad z \qquad c \quad z$$

Then

$$v_\perp(\infty) = \frac{eE}{m} \sqrt{\frac{\pi}{2\alpha\omega_c v_z}} \,|\, [\cos(\phi - 2\omega_c/\alpha v_z) - \cos\phi \quad \sin(\phi - 2\omega_c/\alpha v_z) + \sin\phi] \; .$$

The corresponding energy gain is

$$\Delta W_\perp = \frac{1}{2} m v_\perp^2 (\infty) = \frac{\pi e^2 E^2}{2m\alpha\omega_c v_z} [1 - \cos 2(\phi - \omega_c/\alpha v_z)].$$

Note that the energy gain is always positive (as required for $v_\perp(-\infty) = 0$), and can take on any value between zero and $\pi e^2E^2/ m\alpha\omega_c v_z$ depending on the phase ϕ. For a group of particles incident on the resonance region with random phases, we can consider all phase angles equally likely, so that the average energy gain per particle is given by

$$\overline{\Delta W_\perp} = \langle \Delta W_\perp \rangle_\phi = \frac{\pi e^2 E^2}{2m\alpha\omega_c v_z}. \tag{7}$$

This result is identical to equation (10) of Kuckes paper (where $\omega_c' = \alpha\omega_c$).

Now consider a group of non-interacting particles, all moving with the same velocity, v_z. The power absorbed per unit area normal to z is given by

$$\frac{dP}{dA} = n\overline{\Delta W_\perp} v_z \tag{8}$$

where n is the density of particles. But dA can be written in terms of the magnetic flux ψ as

$$d\psi = B_0 dA,$$

so that equation (8) becomes

$$\frac{dP}{d\psi} = \frac{\pi}{2}\ \frac{neE^2}{\alpha B_0{}^2}.$$

Note that the power absorbed by the particles is independent of their parallel velocity, since more fast particles pass through resonance per unit time but spend less time in resonance than do the slow particles. The effect of the parallel velocity is to doppler shift the resonance away from the plane $\omega = \omega_c$, or in the case of a spectrum of velocities, a doppler broadening occurs. The result can be carried one step further by noting that the differential resonance volume can be written as

$$d^2V = dAdz = \frac{d\psi}{B_0}\ \frac{dB}{\alpha B_0},$$

so that

$$\frac{dP}{d\psi} = \frac{\pi}{2}\, neE^2\, \frac{d^2V}{dBd\psi}\ . \tag{9}$$

This result is identical to equation (4) of PLP 213, but the derivations are quite different. (There is a factor of 2/3 difference, but this comes from the fact that the electric field in PLP 213 was assumed random such that $E_{\perp}^{2} = \frac{2}{3}\,\overline{E^2}$.)

References

1. A. F. Kuckes, Plasma Physics 10, 367, (1968).