Calculations of Ion Heating by ECRH

by

J. C. Sprott

January, 1970

Plasma Studies
University of Wisconsin

Calculations of Ion Heating by ECRH
by
J. C. Sprott

Departments of Physics & Electrical Engineering

Kuswa[1] has observed that when microwaves are used to heat electrons at
the electron cyclotron frequency in a toroidal octupole with low neutral gas
pressure, there is a significant ion heating, much larger than can be ex-
plained by electron-ion coulomb collisions. Consoli, et.al.,[2] have observed
a similar effect when a cloud of cold plasma moving along a field line falls
through a magnetic gradient where electron cyclotron resonance heating occurs.
The electrons gain perpendicular energy as they pass through resonance, and
the $-\mu_e \nabla_\parallel B$ force accelerates the electrons along the field, producing an
electric field that drags along the ions. The theoretical description pro-
posed by Consoli[2] and others, however, is inadequate for explaining the
heating in multipoles and other devices where the plasma is initially spread
out along the magnetic field and completely encompasses the resonance zone.

In this PLP, we will calculate the ion heating produced by a change in
the anisotropy of the electron distribution under certain limiting conditions
that approximate the experimental conditions in multipoles and similar de-
vices. We assume that the plasma is characterized initially by isotropic,
nearly Maxwellian, electron and ion distributions with different temperatures.
An anisotropy is then introduced in the electron distribution by, for example,
preferentially increasing the perpendicular energy by electron cyclotron
resonance heating. The electron density then increases in the region of small
magnetic field creating an electrostatic potential well in which the ions
oscillate. The electric field is solved self-consistently for the special
case in which the form of the ion distribution at the local magnetic field
minimum remains unchanged except for a change in temperature. The result

will be to show that if the anisotropy develops gradually, the ions behave adiabatically and undergo a compressional heating followed by a corresponding cooling when the anisotropy slowly disappears. If the anisotropy appears suddenly, however, an irreversible, non-adiabatic heating occurs.

Let ℓ be the length along a magnetic field line measured from a local field minimum. Assume the electron and ion energy distribution functions at $\ell=0$ are given by

$$
\left\{
\begin{array}{l}
f_e(W_{\perp e}, W_{\| e}, 0) = A_e \exp(-\dfrac{W_{\perp e}(0)}{kT_{\perp e}})\exp(-\dfrac{W_{\| e}(0)}{kT_{\| e}}) \\[4mm]
f_i(W_{\perp i}, W_{\| i}, 0) = A_i \exp(-\dfrac{W_{\perp i}(0)+W_{\| i}(0)}{kT_i}) \ .
\end{array}
\right.
$$

In the above equations and throughout this paper, a quantity such as $W_{\perp eo}(0)$ means the component of electron energy perpendicular to $\vec{B}$; the subscript o means time t=0, while the (0) means position $\ell=0$. The assumed distribution functions are nearly Maxwellian and represent a good approximation to the distribution in toroidal geometries and in open ended systems with a high mirror ratio where the loss cones are small. The electron distribution can be written in terms of the magnetic moment μ_e and total energy $W_e-e\Phi$, where Φ is the electrostatic potential (taken as zero at $\ell=0$), as

$$
f_e(\mu_e, W_e-e\Phi) = A_e \exp(-\dfrac{\mu_e B(0)}{kT_{\perp e}})\exp(-\dfrac{W_e-e\Phi-\mu_e B(0)}{kT_{\| e}}).
$$

If the magnetic moment and energy are conserved, the distribution function written in this form is valid at all ℓ for which B monatonically increases from $\ell=0$, and can therefore be written as

$$
f_e(W_{\perp e}, W_{\| e}, \ell) = A_e \exp[-W_{\perp e}(\dfrac{B(\ell)kT_{\perp e}-B(0)(kT_{\perp e}-kT_{\| e})}{B(\ell)kT_{\perp e}kT_{\| e}})]\exp(-\dfrac{W_{\| e}}{kT_{\| e}})\exp(\dfrac{e\phi(\ell)}{kT_{\| e}}).
$$

The density distribution is determined by integration:

$$n_e(\ell) = \int_0^\infty \int_0^\infty f_e(W_{\perp e}, W_{\parallel e}, \ell) dW_{\perp e} dW_{\parallel e} = \frac{n_e(0)B(\ell)}{B(0)+[B(\ell)-B(0)]\theta} \exp\left(\frac{e\Phi(\ell)}{kT_{\parallel e}}\right) \quad ,$$

where θ is the electron anisotropy:

$$\theta = T_{\perp e}/T_{\parallel e} \quad .$$

Note that for $\theta=1$ (isotropic electrons), n_e is independent of B and has the familiar exponential dependence on Φ. The ions are assumed to remain isotropic, giving the simpler result:

$$n_i(\ell) = n_i(0) \exp\left(-\frac{e\Phi(\ell)}{kT_i}\right) \quad .$$

If we neglect density gradients perpendicular to $\vec{B}$ ($\nabla_\perp n=0$), Poisson's equation can be written in the one-dimensional form,

$$\frac{d^2\Phi}{d\ell^2} = [n_e(\ell) - n_i(\ell)]e/\varepsilon_o \quad .$$

The problem is linearized in the usual way,

$$\exp\left(\frac{e\Phi}{kT}\right) \cong 1 + \frac{e\Phi}{kT} \quad ,$$

to obtain the differential equation,

$$\frac{d^2\Phi}{d\ell^2} - \frac{ne^2}{\varepsilon_o k}\left(\frac{T_i+T_{\parallel e}}{T_i T_{\parallel e}}\right)\Phi = \frac{e}{\varepsilon_o}\left[\frac{n_e(0)B(\ell)}{B(0)+[B(\ell)-B(0)]\theta} - n_i(0)\right] .$$

The first term on the right hand side can be expanded in a taylor series about $\ell=0$:

$$\frac{B(\ell)}{B(0)+[B(\ell)-B(0)]\theta} \cong 1 - (\theta-1)\beta\ell^2 \quad ,$$

where

$$\beta = \frac{1}{2B(0)} \left.\frac{d^2B}{d\ell^2}\right|_{\ell=0} \quad .$$

The differential equation then becomes

$$\frac{d^2\Phi}{d\ell^2} - \frac{\Phi}{\lambda^2} = \frac{e}{\varepsilon_o} [n_e(0) - n_i(0) - n_e(0)(\theta-1)\beta\ell^2],$$

where we recognize the Debye length,

$$\lambda = \sqrt{\frac{\varepsilon_o k}{ne^2} \left(\frac{T_i T_{\parallel e}}{T_i + T_{\parallel e}}\right)} \quad .$$

Note that for an isotropic plasma ($\theta=1$), $n_e(0)=n_i(0)$, and all potentials fall off exponentially in the order of a Debye length. For $\theta \neq 1$, the potential has the parabolic form:

$$\Phi = \frac{k\beta}{e} \left(\frac{T_i T_{\parallel e}}{T_i + T_{\parallel e}}\right)(\theta-1)\ell^2 \quad .$$

The potential vanishes if the electrons are isotropic or if either the electrons or ions are cold.

Now consider the parallel equation of motion of an ion:

$$M\dot{v}_{\parallel i} = -e \frac{d\Phi}{d\ell} - \mu_i \frac{dB}{d\ell} \quad .$$

Near $\ell=0$, both Φ and B are parabolic in ℓ, and so the equation of motion can be written as

$$\ddot{Z} + \omega^2(t)Z = 0,$$

where

$$\omega^2(t) = \frac{2\beta}{M} \left[\left(\frac{kT_i T_{\parallel e}}{T_i + T_{\parallel e}}\right)(\theta-1) + W_{\perp i}(0)\right].$$

This equation is an example of the classic, time dependent, linear, harmonic oscillator, studied by many authors, most recently, Symon.[3] It is well known that if $\omega(t)$ changes slowly enough ($\dot{\omega}/\omega << \omega$), the action integral is invariant and is given by

$$J = \frac{W_{\|i}(0)}{\omega} \, ,$$

so that the parallel energy is

$$W_{\|i}(0) = \frac{\omega}{\omega_o} W_{\|io}(0).$$

If we assume the electrons are initially isotropic, and that the anisotropy slowly increases to θ, then

$$W_{\|i}^2(0) = \frac{W_{\|io}^2(0)}{W_{\perp io}(0)} \left[\left(\frac{kT_i T_{\|e}}{T_i + T_{\|e}} \right)(\theta-1) + W_{\perp i}(0) \right].$$

An exact treatment at this point would require us to calculate the time evolution of the ion distribution function. Such a calculation is not only difficult, but it violates our initial assumption that the form of the ion distribution function remains unchanged. It is more realistic to assume an isotropic ion distribution than to try to calculate the anisotropy, because non-adiabatic scattering is very strong for energetic ions in non-uniform magnetic fields.[4] We will instead consider a "typical" particle for which

$$kT_i = W_{\|i}(0) = W_{\perp i}(0).$$

Substitution gives

$$T_i^2 + (T_{\|e} - T_{io})T_i - T_{\|e}T_{io}\theta = 0.$$

Solving for T_i,

$$T_i = \frac{1}{2}(T_{io} - T_{\|e}) \pm \frac{1}{2}\sqrt{(T_{\|e} - T_{io})^2 + 4T_{\|e}T_{io}\theta} \, .$$

Only the + sign in front of the radical is physical, since we require $T_i = T_{io}$ for $\theta = 1$. For the usual case of $T_{\|e} \gg T_{io}$, we obtain

$$T_i = T_{io}\theta.$$

Other special cases are

$$
\begin{cases}
T_i = T_{io}\sqrt{\theta} & \text{for } T_{\parallel e} = T_{io} \\
\\
T_i = T_{io} + T_{\parallel e}\theta & \text{for } T_{\parallel e} << T_{io}.
\end{cases}
$$

Unfortunately, the result is in terms of the parameter θ which is difficult to measure experimentally and even more difficult to calculate. We note however, that θ can be determined from the electron density distribution along a field line by

$$
\theta = 1 - \frac{\nabla_{\parallel} n_e}{n_e}\ \frac{B}{\nabla_{\parallel} B} ,
$$

or from the potential distribution by

$$
\theta = 1 + \frac{e}{k}\left(\frac{T_i + T_{\parallel e}}{T_i T_{\parallel e}}\right)\frac{B(0)\nabla_{\parallel}\Phi}{\nabla_{\parallel} B} .
$$

The above equation shows that in assuming $|e\Phi/kT| << 1$ we have implicitly assumed that $\theta - 1 << 1$. For $\nabla_{\parallel}$
tential gradient is identical to the plasma potential gradient. We can also place a theoretical upper limit on θ by noting that the anisotropy ceases to increase when the particles mirror at the resonance, so that

$$
\theta \leq \frac{B(0)}{B_{res} - B(0)} .
$$

This heating calculation is probably inadequate to explain the experimental results for two reasons. First, the electron anisotropy is probably not large enough to produce the observed heating, and second, the ions should cool quickly after the heating is turned off as the electrons become isotropic. This heating is, in fact, just caused by the adiabatic compression of the ions as they try to follow the electrons into the low field region.

A more interesting effect arises when the electron heating is strong and the anisotropy develops so quickly that the ions behave non-adiabatically. Symon[3] has shown that an exact invariant exists for this case, and that the mean value of J, averaged over all initial phases, after a change in ω is given in terms of the initial value J_o as

$$J = J_o\sqrt{1+\omega_1^2 C^2} \ ,$$

where ω_1 is the final ω and C is the amplitude of the non-adiabatic jump. For $\omega_1 C \ll 1$, Symon has shown that C is given approximately by

$$\omega_1 C = \int_{-\infty}^{\infty} \frac{\dot{\omega}}{\omega} e^{2i\phi} dt$$

where

$$\phi = \int^t \omega(t) dt \ .$$

We now assume a special form of $\omega(t)$ that we expect to approximate the actual variation when an anisotropy suddenly develops:

$$\omega(t) = \begin{cases} \omega_o & \text{(for } t<0) \\ \omega_1+(\omega_o-\omega_1)e^{-t/\tau} & \text{(for } t>0). \end{cases}$$

Then,

$$\phi = \omega_1 t + (\omega_1-\omega_o)\tau \ e^{-t/\tau}.$$

Also,

$$\frac{\dot{\omega}}{\omega} = \frac{(\omega_1-\omega_o)e^{-t/\tau}}{\tau[\omega_1+(\omega_o-\omega_1)e^{-t/\tau}]} \cong \frac{\omega_1-\omega_o}{\omega_o\tau} \qquad \text{for } t>0),$$

so

$$\omega_1 C \cong \frac{\omega_1-\omega_o}{\omega_o} e^{2i(\omega_1-\omega_o)\tau} \int_o^{\infty} e^{-t/\tau} e^{2i\omega_o t} dt$$

$$= \frac{\omega_1 - \omega_o}{\omega_o} \left[\frac{1 + 2i\omega_o \tau}{1 + 4\omega_o^2 \tau^2}\right] e^{2i(\omega_1 - \omega_o)\tau} .$$

Squaring gives,

$$\omega_1^2 c^2 = \frac{(\omega_1 - \omega_o)^2}{\omega_o^2 (1 + 4\omega_o^2 \tau^2)} \cong \frac{\dot{\omega}_o^2}{4\omega_o^4} ,$$

where the last approximation is valid for $\omega_o^2 \tau^2 \gg 1$ (i.e., where the anisotropy builds up over many bounce periods). The increase in T_i is then given by

$$T_i^2 = T_{io}^2 \frac{\omega_1^2}{\omega_o^2} \left[1 + \frac{\dot{\omega}_o^2}{4\omega_o^4}\right] .$$

Straightforward differentiation of $\omega(t)$, assuming $\frac{d\theta}{dt} = \frac{1}{T_{\parallel eo}} \frac{dT_{\perp e}}{dt}$ gives

$$\frac{\dot{\omega}_o^2}{4\omega_o^4} = \frac{Mk^2}{32\beta(kT_{io})^3} \left[\left(\frac{T_i}{T_i + T_{\parallel e}}\right) \frac{dT_{\perp e}}{dt} + \frac{dT_i}{dt}\right]_o^2 .$$

We could at this point approximate the last term by

$$\frac{dT_i}{dt} = \frac{T_i - T_{io}}{\tau}$$

and solve the problem exactly, but it is easy to argue on physical grounds that $dT_i/dt \ll dT_{\perp e}/dt$, since the ions are heated indirectly by the electrons. With this approximation, we can solve the quadratic equation for T_i as before to obtain

$$T_i = \frac{1}{2}[T_{io}(1+A) - T_{\parallel e}] + \frac{1}{2}\sqrt{[T_{\parallel e} - T_{io}(1+A)]^2 + 4T_{\parallel e}T_{io}(1+A)\theta} ,$$

where

$$A = \frac{M}{32\beta kT_{io}} \left[\frac{1}{T_i + T_{\parallel e}} \frac{dT_{\perp e}}{dt}\right]_o^2 .$$

For $T_{\parallel e} \gg T_{io}$, T_i is given by

$$T_i = T_{io}\theta(1+A).$$

The first term is the reversible adiabatic heating, and the second term is the irreversible non-adiabatic heating. If the anisotropy slowly disappears after the electron heating is turned off, the resulting ion temperature would be given approximately by

$$T_i \cong \frac{M}{32\beta k} \left(\frac{1}{T_{io}+T_{\parallel eo}}\right)^2 \left(\frac{dT_{\perp e}}{dt}\right)_o^2 \, ,$$

which is to say that the transit time of an ion between its mirror points is the order of the time required for T_e to change appreciably as a result of electron cyclotron heating. Since dT_e/dt is proportional to the microwave power P_o, we can venture the prediction that T_i is proportional to P_o^2. Furthermore, we expect T_i to be fairly independent of microwave pulse length above a certain value since the heating is caused mainly by the turn-on transient. Note also that T_i is independent of plasma density, in contrast to other coupling mechanisms such as coulomb collisions, but that T_i increases in proportion to the ion mass, and is largest when the electrons and ions are initially cold.

The above calculations represent only the lowest order estimate of ion heating by microwaves since nearly all quantities are expanded in a power series, and only the first non-vanishing terms are kept. In particular, it would be interesting to include higher terms in the expansions of $e\Phi/kT$, $B(\ell)$, $\theta(t)$, and $\dot{\omega}/\omega^2$. Also, one should consider other electron distribution functions, such as those experimentally measured, or loss cone distributions for mirror machines. Finally, it would be useful to calculate the time evolution of the ion distribution function including non-adiabatic and coulomb scattering. However, the simple calculation performed here will hopefully provide a starting point for understanding and interpreting experimental observations, and for developing a more exact theory of ion heating by ECRH.

References

1. Glenn W. Kuswa, Bull. Am. Phys. Soc. <u>14</u>, 1033 (1969).

2. T. Consoli, in Proc. of the Third Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Novosibirsk, U.S.S.R. (1968), paper CN-24/J-1.

3. K. R. Symon, to be published (COO-1726-40).

4. J. E. Howard, Ph.D. Thesis, Univ. of Wis. (1969).